Calculus 2, part 1 of 2: Integrals with applications[1]

Single variable calculus

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A short table of contents

[1]Recorded April–September 2024. Published on www.udemy.com on 2024-09-XX.

An extremely detailed table of contents; the videos (titles in green) are numbered

In blue: problems solved on an iPad (the solving process presented for the students; active problem solving)

In red: solved problems demonstrated during a presentation (a walk-through; passive problem solving)

In magenta: additional problems solved in written articles (added as resources).

In dark blue: *Read along with this section*: references for further reading and more practice problems in the *Calculus* book (chapters 5, 6, and 7) by Gilbert Strang and Edwin *Jed* Herman:

`https://math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)`

`https://math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)/zz%3A_Back_Matter/30%3A_Detailed_Licensing`

This book is added as a resource to Video 1, with kind permission of the LibreTexts Office (given on July 20th, 2023).

S1 Introduction to the course

You will learn: about the content of this course and about importance of Integral Calculus. The purpose of this section is *not* to teach you all the details (this comes later in the course) but to show you the big picture.

1 Introduction to the course.

Extra material: Gilbert Strang & Edwin *Jed* Herman: **Calculus**, OpenStax, as described above.

Extra material: the formula sheet from the course *Precalculus 3: Trigonometry.*

Extra material: the formula sheet from the course *Precalculus 4: Exponentials and logarithms.*

Extra material: an article with main integration techniques and where to find them.

Extra material: this list with all the movies and problems.

2 Two types of integrals, two ways to go.

3 My choices versus the choices in the book.

4 The spoiler you need to follow both paths.

5 Main integration techniques and where to find them.

6 Plenty of applications.

S2 Basic formulas for differentiation in reverse

You will learn: the concept of antiderivative (primitive function, indefinite integral); formulas for the derivatives of basic elementary functions in reverse.

Read along with this section: **Calculus book**: Chapter 4.10 *Antiderivatives*, pages: from 550 (4.10.1) to 564 (4.10E.4).

7 Reverting differentiation in simple cases.

Basic integration formulas:

* $\int e^x\,dx = e^x + C$
* $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$
* $\int x^\alpha\,dx = \dfrac{x^{\alpha+1}}{\alpha+1} + C, \quad \alpha \neq -1$
* $\int \cos x\,dx = \sin x + C$

* $\int \sin x\,dx = -\cos x + C$
* $\displaystyle\int \frac{1}{\cos^2 x}\,dx = \tan x + C$
* $\displaystyle\int \frac{1}{1+x^2}\,dx = \arctan x + C$
* $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$
* $\int a^x\,dx = \dfrac{a^x}{\ln a} + C, \quad a > 0, \;\; a \neq 1.$

8 Some important facts about primitive functions.

9 Integrals of hyperbolic functions and some related stuff.

More integration formulas:

* $\int \sinh x\,dx = \cosh x + C$
* $\int \cosh x\,dx = \sinh x + C$
* $\displaystyle\int \frac{1}{\cosh^2 x}\,dx = \tanh x + C$
* $\displaystyle\int \frac{1}{\sqrt{1+x^2}}\,dx = \ln(x + \sqrt{x^2+1}) + C.$

10 Linearity of integration.

An important fact: For each pair of constants $\alpha,\ \beta \in \mathbb{R}$ and for each pair continuous functions $f,\ g : [a,b] \to \mathbb{R}$ we have

$$\int [\alpha f(x) + \beta g(x)]\,dx = \alpha \cdot \int f(x)\,dx + \beta \cdot \int g(x)\,dx.$$

This rule can be extended (using induction) to any number of functions.

11 Linearity of integration, Exercise 1.

Exercise 1: Compute $\int (6x^2 - 3x + 5)\,dx$.

Extra material: notes with solved Exercise 1.

12 Linearity of integration, Exercise 2.

Exercise 2: Compute $\int (2x^2 + 1)^3\,dx$.

Extra material: notes with solved Exercise 2.

13 Linearity of integration, Exercise 3.

Exercise 3: Compute $\int (1 + \sqrt{x})^4\,dx$.

Extra material: notes with solved Exercise 3.

14 Linearity of integration, Exercise 4.

Exercise 4: Compute $\displaystyle\int \frac{(x+1)(x^2-3)}{3x^2}\,dx$.

Extra material: notes with solved Exercise 4.

15 Linearity of integration, Exercise 5.

Exercise 5: Compute $\displaystyle\int \frac{(x-\sqrt{x})(1+\sqrt{x})}{\sqrt[3]{x}}\,dx$.

Extra material: notes with solved Exercise 5.

16 Linearity of integration, Exercise 6.

Exercise 6: Compute $\displaystyle\int \frac{\sqrt{x^4 + x^{-4} + 2}}{x^3}\,dx$.

Extra material: notes with solved Exercise 6.

17 Linearity of integration, Exercise 7.

Exercise 7: Compute $\displaystyle\int \left(2^x + \sqrt{\tfrac{1}{x}}\right)\,dx$.

Extra material: notes with solved Exercise 7.

18 Linearity of integration, Exercise 8.

Exercise 8: Compute $\int (2^x + 3^x)^2\,dx$.

Extra material: notes with solved Exercise 8.

19 Linearity of integration, Exercise 9.

Exercise 9: Compute $\displaystyle\int \frac{2^{x+1} - 5^x}{10^x}\,dx$.

Extra material: notes with solved Exercise 9.

20 Linearity of integration, Exercise 10.

Exercise 10: Compute $\displaystyle\int \left(4\cos x - \frac{5}{\sqrt{9 - 9x^2}}\right)\,dx$.

Extra material: notes with solved Exercise 10.

21 Linearity of integration, Exercise 11.

Exercise 11: Compute $\displaystyle\int \frac{x^2}{1 + x^2}\,dx$.

Extra material: notes with solved Exercise 11.

22 Linearity of integration, Exercise 12.

Exercise 12: Compute $\displaystyle\int \frac{x^5 - x + 1}{x^2 + 1}\,dx$.

Extra material: notes with solved Exercise 12.

23 Linearity of integration, Exercise 13.

Exercise 13: Compute $\displaystyle\int \frac{\sqrt{1 + x^2} - \sqrt{1 - x^2}}{\sqrt{1 - x^4}}\,dx$.

Extra material: notes with solved Exercise 13.

24 Linearity of integration, Exercise 14.

Exercise 14: Derive the formula for the derivative of $f(x) = \cot x$. Compute then $\displaystyle\int \frac{1}{\sin^2 x \cos^2 x}\,dx$.

Extra material: notes with solved Exercise 14.

25 Linearity of integration, Exercise 15.

Exercise 15: Compute $\int \cot^2 x\,dx$.

Extra material: notes with solved Exercise 15.

26 A soft introduction to variable substitution.

Given $a \neq 0$ and $b \in \mathbb{R}$; f is any continuous function. If $\int f(t)\,dt = F(t) + C$, then

$$\int f(ax + b)\,dx = \frac{1}{a}F(ax + b) + C.$$

The most common cases are when $a = 1$ or $b = 0$. We get then the following:

$$\int f(x+b)\,dx = F(x+b) + C, \qquad \int f(ax)\,dx = \frac{1}{a}F(ax) + C.$$

Some examples:

* $\int e^{-3x}\,dx = -\frac{1}{3}e^{-3x} + C$
* $\displaystyle\int \frac{1}{x-b}\,dx = \ln|x-b| + C$
* $\displaystyle\int \frac{1}{(x-b)^k}\,dx = \int (x-b)^{-k}\,dx = \frac{(x-b)^{-k+1}}{-k+1} + C = -\frac{1}{(k-1)(x-b)^{k-1}} + C, \quad k \neq 1$
* $\int \cos mx\,dx = \frac{1}{m}\cdot \sin mx + C, \quad m \neq 0$
* $\int \sin mx\,dx = -\frac{1}{m}\cdot \cos mx + C, \quad m \neq 0$
* $\displaystyle\int \frac{1}{b^2+x^2}\,dx = \frac{1}{b^2}\cdot \int \frac{1}{1+(\frac{x}{b})^2}\,dx = \tfrac{1}{b^2}\cdot b\cdot \arctan\tfrac{x}{b} + C = \tfrac{1}{b}\cdot \arctan\tfrac{x}{b} + C, \quad b \neq 0$
* $\displaystyle\int \frac{1}{\sqrt{b^2-x^2}}\,dx = \frac{1}{b}\cdot \int \frac{1}{\sqrt{1-(\frac{x}{b})^2}}\,dx = \tfrac{1}{b}\cdot b\cdot \arcsin\tfrac{x}{b} + C = \arcsin\tfrac{x}{b} + C, \quad b > 0.$

27 Easy variable substitution, Exercise 16.

Exercise 16: Compute $\displaystyle\int \frac{1}{2+3x^2}\,dx$.

Extra material: notes with solved Exercise 16.

28 Easy variable substitution, Exercise 17.

Exercise 17: Compute $\displaystyle\int \frac{(e^x-1)(e^{2x}+1)}{e^x}\,dx$.

Extra material: notes with solved Exercise 17.

29 Easy variable substitution, Exercise 18.

Exercise 18: Compute $\int \sqrt[3]{1-3x}\,dx$.

Extra material: notes with solved Exercise 18.

30 Easy variable substitution and some uneasy trigonometry, Problem 1.

Problem 1: Compute $\int \sin x \sin 2x\,dx$.

Extra material: notes with solved Problem 1.

31 Easy variable substitution and some uneasy trigonometry, Problem 2.

Problem 2: Compute $\int \sin x \sin 2x \sin 3x\,dx$.

Extra material: notes with solved Problem 2.

32 Easy variable substitution, Exercise 19.

Exercise 19: Let $c \neq 0$. Compute $\displaystyle\int \frac{ax+b}{cx+d}\,dx$.

Extra material: notes with solved Exercise 19.

33 Easy variable substitution, Exercise 20.

Exercise 20: Compute $\displaystyle\int \frac{2x^2-3x+1}{x+1}\,dx$.

Extra material: notes with solved Exercise 20.

34 Logarithmic derivative and its charm.

An important formula:

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C.$$

Some examples:

- $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\ln|\cos x| + C$
- $\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \ln|\sin x| + C.$

35 Logarithmic derivative, Exercise 21.

Exercise 21: Compute $\int \frac{e^{2x}}{e^{2x}+1}\,dx.$

Extra material: notes with solved Exercise 21.

36 Logarithmic derivative: three difficult and important examples, Problem 3.

Problem 3: Compute the following three integrals:

$$\int \frac{dx}{\sin x \cos x}, \qquad \int \frac{dx}{\sin x}, \qquad \int \frac{dx}{\cos x}.$$

Extra material: notes with solved Problem 3.

37 The last one, Exercise 22.

Exercise 22: Compute $\int \frac{3x^2 - 8x + 5}{x^2+1}\,dx.$

Extra material: notes with solved Exercise 22.

S3 Integration by parts: Product Rule in reverse

You will learn: understand and apply the technique of integration called *integration by parts*; some very typical and intuitively clear examples (sine or cosine times a polynomial, the exponential function times a polynomial), less obvious examples (sine or cosine times the exponential function), mind-blowing examples (arctangent and logarithm), and other examples.

Read along with this section: **Calculus book**: Chapter 7.1 *Integration by Parts*, pp: from 877 (7.1.1) to 892 (7.1E.7).

38 Integration by parts: how it works and when to use it.

39 Integration by parts: Example 1.

Example 1: Compute $\int xe^x\,dx.$

Extra material: notes with solved Example 1.

40 Integration by parts: Example 2.

Example 2: Compute $\int x \sin x\,dx.$

Extra material: notes with solved Example 2.

41 Integration by parts: Example 3.

Example 3: Compute $\int x \cos x\,dx.$

Extra material: notes with solved Example 3.

42 Integration by parts: Example 4.

Example 4: Compute $\int e^x \cos x\,dx.$

Extra material: notes with solved Example 4.

43 Integration by parts: Example 5.

Example 5: Compute $\int e^x \sin x \, dx$.

Extra material: notes with solved Example 5.

44 Integration by parts: Example 6.

Example 6: Compute $\int \ln x \, dx$.

Extra material: notes with solved Example 6.

45 Integration by parts: Example 7.

Example 7: Compute $\int \arctan x \, dx$.

Extra material: notes with solved Example 7.

46 What happens when the degree of the polynomial is higher.

47 Integration by parts, Exercise 1.

Exercise 1: Compute $\int x^3 \cos x \, dx$.

Extra material: notes with solved Exercise 1.

48 Integration by parts, Exercise 2.

Exercise 2: Compute $\int x^3 e^{-2x} \, dx$.

Extra material: notes with solved Exercise 2.

49 Integration by parts, Exercise 3.

Exercise 3: Compute $\int x^n \ln x \, dx$ for some $n \neq -1$.

Extra material: notes with solved Exercise 3.

50 Integration by parts, Exercise 4.

Exercise 4: Compute $\int e^{2x} \sin 3x \, dx$.

Extra material: notes with solved Exercise 4.

51 Integration by parts, Exercise 5.

Exercise 5: Compute $\int \ln(1+x^2) \, dx$.

Extra material: notes with solved Exercise 5.

52 Integration by parts, Exercise 6.

Exercise 6: Compute $\displaystyle\int x \cdot \ln \frac{1+x}{1-x} \, dx$.

Extra material: notes with solved Exercise 6.

Extra material: an article with more solved problems on integration by parts.

- ⋆ **Extra problem 1**: Find the antiderivative to $f(x) = (x+1) \cdot \cos 3x$.
- ⋆ **Extra problem 2**: Find the antiderivative to $f(x) = x^2 \sin x$.
- ⋆ **Extra problem 3**: Find the antiderivative to $f(x) = x^2 \cdot \cos 3x$.
- ⋆ **Extra problem 4**: Find the antiderivative to $f(x) = e^{-2x} \sin x$.
- ⋆ **Extra problem 5**: Find the antiderivative to $f(x) = \dfrac{\ln x}{x^2}$.
- ⋆ **Extra problem 6**: Find the antiderivative to $f(x) = \sqrt{x} \cdot \ln x$.
- ⋆ **Extra problem 7**: Find the antiderivative to $f(x) = \sqrt{1-x^2} \cdot \dfrac{1}{x^2}$.

S4 Change of variables: Chain Rule in reverse

You will learn: how to perform variable substitution in integrals and how to recognise that one should do just this.

Read along with this section: **Calculus book**: Chapter 5.5 *Substitution*, pages: from 657 (5.5.1) to 673 (5.5E.8); Chapter 5.6 *Integrals Involving Exponential and Logarithmic Functions*, pages: from 674 (5.6.1) to 689 (5.6E.7); Chapter 5.7 *Integrals Resulting in Inverse Trigonometric Functions*, pages: from 690 (5.7.1) to 700 (5.7E.6).

53 Integration by substitution: how it works and when to use it, Example 1.

Example 1: How to solve the problem from V34:

$$\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + C$$

with the new method.

54 Easy substitutions from Section 2, Example 2.

Example 2: Some exercises from Section 2:

* $\int e^{-3x}\, dx = -\frac{1}{3}e^{-3x} + C$
* $\int \cos mx\, dx = \frac{1}{m} \cdot \sin mx + C, \quad m \neq 0$
* $\displaystyle\int \frac{1}{2+3x^2}\, dx = \frac{\sqrt{6}}{6} \arctan(\tfrac{\sqrt{6}}{2}x) + C.$

Extra material: notes with solved Example 2.

55 Recognising (almost) derivatives, Example 3.

Example 3: Compute $\int \sin^3 x \cos x\, dx$.

Extra material: notes with solved Example 3.

56 Recognising (almost) derivatives, Example 4.

Example 4: Compute $\displaystyle\int \frac{\ln^5 x}{x}\, dx$.

Extra material: notes with solved Example 4.

57 Recognising (almost) derivatives, Example 5.

Example 5: Compute $\displaystyle\int \frac{\sin x}{(1-3\cos x)^3}\, dx$.

Extra material: notes with solved Example 5.

58 Recognising (almost) derivatives, Example 6.

Example 6: Compute $\displaystyle\int \frac{e^x}{2+e^x}\, dx$.

Extra material: notes with solved Example 6.

59 Recognising (almost) derivatives, Example 7.

Example 7: Compute $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\, dx$.

Extra material: notes with solved Example 7.

60 Recognising (almost) derivatives, Example 8.

Example 8: Compute $\displaystyle\int \frac{\ln(3x+1)}{3x+1}\, dx$.

Extra material: notes with solved Example 8.

61 Recognising (almost) derivatives, Example 9.

Example 9: Compute $\displaystyle\int \frac{x^4}{(x^5+1)^4}\,dx$.

Extra material: notes with solved Example 9.

62 Recognising (almost) derivatives, Example 10.

Example 10: Compute $\displaystyle\int \frac{x}{3-2x^2}\,dx$.

Extra material: notes with solved Example 10.

63 Recognising (almost) derivatives, Example 11.

Example 11: Compute $\displaystyle\int \frac{x}{4+x^4}\,dx$.

Extra material: notes with solved Example 11.

64 Recognising (almost) derivatives, Example 12.

Example 12: Compute $\displaystyle\int \frac{x^2}{\sqrt{1-x^3}}\,dx$.

Extra material: notes with solved Example 12.

65 Recognising (almost) derivatives, Example 13.

Example 13: Compute $\displaystyle\int \frac{1}{x\cdot\ln x\cdot\ln(\ln x)}\,dx$.

Extra material: notes with solved Example 13.

66 Back to the integral from V49 for $n=-1$, Example 14.

Example 14: Compute $\int x^n \ln x\,dx$ for $n=-1$.

Extra material: notes with solved Example 14.

67 Back to the integral from V30, Problem 1.

Problem 1: Compute $\int \sin x \sin 2x\,dx$ using variable substitution.

Extra material: notes with solved Problem 1.

68 Different results can happen: how to handle them, Problem 2.

Problem 2: Compute $\int \sin x \cos x\,dx$ in three different ways: two ways with help of variable substitution, and one using the formula for the sine of a double angle; explain why it is OK that you get three different answers and show how to demonstrate that they are all correct.

69 **Optional**: Back to the integral of cosecant from V36, Problem 3.

Problem 3: Compute the integral of cosecant using substitution

$$t = \frac{1}{\sin x} + \cot x.$$

The result will look different compared to the one obtained in V36. Show that both results are equal.

Extra material: notes with solved Problem 3.

70 A less obvious case, Problem 4.

Problem 4: Compute $\displaystyle\int \frac{1}{e^x+e^{-x}}\,dx$.

Extra material: notes with solved Problem 4.

71 A less obvious case, Problem 5.

Problem 5: Compute $\int \arcsin x\,dx$.

Extra material: notes with solved Problem 5.

72 Three examples related to the arctangent, Problem 6.

Problem 6: Compute the following integrals:

* $\displaystyle\int \frac{1}{x^2+9}\,dx$
* $\displaystyle\int \frac{1}{x^2+2x+2}\,dx$
* $\displaystyle\int \frac{2x+5}{x^2-4x+5}\,dx.$

Extra material: notes with solved Problem 6.

73 Back to arcsine, Problem 7.

Problem 7: Compute $\displaystyle\int \frac{1}{\sqrt{2-3x^2}}\,dx$. Compare to V54 and V62.

Extra material: notes with solved Problem 7.

74 A strange one, Problem 8.

Problem 8: Compute $\displaystyle\int \frac{1}{x\sqrt{x-1}}\,dx.$

Extra material: notes with solved Problem 8.

75 Three examples with the square root of x.

Compute the integral $\displaystyle\int \frac{dx}{\sqrt{x}+1}$. Compare the solution to Extra Problems 9 and 10 from the article.

Extra material: notes from the iPad.

Extra material: an article with more solved problems on change of variables in integrals. In all the problems beneath, find the primitive functions to the given functions f:

- ⋆ **Extra problem 1**: $f(x) = x^2\sqrt{2x^3-3}$,
- ⋆ **Extra problem 2**: $f(x) = \dfrac{\arctan x}{1+x^2}$,
- ⋆ **Extra problem 3**: $f(x) = \dfrac{1}{x+x\cdot\ln x}$,
- ⋆ **Extra problem 4**: $f(x) = \dfrac{2x}{\sqrt{1-x^4}}$,
- ⋆ **Extra problem 5**: $f(x) = \dfrac{1}{\sqrt{x-x^2}}$,
- ⋆ **Extra problem 6**: $f(x) = \dfrac{e^x-e^{2x}}{1+e^x}$,
- ⋆ **Extra problem 7**: $f(x) = \dfrac{x}{x^2-2x+2}$,
- ⋆ **Extra problem 8**: $f(x) = \dfrac{\cos x\cdot e^{\sin x}}{1+e^{2\cdot\sin x}}$,
- ⋆ **Extra problem 9**: $f(x) = \dfrac{x}{\sqrt{x}+2}$,
- ⋆ **Extra problem 10**: $f(x) = \dfrac{\sqrt{x}}{1+x^3}$.

S5 Integrating rational functions: partial fraction decomposition

You will learn: how to integrate rational functions using partial fraction decomposition.

Read along with this section: **Calculus book**: Chapter 7.4 *Partial Fractions*, pages: from 923 (7.4.1) to 937 (7.4E.5).

76 Five key concepts needed for the integration of rational functions.

77 Prerequisites from Precalculus 2.

Example 0: Compute $\int \frac{1}{x^2-1}\,dx$. Generally, if $a \neq b$ then $\int \frac{dx}{(x+a)(x+b)} = \frac{1}{a-b}\ln\left|\frac{x+b}{x+a}\right| + C$.

78 We have already worked with integrals of rational functions.

79 Integrals leading to the logarithm or to power functions, Formula 1.

Formula 1:

$$\int \frac{1}{(x-a)^n}\,dx = \begin{cases} \ln|x-a| + C\ , & n=1 \\ \frac{(x-a)^{-n+1}}{-n+1} + C = \frac{1}{(-n+1)(x-a)^{n-1}} + C\ , & n \geqslant 2 \end{cases}$$

Example 1: Compute $\int \frac{4}{x^3-x^5}\,dx$. Partial fraction decomposition was performed in V200 in Precalculus 2.

Extra material: notes with solved Example 1.

80 Integrals leading to the logarithm or to power functions, Formula 2.

Formula 2:

$$\int \frac{x}{(x^2+1)^n}\,dx = \begin{cases} \frac{1}{2}\ln(x^2+1) + C\ , & n=1 \\ \frac{1}{2}\frac{(x^2+1)^{-n+1}}{-n+1} + C = \frac{1}{2(-n+1)(x^2+1)^{n-1}} + C\ , & n \geqslant 2 \end{cases}$$

Example 2: Compute $\int \frac{x^3}{(x^2+1)^2}\,dx$.

Extra material: notes with solved Example 2.

81 Integrals leading to the arctangent or to power functions, Formula 3.

Formula 3:

$$I_n = \int \frac{1}{(x^2+1)^n}\,dx = \begin{cases} \arctan x + C\ , & n=1 \\ \frac{1}{2n-2}\cdot\frac{x}{(x^2+1)^{n-1}} + \frac{2n-3}{2n-2}\cdot I_{n-1}\ , & n \geqslant 2 \end{cases}$$

Extra material: notes with a derivation of Formula 3.

82 Formula 3, Example 3.

Example 3: Compute $\int \frac{1}{(x^2+1)^4}\,dx$.

Extra material: notes with solved Example 3.

83 Variable substitution and arctangent, Formula 4.

Formula 4:

$$\int \frac{a}{b^2+c^2x^2}\,dx = \frac{a}{bc}\arctan\left(\frac{cx}{b}\right) + C, \qquad a,b,c \neq 0.$$

84 Variable substitution and arctangent, Formula 5.

Formula 5:

$$\int \frac{1}{x^2+2ax+b}\,dx = \frac{1}{\sqrt{b-a^2}}\arctan\left(\frac{x+a}{\sqrt{b-a^2}}\right) + C, \qquad b-a^2 > 0.$$

85 Variable substitution, logarithm, and arctangent, Formula 6.

Formula 6:

$$\int \frac{2ax+b}{x^2+2cx+d}\,dx = a\ln(x^2+2cx+d) + \frac{b-2ac}{\sqrt{d-c^2}}\arctan\left(\frac{x+c}{\sqrt{d-c^2}}\right) + C, \qquad d-c^2>0.$$

Example 4: Compute $\int \frac{3x-1}{x^2+4x+6}\,dx$.

Extra material: notes with solved Example 4.

86 Method of Strategic Substitution in partial fraction decomposition.

Back to Examples 0 and 1 from V77 and V79:

Decompose $\frac{1}{x^2-1}$ and $\frac{4}{x^3-x^5}$.

Extra material: notes with solved Example 0.

87 Integration of rational functions, Exercise 1.

Exercise 1: Compute $\int \frac{4x^2+4x-11}{(2x-1)(2x+3)(2x-5)}\,dx$.

Extra material: notes with solved Exercise 1.

88 Integration of rational functions, Exercise 2.

Exercise 2: Compute $\int \frac{x^2+3x+1}{x^2-4}\,dx$.

Extra material: notes with solved Exercise 2.

89 Integration of rational functions, Exercise 3.

Exercise 3: Compute $\int \frac{x^3+x-1}{(x^2+2)^2}\,dx$.

Extra material: notes with solved Exercise 3.

90 Integration of rational functions, Exercise 4.

Exercise 4: Compute $\int \frac{x^4-5x^3+6x^2-18}{x^3-3x^2}\,dx$.

Extra material: notes with solved Exercise 4.

91 Integration of rational functions, Exercise 5.

Exercise 5: Compute $\int \frac{x}{x^4-3x^2+2}\,dx$.

Extra material: notes with solved Exercise 5.

92 Integration of rational functions, Problem 1.

Problem 1: Compute $\int \frac{x}{x^5+x^4-2x^3-2x^2+x+1}\,dx$.

Extra material: notes with solved Problem 1.

93 Integration of rational functions, Problem 2.

Problem 2: Compute $\int \frac{3x-1}{(x^2+4x+6)^3}\,dx$.

Extra material: notes with solved Problem 2.

94 Integration of rational functions, Problem 3.

Problem 3: Compute $\int \frac{\cos x}{\sin^2 x-\sin x}\,dx$.

Extra material: notes with solved Problem 3.

95 Integration of rational functions, Problem 4.

Problem 4: Compute $\int \frac{e^{2x}}{e^{2x}+2e^x+1}\,dx$.

Extra material: notes with solved Problem 4.

Extra material: an article with some solved problems on application of partial fraction decomposition for computing integrals of rational functions. Problems 1–8 were already presented in *Precalculus 2: Polynomials and rational functions*, the other problems are new.

- ⋆ **Extra problem 1**: Compute
$$\int \frac{x}{x^2+3x+2}\,dx.$$
- ⋆ **Extra problem 2**: Compute
$$\int \frac{x^3-5x-7}{x^2-x-2}\,dx.$$
- ⋆ **Extra problem 3**: Compute
$$\int \frac{1}{x^2+8x+12}\,dx.$$
- ⋆ **Extra problem 4**: Compute
$$\int \frac{9+x}{9-x^2}\,dx.$$
- ⋆ **Extra problem 5**: Compute
$$\int \frac{3x+11}{x^2-x-6}\,dx.$$
- ⋆ **Extra problem 6**: Compute
$$\int \frac{2x+1}{x(x^2+1)}\,dx.$$
- ⋆ **Extra problem 7**: Compute
$$\int \frac{1-x}{x^2+4x+3}\,dx.$$
- ⋆ **Extra problem 8**: Compute
$$\int \frac{x-x^2-3}{(x^2+1)(x-2)}\,dx.$$
- ⋆ **Extra problem 9**: Compute
$$\int \frac{2x^2+8x+11}{x^2+4x+4}\,dx.$$
- ⋆ **Extra problem 10**: Compute
$$\int \frac{x^3+4x^2+2x-5}{x^2+4x+3}\,dx.$$
- ⋆ **Extra problem 11**: Compute
$$\int \frac{8x+5}{2x^2+3x+1}\,dx.$$
- ⋆ **Extra problem 12**: Compute
$$\int \frac{x^3+2x^2-x-1}{x^2+x-2}\,dx.$$
- ⋆ **Extra problem 13**: Compute
$$\int \frac{3x-2}{x^2-3x-4}\,dx.$$
- ⋆ **Extra problem 14**: Compute
$$\int \frac{x^3-4x^2+6x+3}{x^2-4x+6}\,dx.$$

S6 Trigonometric integrals

You will learn: how to compute integrals containing trigonometric functions with various methods, like for example using trigonometric identities, using the universal substitution (tangent of a half angle) or other substitutions that reduce our original problem to the computing of an integral of a rational function.

Read along with this section: **Calculus book**: Chapter 7.2 *Trigonometric Integrals*, pp: 893 (7.2.1) to 907 (7.2E.6).

96 Trigonometric formulas and where to find them.

97 Trigonometric integrals we have seen until now.

98 Power reduction formulas for computing integrals of the sine or the cosine squared.
Example 1: Let $m \neq 0$. Compute $\int \sin^2 mx\,dx$ and $\int \cos^2 mx\,dx$.

99 Integral of the cube of the sine or of the cosine, by change of variables.
Compute $\int \sin^3 x\,dx$ and $\int \cos^3 x\,dx$ using variable substitution.
Show that the same method works for computing $\int \sin^{2n+1} x\,dx$ and $\int \cos^{2n+1} x\,dx$ for any natural $n \geqslant 1$.
Extra material: notes from the iPad.

100 Integral of the fourth power of the sine with help of repeated power reduction.
Show that $\int \sin^4 x\,dx = \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$ using repeated power reduction.

101 Recursive formulas for integrals of powers of the sine or of the cosine.
Let $n \geqslant 2$ be a natural number, $S_k = \int \sin^k x\,dx$ and $C_k = \int \cos^k x\,dx$ for each $k \in \mathbb{N}$. Show that:

* $S_n = \dfrac{n-1}{n} \cdot S_{n-2} - \dfrac{1}{n} \cdot \cos x \sin^{n-1} x$,
* $C_n = \dfrac{n-1}{n} \cdot C_{n-2} + \dfrac{1}{n} \cdot \sin x \cos^{n-1} x$.

(Only the first formula is derived in the video; derivation of the second one is left as an exercise.)
Extra material: notes from the iPad.

102 Powers of the sine or of the cosine versus the functions of multiple arguments.

103 Cubes in two ways, Exercise 1.
Exercise 1: Compute $\int \sin^3 x\,dx$ using the formula from V101. Compare your result to the result from V99.
Extra material: notes with solved Exercise 1.

104 Fourth powers in two ways, Exercise 2.
Exercise 2: Compute $\int \sin^4 x\,dx$ using the formula from V101. Compare your result to the result from V100.
Extra material: notes with solved Exercise 2.

105 The product of powers of the sine and the cosine.
Methods of computing $\int \cos^j x \sin^k x\,dx$ where $j, k \in \mathbb{N}$.
It is easy if at least one of the numbers j, k is odd.

106 The product of powers of the sine and the cosine, Exercise 3.
Exercise 3: Compute $\int \sin^3 x \cos^8 x\,dx$.
Extra material: notes with solved Exercise 3.

107 The product of powers of the sine and the cosine, Exercise 4.
Exercise 4: Compute $\int \cos^5 ax\,dx$, where $a \neq 0$.
Extra material: notes with solved Exercise 4.

108 Integrals involving secants and tangents, Exercise 5.

Exercise 5: Motivate or compute the following:

* $\displaystyle\int \sec x \tan x \, dx = \sec x + C$
* $\displaystyle\int \sec x \, dx = \ln|\sec x + \tan x| + C$
* $\displaystyle\int \tan^2 x \, dx$
* $\displaystyle\int \sec^4 x \, dx$
* $\displaystyle\int \sec^3 x \tan^3 x \, dx.$

For the second formula, substitute $t = \sec x + \tan x$.

Extra material: notes with solved Exercise 5.

109 Recursive formulas for integrals of powers of the tangent or of the secant.

Let $n \geqslant 2$ be a natural number, $T_k = \int \tan^k x \, dx$ and $E_k = \int \sec^k x \, dx$ for each $k \in \mathbb{N}$. Show that:

* $T_n = -T_{n-2} + \dfrac{1}{n-1} \cdot \tan^{n-1} x,$
* $E_n = \dfrac{n-2}{n-1} \cdot E_{n-2} + \dfrac{1}{n-1} \cdot \sec^{n-2} x \tan x.$

(Only the first formula is derived in the video; derivation of the second one is left as an exercise; In the Calculus book on page 899 you can see several examples of application of these formulas; use them as exercises.)

Extra material: notes from the iPad.

110 Product to sum formulas are good for integrals.

$$\int \cos mx \cos nx \, dx = \frac{1}{2(m-n)} \sin(m-n)x + \frac{1}{2(m+n)} \sin(m+n)x + C$$

$$\int \sin mx \sin nx \, dx = \frac{1}{2(m-n)} \sin(m-n)x - \frac{1}{2(m+n)} \sin(m+n)x + C$$

$$\int \sin mx \cos nx \, dx = -\frac{1}{2(m-n)} \cos(m-n)x - \frac{1}{2(m+n)} \cos(m+n)x + C$$

for $m \neq n$ and $m \neq -n$ (otherwise we already know how to do the computations; see V68 and V98).

111 Rational expressions in two variables.

Some examples of rational expressions in two variables: $R(u,v) = \dfrac{p(u,v)}{d(u,v)}$. Even and odd functions.

112 World's sneakiest substitution: the universal substitution.

Examples: Remake the following integrals into integrals of rational functions:

* $\displaystyle\int \frac{1}{\sin x(2+\cos x)} \, dx$
* $\displaystyle\int \frac{\cos^3 x}{\sin^4 x + \cos^2 x} \, dx$
* $\displaystyle\int \frac{\sin x}{\sin^3 x + \cos^3 x} \, dx.$

Compute the first integral. It will come back in V117, where we will compute it using another method.
Extra material: notes from the iPad.

113 The universal substitution, Exercise 6.

Exercise 6: Compute $\int \frac{1}{\sin x}\, dx$. Compare the solution to the one given in V36.

Extra material: notes with solved Exercise 6.

114 The universal substitution, Exercise 7.

Exercise 7: Compute $\int \frac{1}{\cos x}\, dx$. Compare the solution to the one given in V36.

Extra material: notes with solved Exercise 7.

115 The universal substitution, Exercise 8.

Exercise 8: Compute $\int \frac{1}{2+\cos x}\, dx$.

Extra material: notes with solved Exercise 8.

116 The universal substitution, Exercise 9.

Exercise 9: Compute $\int \frac{1}{\sin x+\cos x}\, dx$.

Extra material: notes with solved Exercise 9.

117 Rational expressions odd w.r.t. the sine, Exercise 10.

Exercise 10: Compute $\int \frac{1}{(2+\cos x)\sin x}\, dx$.

Extra material: notes with solved Exercise 10.

118 Rational expressions odd w.r.t. the sine, Exercise 11.

Exercise 11: Compute $\int \frac{\sin^7 x}{\cos^4 x}\, dx$.

Extra material: notes with solved Exercise 11.

119 Rational expressions odd w.r.t. the sine, Exercise 12.

Exercise 12: Show how to compute $\int \frac{\cos^4 x}{\sin^7 x}\, dx$.

Extra material: notes with solved Exercise 12.

120 Rational expressions odd w.r.t. the cosine, Exercise 13.

Exercise 13: Compute $\int \frac{1}{\sin x \cos x}\, dx$.

Extra material: notes with solved Exercise 13.

121 Rational expressions even w.r.t. both variables, Exercise 14.

Exercise 14: Compute $\int \frac{\sin x}{\sin^3 x+\cos^3 x}\, dx$.

Extra material: notes with solved Exercise 14.

122 Rational expressions even w.r.t. both variables, Exercise 15.

Exercise 15: Compute $\int \frac{1}{\sin^4 x+\cos^4 x}\, dx$.

Extra material: notes with solved Exercise 15.

123 The last one, just for fun, Problem 1.

Problem 1: Compute $\int(\sin^6 x+\cos^6 x+3\sin^2 x\cos^2 x)\, dx$.

Extra material: notes with solved Problem 1.

S7 Direct and inverse substitution, and more integration techniques

You will learn: Euler substitutions; the difference between direct and inverse substitution; triangle substitutions (trigonometric substitutions); some alternative methods (by undetermined coefficients) in cases where we earlier used integration by parts or variable substitution.

Read along with this section: **Calculus book**: Chapter 7.3 *Trigonometric Substitution*, pages: from 908 (7.3.1) to 922 (7.3E.5); Chapter 7.5 *Other Strategies for Integration*, pages: from 938 (7.5.1) to 945 (7.5E.5); Chapter 7.6 *Numerical Integration*, pages: from 946 (7.6.1) to 961 (7.6E.5). The *other* strategies chosen by me are different from the *other* strategies chosen by the Authors; you can read Ch7.5 just to get an idea about the richness of the set of integration techniques. Material from Ch7.6 is not covered in my course; it is a part of a curriculum in Numerical Methods that is a course in Applied Mathematics.

124 Nobody will teach you all the integration techniques, but...

125 Euler substitutions, why three cases are enough.

126 Euler substitutions, why they work.

127 Euler's substitution 1, an explanation.

128 Euler's substitution 1, an example.

Example 1: Compute $\int \frac{1}{\sqrt{x^2+b}}\,dx$ for $b \neq 0$. (Look back at V9.) Compute $\int \frac{1}{\sqrt{x^2-6x+15}}\,dx$.

Extra material: notes with solved example.

129 Euler's substitution 2, an explanation.

130 Euler's substitution 2, an example.

Example 2: Perform Euler's second substitution on $\int \frac{dx}{x+\sqrt{x^2-x+1}}$. You don't need to compute the integral.

Extra material: notes with solved example.

131 Euler's substitution 3, an explanation.

132 Euler's substitution 3, an example.

Example 3: Perform Euler's third substitution on $\int \frac{x^2}{\sqrt{-x^2+3x-2}}\,dx$. You don't need to compute the integral.

Extra material: notes with solved example.

133 **Optional**: A geometrical interpretation of Euler's substitutions 2 and 3.

Extra material: (**optional**) an article by Jan L. Cieśliński and Maciej Jurgielewicz.

134 Euler substitutions, Problem 1.

Problem 1: Compute $\int \frac{dx}{x\sqrt{x^2+x+1}}$.

Extra material: notes with solved Problem 1.

135 Euler substitutions, Problem 2.

Problem 2: Compute $\int \frac{dx}{x\sqrt{x^2+4x-4}}$.

Extra material: notes with solved Problem 2.

136 Euler substitutions, Problem 3.

Problem 3: Compute $\int \frac{dx}{x\sqrt{-x^2+x+2}}$.

Extra material: notes with solved Problem 3.

137 Rational expressions of rational powers.

138 Rational expressions of rational powers, Problem 4.

Problem 4: Compute $\displaystyle\int \frac{\sqrt{x}}{\sqrt[3]{x}+4}\,dx$.

Extra material: notes with solved Problem 4.

139 Rational expressions of rational powers, Problem 5.

Problem 5: Compute $\displaystyle\int \sqrt[4]{3x-7}\,dx$.

Extra material: notes with solved Problem 5.

140 Rational expressions of rational powers, Problem 6.

Problem 6: Compute $\displaystyle\int \frac{dx}{\sqrt[3]{4-5x}}$.

Extra material: notes with solved Problem 6.

141 Rational expressions of rational powers, Problem 7.

Problem 7: Compute $\displaystyle\int x\sqrt{2x-10}\,dx$.

Extra material: notes with solved Problem 7.

142 Rational expressions of rational powers, an atypical one, Problem 8.

Problem 8: Compute $\displaystyle\int \sqrt[3]{\frac{x+1}{x-1}}\cdot\frac{dx}{x+1}$.

Extra material: notes with solved Problem 8.

143 Direct (u) versus inverse substitution.

144 Back to trigonometric substitutions from Section 6: reference triangles.

145 Three triangle substitutions.

146 Triangle substitutions, Case 1.

147 Triangle substitution 1, an example.

Example: Let $a>0$. Compute $\displaystyle\int \sqrt{a^2-x^2}\,dx$. Compute $\displaystyle\int \sqrt{3-2x-x^2}\,dx$.

Extra material: notes with solved Example.

148 Triangle substitution 1, Problem 9.

Problem 9: Compute $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\,dx$.

Extra material: notes with solved Problem 9.

149 Triangle substitution 1, Problem 10.

Problem 10: Compute $\displaystyle\int \frac{dx}{x^2\sqrt{4-x^2}}$.

Extra material: notes with solved Problem 10.

150 Triangle substitutions, Case 2.

151 Triangle substitution 2, an example.

Example: Let $a>0$. Compute $\displaystyle\int \sqrt{x^2+a^2}\,dx$. Compute $\displaystyle\int \sqrt{x^2-2x+5}\,dx$.

Extra material: notes with solved Example.

152 Triangle substitution 2, Problem 11.

Problem 11: Compute $\displaystyle\int \frac{dx}{x^2\sqrt{4+x^2}}$.

Extra material: notes with solved Problem 11.

153 Triangle substitution 2, Problem 12.

Problem 12: Compute $\displaystyle\int \frac{dx}{(x^2+9)^{3/2}}$.

Extra material: notes with solved Problem 12.

154 Triangle substitutions, Case 3.

155 Triangle substitution 3, an example.

Example: Let $a > 0$. Compute $\displaystyle\int \sqrt{x^2 - a^2}\,dx$. Compare your result to the one from V151.

Extra material: notes with solved Example.

156 Triangle substitution 3, Problem 13.

Problem 13: Compute $\displaystyle\int \frac{\sqrt{x^2-4}}{x}\,dx$.

Extra material: notes with solved Problem 13.

157 Undetermined coefficients.

158 Some remarks about equality of certain functions.

Extra material: an article about polynomials from the course *Precalculus 2: Polynomials and rational functions*.

Let $P_1(x), P_2(x), R_1(x), R_2(x), S_1(x), S_2(x)$ denote polynomials and $\alpha \neq 0$ be any number. Then:

* $P_1(x) = P_2(x)$ iff they have the same degree and corresponding coefficients.
* $e^{\alpha x}P_1(x) = e^{\alpha x}P_2(x) \quad \Leftrightarrow \quad P_1(x) = P_2(x)$.
* $R_1(x)\cos\alpha x + R_2(x)\sin\alpha x = S_1(x)\cos\alpha x + S_2(x)\sin\alpha x \quad \Leftrightarrow \quad [\ R_1(x) = S_1(x) \ \wedge \ R_2(x) = S_2(x)\ ]$.

159 Undetermined coefficients instead of integration by parts, an example.

Example: Compute the integral $\int x^3 e^{-2x}\,dx$ from V48 with help of the new method.

Extra material: notes with solved Example.

160 Undetermined coefficients instead of integration by parts, another example.

Example: Compute the integral $\int (x^3 - x + 1)\sin 2x\,dx$ with help of the new method.

Extra material: notes with solved Example.

161 **Optional**: Our most complicated method.

Extra material: an article with a motivation of the method for computing $\displaystyle\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}\,dx$ (for $a \neq 0$).

162 **Optional**: Our most complicated method, an example.

Example: Compute the integral $\displaystyle\int \frac{x^3 - x + 1}{\sqrt{x^2+2x+2}}\,dx$.

Extra material: notes with solved Example.

163 **Optional**: Our most complicated method, Problem 14.

Problem 14: Compute the integral $\displaystyle\int \frac{x^3 - 6x^2 + 11x - 6}{\sqrt{x^2+4x+3}}\,dx$.

Extra material: notes with solved Problem 14.

S8 Problem solving

You will learn: you will get an opportunity to practice the integration techniques you have learnt until now; you will also get a very brief introduction to initial value problems (topic that will be continued in a future ODE course, *Ordinary Differential Equations*).

Read along with this section: **Calculus book**: go ahead and try to compute all the integrals from Chapters 5, 6, and 7 (for now: don't mind the evaluation of Riemann integrals, just compute antiderivatives of the integrands); Chapter 4.10 *Antiderivatives*, pages: from 550 (4.10.1) to 564 (4.10E.4).

164 Practice, practice, practice.

Extra material: an article with main integration techniques and where to find them.

Extra material: an article with more solved problems on finding primitive functions. In all the problems beneath, find the integral (primitive function, antiderivative) of function f, where:

- ⋆ **Extra problem 1**: $f(x) = \sqrt{x}e^{\sqrt{x}}$,
- ⋆ **Extra problem 2**: $f(x) = \dfrac{(2\sin x + 1)\cos x}{\sin x(\sin^2 x + 1)}$,
- ⋆ **Extra problem 3**: $f(x) = \dfrac{\sin(\ln x) \cdot \ln x}{x}$,
- ⋆ **Extra problem 4**: $f(x) = \dfrac{1}{\sqrt{x+1}+1}$,
- ⋆ **Extra problem 5**: $f(x) = \dfrac{\arctan x}{x^2}$,
- ⋆ **Extra problem 6**: $f(x) = \sin x \cos x \ln(\cos x)$

165 Integrals, Problem 1.

Problem 1: Compute $\displaystyle\int e^{4x}\sqrt{1+e^{2x}}\,dx$.

Extra material: notes with solved Problem 1.

166 Integrals, Problem 2.

Problem 2: Compute $\displaystyle\int \frac{dx}{(4x-x^2)^{3/2}}$.

Extra material: notes with solved Problem 2.

167 Integrals, Problem 3.

Problem 3: Compute $\displaystyle\int \frac{dx}{\sqrt{4-2x-x^2}}$.

Extra material: notes with solved Problem 3.

168 Integrals, Problem 4.

Problem 4: Compute $\displaystyle\int \sin^4 x\cos^5 x\,dx$.

Extra material: notes with solved Problem 4.

169 Integrals, Problem 5.

Problem 5: Compute $\displaystyle\int \frac{\tan x + 1}{\cos^2 x}\,dx$ and $\displaystyle\int \frac{\cos^2 x}{1-\sin x}\,dx$.

Extra material: notes with solved Problem 5.

170 Integrals, Problem 6.

Problem 6: Compute $\displaystyle\int \frac{x^2+1}{(x-1)^2}\,dx$.

Extra material: notes with solved Problem 6.

171 Integrals, Problem 7.

Problem 7: Compute $\displaystyle\int \frac{x^4+1}{x^6+1}\,dx.$

Extra material: notes with solved Problem 7.

172 Integrals, Problem 8.

Problem 8: Compute $\displaystyle\int \frac{e^{3\ln 2x}+5e^{2\ln 2x}}{e^{4\ln x}+5e^{3\ln x}-7e^{2\ln x}}\,dx$ and $\displaystyle\int x^2 2^{x^3+1}\,dx.$

Extra material: notes with solved Problem 8.

173 Integrals, Problem 9.

Problem 9: Compute $\displaystyle\int \cos(\ln x)\,dx.$

Extra material: notes with solved Problem 9.

174 Integrals, Problem 10.

Problem 10: Compute $\displaystyle\int \frac{\arctan e^x}{e^x}\,dx.$

Extra material: notes with solved Problem 10.

175 Integrals, Problem 11.

Problem 11: Compute $\displaystyle\int (\ln x)^4\,dx.$

Extra material: notes with solved Problem 11.

176 Integrals, Problem 12.

Problem 12: Compute $\displaystyle\int (\arcsin x)^2\,dx.$

Extra material: notes with solved Problem 12.

177 Integrals, Problem 13.

Problem 13: Compute $\displaystyle\int \frac{x+1}{\sqrt{1-x^2}}\,dx.$

Extra material: notes with solved Problem 13.

178 Integrals, Problem 14.

Problem 14: Compute $\displaystyle\int \frac{(x-1)^3}{\sqrt{x}}\,dx.$

Extra material: notes with solved Problem 14.

179 Integrals, Problem 15.

Problem 15: Compute $\displaystyle\int x\cdot\arctan x\,dx.$

Extra material: notes with solved Problem 15.

180 Integrals, Problem 16.

Problem 16: Compute $\displaystyle\int \arcsin\sqrt{\frac{x}{1+x}}\,dx.$

Extra material: notes with solved Problem 16.

181 Integrals, Problem 17.

Problem 17: Compute $\displaystyle\int \frac{x\cdot e^{\arctan x}}{(1+x^2)^{3/2}}\,dx.$

Extra material: notes with solved Problem 17.

182 A very brief introduction to Initial-Value Problems (IVP).

183 IVP: verifying solutions, an example.

Example: Verify that functions $y = 2e^{-2x} + Ce^x$ (where $C \in \mathbb{R}$ is any number) are solutions to the ODE: $y' - y = -6e^{-2x}$, and verify that the function $y = 2e^{-2x} + e^x$ is a solution to the IVP: $y' - y = -6e^{-2x},\ y(0) = 3$.

Extra material: notes with solved Example.

184 IVP: finding solutions, an example.

Example: Solve the IVP: $y' = 3e^x + x^2 - 4, \ \ y(0) = 5$.

Extra material: notes with solved Example.

185 Position, velocity, acceleration.

Example: An object is moving along a straight line (the s-axis) so that its position s is a function of time $s = s(t)$ (unit: meter or feet). The velocity of the object at the time t is then measured as $v(t) = s'(t)$ (unit: meter or feet per second). The speed is equal to $|v(t)| = |s'(t)|$ ($v(t)$ is positive if the object is moving forward and negative if the object is moving backward). The acceleration is computed as $a(t) = s''(t)$ (unit: meter or feet per second squared; the acceleration is positive if the velocity is increasing and negative if it is decreasing). Now we consider the following problem:

* Determine the velocity $v(t)$ at time t of an object moving along the s-axis so that at time t its position is given by
$$s(t) = v_0 t + \tfrac{1}{2} a t^2,$$
where v_0 and a are constants.
* Draw the graph of $v(t)$, and show that the area under the graph and above the t-axis, over $[t_1, t_2]$, is equal to the distance the object travels in that time interval.

Discussion: We have differentiated the position s to get the velocity v and then used the area under the velocity graph to recover information about the position. This illustrates the connection between finding areas and finding functions that have given derivatives (i.e., finding antiderivatives), which will be discussed in Section 11.

186 Position, velocity, acceleration, Problem 18.

Problem 18: An object moves along the s-axis in such a way that its position at time t is given by
$$s(t) = 2t^3 - 15t^2 + 24t.$$

* Find the velocity and acceleration of the object at time t.
* In which direction and how fast is the object moving at $t = 2$? Is it speeding up or slowing down at that time?
* When is the object instantaneously at rest? When is its speed instantaneously not changing?
* When is the object moving to the left (down)? to the right (up)?
* When is the object speeding up? slowing down?

(This problem comes from Adams Calculus, 10th edition, page 158.)

Extra material: notes with solved Problem 18.

187 IVP: Falling under gravity, Problem 19.

Problem 19: [In both cases we assume that the only force acting on the objects is gravity.]

a) A ball is thrown down with an initial speed of $20\,\frac{\mathrm{ft}}{\mathrm{s}}$ from the top of a cliff, and it strikes the ground at the bottom of the cliff after $5\,\mathrm{s}$. How high is the cliff?

b) A rock falls from the top of a cliff and hits the ground at the base of the cliff at a speed of $160\,\frac{\mathrm{ft}}{\mathrm{s}}$. How high is the cliff?

Extra material: notes with solved Problem 19.

188 Direct versus inverse problems.

S9 Riemann integrals: definition and properties

You will learn: how to define Riemann integrals (definite integrals) and how they relate to the concept of area; partitions, Riemann (lower and upper) sums; integrable functions; properties of Riemann integrals; a proof of uniform continuity of continuous functions on a closed bounded interval; a proof of integrability of continuous functions (and of functions with a finite number of discontinuity points); monotonic functions; a famous example of a function that is not integrable; a formulation, proof and illustration of The Mean Value Theorem for integrals; mean value of a function over an interval.

Read along with this section: **Calculus book**: Chapter 5.1 *Approximating Areas*, pages: from 570 (5.1.1) to 594 (5.1E.10); Chapter 5.2 *The Definite Integral*, pages: from 595 (5.2.1) to 618 (5.2E.9). **For those of you who want to know all the details**: Lecture notes by Professor John K. Hunter from UC Davis (Professor's permission kindly granted via e-mail on July 30, 2024).

189 From Geometry to Calculus, one more time.

Lecture notes (Chapter 11) by Professor John K. Hunter from UC Davis.

190 The concept of area.

191 Our early example.

Example: compute the area between the x-axis and the graph of $f(x) = x^2$ above the interval $[0,1]$ using approximation by rectangles (examples presented in V 34&101 in *Calculus* 1*, part* 1 *of* 2*: Limits and continuity*).

192 Integrability.

Concepts introduced (visually) in this video: a *partition* of interval, *lower sum*, *upper sum*, *integrable function*.

193 Refinements of partitions, and relations between upper and lower Riemann sums.

Given interval $[a,b]$ and a bounded $f:[a,b]\to\mathbb{R}$.

1. If Π is *any* partition of $[a,b]$, then:
$$L(f,\Pi)\leqslant U(f,\Pi),$$
i.e., the lower sum can never be greater than the upper sum (for the same partition).
2. If Π is *any* partition of $[a,b]$, $x_i^*\in[x_{i-1},x_i]$ for $i=1,2,\ldots,n$, and $S_n=\sum\limits_{i=1}^{n} f(x_i^*)\Delta x_i$, then
$$L(f,\Pi)\leqslant S_n\leqslant U(f,\Pi),$$
i.e., each Riemann sum is between the lower and the upper sum for a given partition.
3. If Π' and Π'' are two partitions of $[a,b]$ such that $\Pi'\subset\Pi''$ (i.e., Π'' is a refinement of Π') then
$$L(f,\Pi')\leqslant L(f,\Pi''),\qquad U(f,\Pi')\geqslant U(f,\Pi''),$$
i.e., the lower sums increase (don't decrease) and the upper sums decrease (don't increase) after refining the partition.
4. If Π' and Π'' are *any* partitions of $[a,b]$, then
$$L(f,\Pi')\leqslant U(f,\Pi''),$$
i.e., no lower sum can be greater than any upper sum (regardless partitions).
5. If $m\leqslant f(x)\leqslant M$ for all $x\in[a,b]$, then $m(b-a)\leqslant L(f,\Pi)\leqslant U(f,\Pi)\leqslant M(b-a)$ for all partitions Π of $[a,b]$. This means that the set of all the lower sums and the set of all the upper sums are bounded and, as they are subsets of $\mathbb{R}$, the following numbers are well defined (by the Axiom of Completeness), and we can call them *the lower and the upper Darboux integral*, respectively:
$$\underline{\int_a^b} f(x)\,dx=\sup_{\Pi} L(f,\Pi),\qquad \overline{\int_a^b} f(x)\,dx=\inf_{\Pi} U(f,\Pi).$$

6. As a result, we have:
$$m(b-a) \leqslant \underline{\int_a^b} f(x)\,dx \leqslant \overline{\int_a^b} f(x)\,dx \leqslant M(b-a).$$
They don't need to be equal (see V195), but if they *are* equal, then function f is called *integrable*.

Definition: A bounded function $f:[a,b]\to\mathbb{R}$ is called *Riemann integrable* if
$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx.$$
This common value is called *Riemann integral of f over* $[a,b]$ and is denoted $\int\limits_a^b f(x)\,dx$.

194 Integrable functions, an example.

Example: Let $a<b$ and $c\in\mathbb{R}$. Function $f:[a,b]\to\mathbb{R}$ defined as $f(x)=c$ for all $x\in D_f$ is Riemann integrable.

195 Finally, an example of a function that is *not* integrable.

The characteristic function of $[0,1]\cap\mathbb{Q}$:
$$\chi(x)=\begin{cases}1\,, & x\in[0,1]\cap\mathbb{Q}\\ 0\,, & x\in[0,1]\smallsetminus\mathbb{Q}\end{cases}$$
is *not* Riemann integrable.

196 More practical tests for integrability (Cauchy, sequential).

Two definitions and one practical test for integrability. Let $f:[a,b]\to\mathbb{R}$ be bounded.

* If $A\subset[a,b]$ then $\Omega(f,A)=\operatorname{osc}(f,A)=\delta(f(A))=\sup\limits_{x,x'\in A}|f(x)-f(x')|$ is called *the oscillation of function f on the set A* or *the diameter of the image $f(A)$*.
* If Π is a partition of $[a,b]$, then $\Omega(f,\Pi)=U(f,\Pi)-L(f,\Pi)$ is called *the oscillatory sum* of function f on the interval $[a,b]$ with partition Π. If we denote for this partition: $\Omega_i=\Omega(f,[x_{i-1},x_i])$ for $i=1,2,\dots n$, we obviously have:
$$0\leqslant U(f,\Pi)-L(f,\Pi)=\Omega(f,\Pi)=\sum_{i=1}^n(M_i-m_i)\Delta x_i=\sum_{i=1}^n\Omega_i\Delta x_i.$$
* **Integrability criterion (Cauchy)**: Bounded function $f:[a,b]\to\mathbb{R}$ is Riemann integrable iff
$$\forall\,\varepsilon>0\quad\exists\,\Pi\quad\Omega(f,\Pi)<\varepsilon,$$
or, equivalently (see *the sequential characterisation of integrability* in the notes from UC Davis, **Theorem 11.26**, p.12 in the pdf, p.216 in the Notes), if there exists a sequence of partitions (Π_n) such that
$$\lim_{n\to\infty}\Omega(f,\Pi_n)=0.$$

197 Application of the new test to the example from V191.

Example: Apply the test from V196 to prove that $f(x)=x^2$ is Riemann integrable on $[0,1]$.

198 Continuous functions on compact intervals are uniformly continuous.

Extra material: an article with a proof of the theorem about uniform continuity of continuous functions on compact intervals.

199 Continuous functions on compact intervals are Riemann integrable.

Theorem: If $f:[a,b]\to\mathbb{R}$ is continuous, then it is integrable on $[a,b]$.

200 Monotone functions on compact intervals are Riemann integrable.

Theorem: If $f : [a, b] \to \mathbb{R}$ is monotone, then it is integrable on $[a, b]$.

201 Some properties of oscillations and oscillatory sums.

Let $f, g : D \to \mathbb{R}$ be bounded, $A \subset D$, and $\alpha, \beta \in \mathbb{R}$.

* Oscillation of any linear combination is less than or equal to the linear combination of oscillations, i.e.:

$$\Omega(\alpha f + \beta g, A) \leqslant |\alpha| \cdot \Omega(f, A) + |\beta| \cdot \Omega(g, A).$$

* $\Omega(fg, A) \leqslant F \cdot \Omega(g, A) + G \cdot \Omega(f, A)$, where $|f(x)| \leqslant F$ and $|g(x)| \leqslant G$ for all $x \in A$.
* If $|g(x)| \geqslant c > 0$ for all $x \in A$, then $\Omega(\frac{1}{g}, A) \leqslant \frac{\Omega(g,A)}{c^2}$.

Let Π be any partition of $D = [a, b]$, and f and g be bounded. As a consequence of the properties above, by taking $A = [x_{i-1}, x_i]$ for $i = 1, 2, \dots, n$ and putting the inequalities together (after multiplying each of them by the corresponding $\Delta x_i > 0$), we get the following properties of the oscillatory sums:

* $\Omega(\alpha f + \beta g, \Pi) \leqslant |\alpha| \cdot \Omega(f, \Pi) + |\beta| \cdot \Omega(g, \Pi)$.
* $\Omega(fg, \Pi) \leqslant F \cdot \Omega(g, \Pi) + G \cdot \Omega(f, \Pi)$, where $|f(x)| \leqslant F$ and $|g(x)| \leqslant G$ for all $x \in D$.
* If $|g(x)| \geqslant c > 0$ for all $x \in D$, then $\Omega(\frac{1}{g}, \Pi) \leqslant \frac{\Omega(g,\Pi)}{c^2}$.

202 Some properties of Riemann integrals.

Lemma: If $\Pi' \subset \Pi''$ are two partitions of the interval $[a, b]$ (Π'' is a refinement of Π') and $f : [a, b] \to \mathbb{R}$ is bounded, then

$$\Omega(f, \Pi') \geqslant \Omega(f, \Pi'').$$

Let $f, g : [a, b] \to \mathbb{R}$ be bounded and integrable, and $\alpha, \beta \in \mathbb{R}$ be such that $\alpha^2 + \beta^2 \neq 0$.

* Functions $\alpha f + \beta g$, fg, $\frac{f}{g}$ are then also integrable (the last one if $|g(x)| \geqslant c > 0$ for all $x \in [a, b]$).
* **Linearity of Riemann integrals**:

$$\int_a^b (\alpha f(x) + \beta g(x))\, dx = \alpha \int_a^b f(x)\, dx + \beta \int_a^b g(x)\, dx.$$

* Function $|f|$ is then also integrable. **Illustrate with examples that the converse is not true.**

203 Monotonicity of integrals.

* If $f, g : [a, b] \to \mathbb{R}$ are integrable and such that $f(x) \leqslant g(x)$ for all $x \in [a, b]$ then $\int_a^b f(x)\, dx \leqslant \int_a^b g(x)\, dx$.
* If $f : [a, b] \to \mathbb{R}$ is integrable then (by V202) so is $|f|$. Moreover: $\left|\int_a^b f(x)\, dx\right| \leqslant \int_a^b |f(x)|\, dx$.

204 Additivity of integration w.r.t. the interval.

* If $f : [a, b] \to \mathbb{R}$ is integrable, and $[\alpha, \beta] \subset [a, b]$, then $f|_{[\alpha,\beta]}$ is also integrable.
* Given $f : [a, b] \to \mathbb{R}$ and $c \in (a, b)$. If $f|_{[a,c]}$ and $f|_{[c,b]}$ are integrable, then so is f on $[a, b]$. Moreover:

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

* Motivate the definition: $\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx$. With this definition, the formula above is also valid in cases when c is outside the interval $[a, b]$ and f is integrable on some closed interval containing all a, b, c.

205 Integrability of piecewise continuous functions.

- If a bounded $f : [a, b] \to \mathbb{R}$ has a finite number of discontinuity points, it is integrable.
- Evaluate $\int_0^{3.5} \lfloor x \rfloor \, dx$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

206 Mean Value Theorem for integrals 1.

Theorem (MVT for Integrals): If $f : [a, b] \to \mathbb{R}$ is continuous, then

$$\exists \ c \in [a, b] \quad \int_a^b f(x) \, dx = f(c) \cdot (b - a).$$

Prove the theorem and show how it is related to MVT from Calc1p2, V140.

207 Mean Value Theorem for integrals 2.

Theorem (MVT 2 for Integrals): If $f, g : [a, b] \to \mathbb{R}$ are such that f is continuous and g is integrable and has a constant sign on entire interval $[a, b]$, then

$$\exists \ c \in [a, b] \quad \int_a^b f(x)g(x) \, dx = f(c) \cdot \int_a^b g(x) \, dx.$$

Prove the theorem and show how it is related to MVT 2 from Calc1p2, V144.

208 Mean value of a continuous function over a compact interval.

Mean value of a continuous $f : [a, b] \to \mathbb{R}$ over the interval $[a, b]$ is computed as

$$\bar{f} = \frac{1}{b - a} \cdot \int_a^b f(x) \, dx.$$

Example: Find the average value of function $f(x) = 1 + \sin x$ over $[-\pi, \pi]$.

S10 Integration by inspection

You will learn: how to determine the value of the integrals of some functions that describe known geometrical objects (discs, rectangles, triangles); properties of integrals of even and odd functions over intervals that are symmetric around the origin; integrals of periodic functions.

Read along with this section: **Calculus book**: Chapter 5.2 *The Definite Integral*, pages: from 595 (5.2.1) to 618 (5.2E.9); (partly) Chapter 5.4 *Integration Formulas and The Net Change Theorem* (Examples 5.4.6 and 5.4.7), pages: from 644 (5.4.6) to 646 (5.4.8).

209 Finding values of some integrals with help of geometry.

- Evaluate $\int_2^6 (2 - |x - 4|) \, dx$ with help of a geometrical interpretation of the integral.
- Evaluate $\int_6^{12} \sqrt{-72 + 18x - x^2} \, dx$ with help of a geometrical interpretation of the integral.

210 Integrals of odd functions over compact and symmetric-to-zero intervals.

- If $a > 0$ and $f : [-a, a] \to \mathbb{R}$ is any continuous and *odd* function, then $\int_{-a}^a f(x) \, dx = 0$.

* Examples: $\int\limits_{-4}^{4} \sin x \, dx = 0$, $\int\limits_{-1.6}^{1.6} (x - 5x^3 + 2x^5) \, dx = 0$.

211 Integrals of even functions over compact and symmetric-to-zero intervals.

* If $a > 0$ and $f : [-a, a] \to \mathbb{R}$ is any continuous and *even* function, then $\int\limits_{-a}^{0} f(x) \, dx = \int\limits_{0}^{a} f(x) \, dx$.
* Example: If $r > 0$ then $\int\limits_{0}^{r} \sqrt{r^2 - x^2} \, dx = \frac{1}{4}\pi r^2$.
* Fact: Let f be a differentiable function defined on the interval $(-a, a)$ for some $a > 0$. If f is even then f' is odd. If f is odd then f' is even. A subtlety: an antiderivative of an even function doesn't need to be odd (for this we need $C = 0$, where C is the y-intercept of the antiderivative).

212 Integrals of periodic functions.

* If $f : \mathbb{R} \to \mathbb{R}$ is continuous and P-periodic, then $\int\limits_{a}^{a+P} f(x) \, dx = \int\limits_{0}^{P} f(x) \, dx$ for each $a \in \mathbb{R}$.
* If $f : \mathbb{R} \to \mathbb{R}$ is continuous, P-periodic, and **odd**, then $\int\limits_{0}^{P} f(x) \, dx = 0$.

213 Some nice properties of the mean value.

Problem 1: Suppose that $a < b$ and f is continuous on $[a, b]$. Let $\bar{f}$ denote the mean value of f over $[a, b]$.

* Show that $\int\limits_{a}^{b} (f(x) - \bar{f}) \, dx = 0$.
* Find the constant k that minimizes the integral $\int\limits_{a}^{b} (f(x) - k)^2 \, dx$.

Extra material: notes with solved Problem 1.

214 **Optional**: What about integrals equal to zero?

S11 Fundamental Theorem of Calculus

You will learn: formulation, proof and interpretation of The Fundamental Theorem of Calculus; how to use the theorem for: 1. evaluating Riemann integrals, 2. computing limits of sequences that can be interpreted as Riemann sums of some integrable functions, 3. computing derivatives of functions defined with help of integrals; some words about applications of The Fundamental Theorem of Calculus in Calculus 3 (Multivariable Calculus).

Read along with this section: **Calculus book**: Chapter 5.3 *The Fundamental Theorem of Calculus*, pages: from 619 (5.3.1) to 638 (5.3E.7); Chapter 5.4 *Integration Formulas and The Net Change Theorem*, pages: from 639 (5.4.1) to 656 (5.4E.10).

215 The connection between finding areas and finding antiderivatives.

216 Algebraic and transcendental functions.

217 What came first: the logarithm or the exponential?

218 Function of the upper limit of integration: its continuity.

If $f : [a, b] \to \mathbb{R}$ is integrable, then $F : [a, b] \to \mathbb{R}$ defined as $F(x) = \int\limits_{a}^{x} f(t) \, dt$ is continuous.

219 Fundamental Theorem of Calculus, part 1.

Theorem: If $f : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$ and continuous at some point $x_0 \in [a, b]$, then the function $F : [a, b] \to \mathbb{R}$ defined as $F(x) = \int_a^x f(t)\, dt$ is differentiable at x_0 and $F'(x_0) = f(x_0)$.

Corollary: If $f : [a, b] \to \mathbb{R}$ is continuous on the entire interval, then $F(x) = \int_a^x f(t)\, dt$ is its primitive function (i.e., each continuous function on an interval has a primitive function).

220 Fundamental Theorem of Calculus, part 2 (the evaluation theorem).

Corollary: If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and G is a primitive function of f then

$$\int_a^b f(x)\, dx = G(b) - G(a) := [G(x)]_a^b$$

Another notation is $G(x)|_a^b$.

Examples:

- $\int_0^1 x^2\, dx$, $\int_1^e \frac{1}{x}\, dx$,
- $\int_0^\pi \sin x\, dx$; compute the mean value of $f(x) = \sin x$ over $[0, \pi]$,
- **A warning**: The formula cannot be applied to $\int_{-1}^1 \frac{1}{x^2}\, dx$ (this comes back in V255).

Extra material: an article with solved Problem: Compute the integral

$$\int_0^4 \frac{|x-1|}{|x-2| + |x-3|}\, dx.$$

221 Area of a disc; another method for computing integrals from V147, V151, and V155.

- Using the results obtained in V147 and V211, compute the area of the disc with radius r.
- In V147, V151, and V155 we used *triangle substitutions* for computing integrals of the square roots of quadratic expressions. Compute the disc-related integral with help of *integration by parts* instead (the other two types can be computed in a very similar way).

Extra material: notes from the iPad.

222 Evaluating integrals, another example.

Example: Evaluate the integral

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\, dx.$$

Extra material: notes with solved Example.

223 Important integrals for future applications (Fourier series).

Example: With help of the results from V98 and V110, evaluate the integrals of

$$\sin^2 mx, \quad \cos^2 mx, \quad \cos mx \cos nx, \quad \sin mx \sin nx, \quad \sin mx \cos nx, \quad \sin mx, \quad \cos mx$$

over the interval $[-\pi, \pi]$; numbers n and m are non-zero integers.

Extra material: notes with solved Example.

224 Computing average values of functions on intervals, an example.

Example: Compute the average value over $[-\pi, \pi]$, $[-\pi/2, \pi/2]$, and $[0, \pi]$ of:

$$\sin x, \quad \cos x, \quad \sin^2 x, \quad \cos^2 x.$$

Extra material: notes with solved Example.

225 Integration by parts for Riemann integrals, two ways to go.

Example: Compute the following integral, using two ways to go (see V41):

$$\int_0^{\pi/2} x \cos x \, dx.$$

Extra material: notes with solved Example.

226 Integration by substitution for Riemann integrals, two ways to go.

Example: Compute the following integral, using two ways to go:

$$\int_1^4 \frac{1}{1+\sqrt{x}} \, dx.$$

Extra material: notes with solved Example.

227 An illustration for integration by substitution.

Illustrate the formula for variable substitution:

* showing how the integral reacts when we scale the variable by factor $k > 0$ and stretch/shrink the interval of integration using scaling by the reciprocal of k; illustrate the principle with some images from Precalculus 1 (V98) and Precalculus 3 (V101).
* using the example from V226.

228 Applications of properties of integrals, an exercise.

Exercise: Use various properties of integration (linearity, additivity w.r.t. the interval of integration) and variable substitution for solving following exercises:

* Prove the property (stated in V210) of integrals of odd functions on compact, symmetric-to-zero intervals:

  If $a > 0$ and $f : [-a, a] \to \mathbb{R}$ is any continuous and *odd* function, then $\displaystyle\int_{-a}^{a} f(x) \, dx = 0$.
* Prove the property of integrals of periodic functions on intervals of one-period length, formulated in V212:

  If $f : \mathbb{R} \to \mathbb{R}$ is continuous and P-periodic, then $\displaystyle\int_a^{a+P} f(x) \, dx = \int_0^P f(x) \, dx$ for each $a \in \mathbb{R}$.
* Given the value $\int_0^5 f(x) \, dx = 10$, evaluate $\int_0^{5/2} (x + f(2x)) \, dx$.

Extra material: notes with solved Exercise.

229 Limits of some type of sequences, Example 1.

Example 1: Compute the following limit by interpreting the sequence as a Riemann sum of some continuous function:

$$\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right).$$

Extra material: notes with solved Example 1.

230 Limits of some type of sequences, Example 2.

Example 2: Compute the following limit by interpreting the sequence as a Riemann sum of some continuous function:

$$\lim_{n\to\infty}\left(\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\cdots+\frac{n}{n^2+n^2}\right).$$

Extra material: notes with solved Example 2.

231 Limits of some type of sequences, Example 3.

Example 3: Let $p>0$. Compute the following limit by interpreting the sequence as a Riemann sum of some continuous function:

$$\lim_{n\to\infty}\frac{1^p+2^p+\cdots+n^p}{n^{p+1}}.$$

Extra material: notes with solved Example 3.

232 Limits of some type of sequences, Example 4.

Example 4: Compute the following limit by interpreting the sequence as a Riemann sum of some continuous function:

$$\lim_{n\to\infty}\frac{1}{n}\cdot\sum_{k=1}^{n}\sin\tfrac{k}{n}\pi.$$

Extra material: notes with solved Example 4.

233 Limits of some type of sequences, Example 5.

Example 5: Compute the following limit by interpreting the sequence as a Riemann sum of some continuous function:

$$\lim_{n\to\infty} n^2\cdot\sum_{k=1}^{n}\frac{k}{(n^2+k^2)^2}.$$

Extra material: notes with solved Example 5.

234 Differentiating functions defined with help of integrals, Example 1.

Example 1: Compute the following derivatives:

$$\frac{d}{dx}\left(\int_a^x \sin t\,dt\right),\quad \frac{d}{dx}\left(\int_a^x \sin t^2\,dt\right),\quad \frac{d}{dx}\left(\int_x^b \sin t^2\,dt\right).$$

Extra material: notes with solved Example 1.

235 Differentiating functions defined with help of integrals, Example 2.

Example 2: Compute the following derivatives:

$$\frac{d}{dx}\left(\int_0^{x^2}\sqrt{1+t^2}\,dt\right),\quad \frac{d}{dx}\left(\int_{x^2}^{x^3}\frac{1}{\sqrt{1-t^4}}\,dt\right),\quad \frac{d}{dx}\left(\int_{\sin x}^{\cos x}\cos(\pi t^2)\,dt\right).$$

Extra material: notes with solved Example 2.

Extra material: an article with more solved problems on applications of The Fundamental Theorem of Calculus.

- ⋆ **Extra problem 1**: Compute $g'(x)$ and $g''(x)$ if

$$g(x)=\int_2^x \arctan(e^t+t)\,dt.$$

- ⋆ **Extra problem 2**: Show that F is decreasing on the entire $\mathbb{R}$:

$$F(x)=\int_0^x e^t(-t^2+5t-7)\,dt.$$

⋆ **Extra problem 3**: Find all the local extremums for

$$F(x) = \int_0^x \frac{t^2 - 1}{e^t}\, dt.$$

⋆ **Extra problem 4**: Determine an equation of the tangent line to the curve $y = F(x)$ at $(0, F(0))$ if

$$F(x) = \int_0^x \frac{t - 1}{e^t + 1}\, dt.$$

⋆ **Extra problem 5**: Determine the expression for $\dfrac{dy}{dx}$ if

$$x \cdot \sin y = \int_0^x \arcsin t\, dt.$$

⋆ **Extra problem 6**: Determine the expression for $\dfrac{dy}{dx}$ for

$$y \cdot e^y = \int_0^x t\sqrt{1 + t^2}\, dt.$$

⋆ **Extra problem 7**: Show that f is convex on the entire $\mathbb{R}$:

$$f(x) = \int_1^{x^2} e^t\, dt.$$

⋆ **Extra problem 8**: Compute the derivatives of the following functions. For the first two of them it is enough to give the answer, for the last one you need to show a computation:

$$F_1(x) = \int_{-1}^x \frac{te^t}{2 + \sin t}\, dt, \qquad F_2(x) = \int_0^x e^t \cos(t^2 + t)\, dt, \qquad F_3(x) = \int_0^{x^2} \frac{2^t}{3t + 2}\, dt.$$

⋆ **Extra problem 9**: Compute the derivatives of the following functions. For the first three of them it is enough to give the answer, for the last one you need to show a computation:

$$F_1(x) = \int_2^x \frac{\sin t}{t + 1}\, dt, \qquad F_2(x) = \int_0^x e^{-t^2} \cdot \arctan t\, dt, \qquad F_3(x) = \int_{-1}^x \frac{e^t + t^3}{t^4 + 5}\, dt, \qquad F_4(x) = \int_x^{\sin x} e^{t^2}\, dt.$$

⋆ **Extra problem 10**: Compute the derivatives of the following functions. For the first three of them it is enough to give the answer, for the last one you need to show a computation:

$$F_1(x) = \int_3^x \frac{\cos t^2}{t^3 + 1}\, dt, \quad F_2(x) = \int_0^x \arcsin t \cdot e^t\, dt, \quad F_3(x) = \int_1^x \cos(\ln(t + \arctan t))\, dt, \quad F_4(x) = \int_{2x}^{e^x} \arctan(t^2 + t)\, dt.$$

236 **Future**: Fundamental Theorems in Multivariable Calculus.

S12 Area between curves

You will learn: compute the area between two curves (graphs of continuous functions), in particular between graphs of continuous functions and the x-axis.

Read along with this section: **Calculus book**: Chapter 6.1 *Areas Between Curves*, pages: from 706 (6.1.1) to 724 (6.1E.10).

237 Area between the graph of a function and the x-axis.

Problem 1: Compute the area of the domain enclosed by the curve $y = f(x)$ and the x-axis for $x \in [a, b]$, where:

- $f(x) = \sin x, \quad a = 0, \quad b = \pi$
- $f(x) = \sin x, \quad a = -2\pi, \quad b = 2\pi$
- $f(x) = \dfrac{1}{x}, \quad a = 1, \quad b = e$
- $f(x) = \ln x, \quad a = \frac{1}{e}, \quad b = e$
- $f(x) = \dfrac{1}{x^2+1}, \quad a = 0, \quad b = \sqrt{3}.$

238 Another (than in V36, V108, V114) method for finding the antiderivative of secant.

Problem 2: Compute the area of the domain enclosed by the curve $y = \sec x$ and the x-axis for $x \in [0, \pi/4]$. In order to find the antiderivative of the secant, use a different method: rewrite the reciprocal of the cosine as $\cos x / \cos^2 x$, apply the Pythagorean identity, and perform partial fraction decomposition. Verify that the result is the same as in V108.

Extra material: notes with solved Problem 2.

239 Area between two graphs.

Problem 3: Find the area of the domain between the curves:

- $y = \sin x$ and $y = \cos x$ for $x \in [0, \pi]$,
- $y = e^x$ and $y = xe^x$ for $x \in [0, 1]$.

Extra material: notes with solved Problem 3.

Extra material: an article with more solved problems on computing area between curves.

- ⋆ **Extra problem 1**: Compute the area of the domain between the x-axis and the curve $y = e^{\sqrt{x}}$ for $1 \leqslant x \leqslant 4$.
- ⋆ **Extra problem 2**: Compute the area between the x-axis and the graph of $y = f(x)$, where $f(x) = x^2 \sin x$ for $0 \leqslant x \leqslant \pi$.
- ⋆ **Extra problem 3**: Compute the finite area enclosed by the curves $y = x^2 + 6x$ and $y = 8 - x^2$.
- ⋆ **Extra problem 4**: Compute the area between the x-axis and the curve $f(x) = x^2 e^{3x}$ for $0 \leqslant x \leqslant 2$.
- ⋆ **Extra problem 5**: Compute the area between the curves $y = e^x$ and $y = e^{x/2}$ for $0 \leqslant x \leqslant 2$.
- ⋆ **Extra problem 6**: Compute the area between the curves $y = x^2 - 2x$ and $y = -x^2 - 2x + 2$ for $-2 \leqslant x \leqslant 2$.
- ⋆ **Extra problem 7**: Compute the area of the domain enclosed between the curves $y = x$ and $y = \dfrac{2}{x+1}$ for $0 \leqslant x \leqslant 2$.
- ⋆ **Extra problem 8**: Compute the area between the curve $y = \dfrac{x}{\sqrt{x+2}}$ and the x-axis for $-1 \leqslant x \leqslant 2$.
- ⋆ **Extra problem 9**: Compute the area between curves $y = 2 + x$ and $y = 3x^2$ for $0 \leqslant x \leqslant 2$.
- ⋆ **Extra problem 10**: Compute the area under the curve $y = \sin\sqrt{x}$ for $0 \leqslant x \leqslant \pi^2$ in the first quadrant.

S13 Arc length

You will learn: compute the arc length of pieces of the graph of differentiable functions.

Read along with this section: **Calculus book**: Chapter 6.4 *Arc Length of a Curve and Surface Area*, pages: from 774 (6.4.1) to 788 (6.4E.4).

240 Arc length: derivation of the formula, some examples.

Example: Compute the length of the curve $y = \sqrt{1-x^2} + \arcsin x$ for $x \in [0,1]$.

Extra material: notes with solved Example.

241 Arc length, Problem 1.

Problem 1: Compute the length of the piece of the graph of $f(x) = \ln x$ for $x \in [1,e]$.

Extra material: notes with solved Problem 1.

242 Arc length: Problem 2.

Problem 2: Compute the length of the arc of the hyperbolic cosine for $x \in [0, \ln 2]$.

Extra material: notes with solved Problem 2.

Extra material: an article with solved Problem: Compute the length of the arc defined by $y = y(x)$ between $x = 1$ and $x = 3$, where

$$y = \frac{x^3}{3} + \frac{1}{4x}.$$

(Hint: Note that $1 + (\frac{dy}{dx})^2$ is the square of a certain expression.)

S14 Rotational volume

You will learn: compute various types of volumes with different methods.

Read along with this section: **Calculus book**: Chapter 6.2 *Determining Volumes by Slicing*, pages: from 725 (6.2.1) to 754 (6.2E.11); Chapter 6.3 *Volumes of Revolution—Cylindrical Shells*, pages: from 755 (6.3.1) to 773 (6.3E.6).

243 Rotational volume, different situations and different methods.

244 The disk method: derivation and an example.

If the curve $y = f(x)$ for $x \in [a,b]$, where $f : [a,b] \to \mathbb{R}$ is a continuous function, rotates about the x-axis, the volume of the solid of revolution formed in this way from the plain region $D = \{(x,y);\ x \in [a,b],\ 0 \leqslant y \leqslant |f(x)|\}$ is computed as

$$V = \pi \int_a^b (f(x))^2 dx.$$

Example: Compute the rotational volume (in the rotation about the x-axis) if $f(x) = \sqrt{1+\sin x}$ on $[0, 2\pi]$.

245 We are finally able to confirm two well-known formulas for volume.

Two examples: Compute the volume of a ball with radius R and of a right circular cone with the base radius r and the height h.

Extra material: notes with solved Examples.

246 The washer method: derivation and an example.

Two continuous functions $f, g : [a,b] \to \mathbb{R}$ are such that $0 \leqslant g(x) \leqslant f(x)$ for $x \in [a,b]$. The domain defined as

$$D = \{(x,y);\ x \in [a,b],\ g(x) \leqslant y \leqslant f(x)\}$$

rotates about the x-axis. The volume of the solid of revolution created in this way is:

$$V = \pi \int_a^b [(f(x))^2 - (g(x))^2]\, dx.$$

Example: The piece of surface between the curves $y = x^2 + 1$ and $y = -x + 3$ rotates about the x-axis. Compute the volume of the solid formed in this way.

247 Cylindrical shells: derivation and an example.

Let $0 \leqslant a < b$. If the curve $y = f(x)$ for $x \in [a, b]$, where $f : [a, b] \to \mathbb{R}$ is a continuous and non-negative function, rotates about the y-axis, the volume of the solid of revolution formed in this way from the plain region

$$D = \{(x, y);\ x \in [a, b],\ 0 \leqslant y \leqslant f(x)\}$$

is computed as

$$V = 2\pi \int_a^b x f(x)\, dx.$$

* We look at one solved example: **Extra problem 4** from the article attached to V248.
* **Motivation** why the method is useful: $y = x(x-1)^2$ for $x \in [0, 1]$ rotates about the y-axis.

248 Cylindrical shells for a domain between two graphs.

Let $0 \leqslant a < b$. Two continuous functions $f, g : [a, b] \to \mathbb{R}$ are such that $0 \leqslant g(x) \leqslant f(x)$ for $x \in [a, b]$. The domain defined as

$$D = \{(x, y);\ x \in [a, b],\ g(x) \leqslant y \leqslant f(x)\}$$

rotates about the y-axis. The volume of the solid of revolution created in this way is:

$$V = 2\pi \int_a^b x[f(x) - g(x)]\, dx.$$

Example: The region between $y = x$ and $y = x^2$ rotates about the y-axis. Compute the volume.

Extra material: an article with more solved problems on rotational volume.

- ⋆ **Extra problem 1**: Curve $y = x + \sqrt{x},\ \ 0 \leqslant x \leqslant 1$, rotates about the x-axis. Compute the volume.
- ⋆ **Extra problem 2**: Show that[2]

  $$\int \frac{e^{2x}}{e^{2x} + 2e^x + 1}\, dx = \frac{1}{e^x + 1} + \ln(e^x + 1) + C \quad (*)$$

  where C is any constant. Then, use this result to compute the volume of the solid that is created by rotation of the following curve about the x-axis:

  $$y = \frac{e^x}{e^x + 1}, \quad 0 \leqslant x \leqslant 1.$$

- ⋆ **Extra problem 3**: The domain enclosed by the curve $f(x) = e^{-x}\sqrt{\sin x}$ (for $0 \leqslant x \leqslant \pi$) and the x-axis rotates about the x-axis. Compute the volume of the rotational solid that is formed in this way.
- ⋆ **Extra problem 4**: Let D be the domain in the first quadrant that is bounded by the graph of $f(x) = e^{4x}$ and the x-axis for $0 \leqslant x \leqslant 1$. Compute the volume of the rotational solid that is created when we rotate D about the y-axis.
- ⋆ **Extra problem 5**: Curve $y = e^x + e^{-x},\ \ 0 \leqslant x \leqslant 1$, rotates about the x-axis. Compute the volume.

[2] This integral was computed in V95.

S15 Surface area

You will learn: compute the area of surfaces obtained after rotation of pieces of the graph of differentiable functions.

Read along with this section: **Calculus book**: Chapter 6.4 *Arc Length of a Curve and Surface Area*, pages: from 774 (6.4.1) to 788 (6.4E.4). For more (physical) applications of Riemann integrals, that are **not** covered in my lectures, you can read the following chapters in the book: Chapter 6.5 *Physical Applications of Integration*, pages: from 789 (6.5.1) to 806 (6.5E.3), Chapter 6.6 *Moments and Centres of Mass*, pages: from 807 (6.5.1) to 828 (6.6E.4).

249 Rotational surface area: derivation and two examples.

If the curve $y = f(x)$ for $x \in [a, b]$, where $f : [a, b] \to \mathbb{R}$ is a continuously differentiable function with positive values, rotates about the x-axis, the (lateral) surface area of the solid of revolution formed in this way from the plain region $D = \{(x, y);\ x \in [a, b],\ 0 \leqslant y \leqslant f(x)\}$ is computed as

$$S = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\, dx.$$

Two examples: Compute the surface area of a ball with radius R and of a right circular cone with the base radius r and the height h.

Extra material: notes with solved Examples.

250 Rotational surface area, Problem 1.

Problem 1: Curve $y = \sin x$ for $x \in [0, \pi]$ is rotated about the x-axis. Compute the lateral surface area of the solid generated in this way. Compute also the volume, because we didn't do it in S14. (Plus: area, arc length.)

Extra material: notes with solved Problem 1.

251 Rotational surface area, Problem 2.

Problem 2: Curve $y = e^x$ for $x \in [0, 1]$ is rotated about the x-axis. Compute the lateral surface area of the solid generated in this way. Compute also the volume, because we didn't do it in S14. (Plus: area, arc length.)

Extra material: notes with solved Problem 2.

S16 Improper integrals of the first kind

You will learn: evaluate integrals over infinite intervals.

Read along with this section: **Calculus book**: Chapter 7.7 *Improper Integrals*, pages: from 962 (7.7.1) to 977 (7.7E.5).

252 Improper integrals, an introduction.

Examples of improper integrals:

* The p-integrals:

$$\int_1^\infty \frac{1}{x^p}\, dx$$

are convergent for $p > 1$ and divergent for $p \leqslant 1$.

* More examples of convergent integrals (such that their value can be determined exactly):

$$\int_0^\infty e^{-x}\, dx, \qquad \int_{-\infty}^\infty \frac{1}{1+x^2}\, dx, \qquad \int_2^\infty \frac{1}{(x+2)(x+3)}\, dx.$$

* The integral of a ZigZag function (and of the sine, and of the cosine) on $[0, \infty)$ is divergent.

* The (improper) integral of the following function over $\mathbb{R}$ is divergent:

$$f(x) = \begin{cases} \frac{1}{x}\,, & x \in (-\infty, -1] \\ x\,, & x \in [-1, 1] \\ \frac{1}{x}\,, & x \in [1, \infty) \end{cases}$$

253 Improper integrals of the first kind, Problem 1.

Problem 1: Compute $\displaystyle\int_{-\infty}^{0} x^2 e^x \, dx$.

Extra material: notes with solved Problem 1.

254 Improper integrals of the first kind, Problem 2.

Problem 2: Compute $\displaystyle\int_{-\infty}^{1} \frac{x}{1+x^4}\, dx$ and $\displaystyle\int_{0}^{\infty} \frac{1}{\sqrt{x+2x^2+x^3}}\, dx$.

Extra material: notes with solved Problem 2.

S17 Improper integrals of the second kind

You will learn: evaluate integrals over intervals that are not closed, where the integrand can be unbounded at (one or both of) the endpoints.

Read along with this section: **Calculus book**: Chapter 7.7 *Improper Integrals*, pages: from 962 (7.7.1) to 977 (7.7E.5).

255 Improper integrals of the second kind.

Examples of improper integrals:

* The q-integrals $\displaystyle\int_{0}^{1} \frac{1}{x^q}\, dx$ are convergent for $q < 1$ and divergent for $q \geqslant 1$. $\displaystyle\int_{0}^{+\infty} \frac{1}{x}\, dx = +\infty$
* An example where both function and interval are bounded, but the integral is improper: $\displaystyle\int_{0}^{1} \frac{\sin x}{x}\, dx$.
* More examples of convergent integrals (such that their value can be determined exactly):

$$\int_{0}^{1} \frac{dx}{\sqrt{1-x^2}}, \qquad \int_{1}^{3} \frac{dx}{\sqrt{x-1}}, \qquad \int_{0}^{1} \ln x \, dx.$$

* The integrals of $g(x) = 1/x$ on $(-1, 1)$ and of $f(x) = \tan x$ on $(-\pi/2, \pi/2)$ are divergent.

256 Improper integrals of the second kind, Problem 1.

Problem 1: Compute $\displaystyle\int_{0}^{4} \frac{dx}{x+\sqrt{x}}$.

Extra material: notes with solved Problem 1.

257 Improper integrals of the second kind, Problem 2.

Problem 2: Compute $\int_{0}^{1} x \ln x \, dx$. Show that $\int_{0}^{1} f(x)\, dx$ for $f : (0, 1] \to \mathbb{R}$ can be rewritten as an improper integral of the first kind.

Extra material: notes with solved Problem 2.

S18 Comparison criteria

You will learn: using comparison criteria for determining convergence of improper integrals by comparing them to some well-known improper integrals.

Read along with this section: **Calculus book**: Chapter 7.7 *Improper Integrals*, pages: from 962 (7.7.1) to 977 (7.7E.5).

258 Comparison criteria for improper integrals of non-negative functions.

Examples: Determine whether the following integrals are convergent by comparing them to other integrals, whose value you are able to compute:

$$\int_0^\infty \frac{1}{e^x + x}\,dx, \quad \int_1^\infty \frac{\cos^4 x}{x^3 + \sin^6 x}\,dx, \quad \int_0^1 \frac{1}{\sqrt{1-x^4}}\,dx, \quad \int_0^\infty \frac{1}{\sqrt{x+x^3}}\,dx,$$

$$\int_{-1}^1 \frac{1}{x\sqrt{1-x^2}}\,dx, \quad \int_0^\infty \frac{1}{xe^x}\,dx, \quad \int_0^\infty \frac{|\sin x|}{x^2}\,dx,$$

Extra material: notes with solved Examples.

259 Comparison criteria, Problem 1.

Problem 1 (Torricelli's Trumpet / Gabriel's Horn): The piece of the hyperbola $y = \frac{1}{x}$ for $x \in [1, \infty)$ rotates about the x-axis. Compute the volume and the surface area of the trumpet that is formed in this way.

Extra material: notes with solved Problem 1.

260 Comparison criteria, Problem 2.

Problem 2 (the one where Calculus 3 will help us, for once): Show that the integral $\int_{-\infty}^\infty e^{-x^2}\,dx$ is convergent.

Try to estimate its value. The exact value is possible to attain with Calculus-3 methods (double integrals).

Extra material: notes with solved Problem 2.

Extra material: an article with more solved problems on improper integrals and their applications.

- ⋆ **Extra problem 1**: Compute the improper integral:

$$\int_{-\infty}^0 \frac{1}{16+x^2}\,dx.$$

- ⋆ **Extra problem 2**: Determine whether the following improper integral is convergent; if it is convergent, compute its value:

$$\int_1^\infty \left(\frac{1}{x} + \frac{1}{x+1} - \frac{2x}{x^2+3}\right)dx.$$

- ⋆ **Extra problem 3**: Determine whether the following improper integral is convergent; if it is convergent, compute its value:

$$\int_0^\infty xe^{-3x}\,dx.$$

- ⋆ **Extra problem 4**: Determine whether the following improper integral is convergent; if it is convergent, compute its value:

$$\int_{-\infty}^0 xe^{2x}\,dx.$$

⋆ **Extra problem 5**: Compute the value of the improper integral:

$$\int_2^\infty \frac{1}{(x+2)(x+3)}\,dx.$$

⋆ **Extra problem 6**: Compute the area of the domain enclosed between the curve $y = \dfrac{3}{2x^2+5x+2}$ and the x-axis for $x \in [1, \infty)$.

⋆ **Extra problem 7**: Compute the area of the domain enclosed between the curves

$$y = \frac{1}{\sqrt{x}}, \quad x \in [1,\infty) \qquad \text{and} \qquad y = \frac{1}{\sqrt{x+1}}, \quad x \in [1,\infty).$$

⋆ **Extra problem 8**: Compute the area between the curves:

$$y = \frac{1}{1+x^2}, \quad x \in [1,\infty) \qquad \text{and} \qquad y = \frac{1}{x^2}, \quad x \in [1,\infty).$$

⋆ **Extra problem 9**: Compute the area of the domain enclosed between the curves

$$y = \frac{1}{x}, \quad 1 \leqslant x < \infty \qquad \text{and} \qquad y = \frac{x}{1+x^2}, \quad 1 \leqslant x < \infty.$$

⋆ **Extra problem 10**: Compute the area of the domain enclosed between the curves

$$y = e^{-x}, \quad 0 \leqslant x < \infty \qquad \text{and} \qquad y = -\frac{1}{x^2+2}, \quad 0 \leqslant x < \infty.$$

⋆ **Extra problem 11**: Determine whether the following improper integral is convergent:

$$\int_0^\infty \frac{\cos x^2}{e^x}\,dx.$$

⋆ **Extra problem 12**: Determine whether the following improper integral is convergent:

$$\int_1^\infty \frac{e^{-x}}{1+\sqrt{x}}\,dx.$$

261 Wrap-up and some words about *Calculus 2, part 2: Sequences and series.*

S19 Extras

You will learn: about all the courses we offer, and where to find discount coupons. You will also get a glimpse into our plans for future courses, with approximate (very hypothetical!) release dates.

B Bonus lecture.

Extra material 1: a pdf with all the links to our courses, and coupon codes.

Extra material 2: a pdf with an advice about optimal order of studying our courses.

Extra material 3: a pdf with information about course books, and how to get more practice.