Calculus 1, part 2 of 2: Derivatives with applications[1]

Single variable calculus

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A short table of contents

[1]Recorded December 2023–March 2024. Published on `www.udemy.com` on 2024-03-XX.

An extremely detailed table of contents; the videos (titles in green) are numbered

In blue: problems solved on an iPad (the solving process presented for the students; active problem solving)

In red: solved problems demonstrated during a presentation (a walk-through; passive problem solving)

In magenta: additional problems solved in written articles (added as resources).

In dark blue: *Read along with this section*: references for further reading and more practice problems in the *Calculus* book (chapters 3 and 4) by Gilbert Strang and Edwin *Jed* Herman:

`https://math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)`

`https://math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)/zz%3A_Back_Matter/30%3A_Detailed_Licensing`

This book is added as a resource to Video 1, with kind permission of the LibreTexts Office (given on July 20th, 2023).

S1 Introduction to the course

You will learn: about the content of this course and about importance of Differential Calculus. The purpose of this section is *not* to teach you all the details (this comes later in the course) but to show you the big picture.

1 Introduction to the course.

Extra material: this list with all the movies and problems.

Gilbert Strang & Edwin *Jed* Herman: **Calculus**, OpenStax, as described above.

Extra material: *Precalculus*: Carl Stitz, Ph.D., Lakeland Community College; Jeff Zeager, Ph.D., Lorain County Community College; version from July 4, 2013.

2 Good news first.

3 Rates of change, slopes, and tangent lines.

4 Derivative at a point and derivative as a function that shows variable slopes.

5 Why derivatives are important.

6 Differential equations: find all the functions that change in a certain way.

7 Elementary functions and their derivatives: more and less intuitive rules.

8 Advanced topics in the Precalculus series.

S2 Definition of the derivative, with some examples and illustrations

You will learn: the formal definition of derivatives and differentiability; terminology and notation; geometrical interpretation of derivative at a point; tangent lines and their equations; how to compute some derivatives directly from the definition and see the result it gives together with the graph of the function in the coordinate system; continuity versus differentiability; higher order derivatives; differentials and their geometrical interpretation; linearization.

Read along with this section: **Calculus book**: Chapters 3.1 *Defining the Derivative* and 3.2 *The Derivative as a Function*, pages: from 253 (3.0.1) to 294 (3.2E.11).

9 Terminology and notation.

Terminology: derivative, differentiation, differential (adj.), infinitesimal, differential (subst.), differentiable, tangent line. **Notation**: Δx, dx, Δf (Δy), df (dy), $\frac{\Delta f}{\Delta x}$ $(\frac{\Delta y}{\Delta x})$, $\frac{df}{dx}$ $(\frac{dy}{dx})$. **Notation for derivative at a point**: *Lagrange*: $f'(x)$, $y'(x)$; *Leibniz*: $\frac{df}{dx}(x)$, $\frac{df(x)}{dx}$, $\frac{d}{dx}f(x)$; *Euler*: $(Df)(x)$; *Newton*: $\dot{y}(x)$.

10 Where to find the Precalculus stuff for repetition: straight lines and rates of change.

11 In what kind of points we are going to consider derivatives.

12 Definition of the derivative at a point, differentiability of functions.

13 How to find equations for tangent lines? Two methods.

14 Derivatives of linear functions, Exercise 1.

Exercise 1: Compute the derivatives (at any point) to $f(x) = c$ and $g(x) = mx + b$, where $c, m, b \in \mathbb{R}$.

Extra material: notes with solved Exercise 1.

15 Derivatives of quadratic functions, Exercise 2.

Exercise 2: Compute the derivative at $x_0 = 1$ to $f(x) = x^2$. Draw the tangent line through $(1, f(1))$ and write its equation.

Extra material: notes with solved Exercise 2.

16 Derivatives of quadratic functions, Exercise 3.

Exercise 3: Compute the derivative at $x_0 = 4$ to $f(x) = -x^2 + 4x - 3$. Draw the tangent line through $(4, f(4))$ and write its equation.

Extra material: notes with solved Exercise 3.

17 Derivative of a cubic polynomial, Exercise 4.

Exercise 4: Compute the derivative at $x_0 = -1$ to $f(x) = \frac{1}{3}x^3$. Draw the tangent line through $(-1, f(-1))$ and write its equation.

Extra material: notes with solved Exercise 4.

18 Derivative of the square root function, Exercise 5.

Exercise 5: Compute the derivative at $x_0 = 0$ to $f(x) = \sqrt{x}$. Give some comments, draw a picture.

Extra material: notes with solved Exercise 5.

19 Another (equivalent) way of defining derivatives, Exercise 6.

Exercise 6: Repeat Exercise 2 from V14 using the new method. Do you get the same result?

Extra material: notes with solved Exercise 6.

20 A function that is not differentiable at some point, Exercise 7.

Exercise 7: Use the definition of derivatives to show that $f(x) = |x|$ is not differentiable at zero, and that it is differentiable at all the other points. Plot $y = f(x)$ and $y = f'(x)$ in the same coordinate system.
(An observation can be made that the graph has a *cusp* at $(0, 0)$; this is how it looks, graphically, when a function is continuous at some point, but not differentiable.)

21 Absolute values and cusps: a generalisation of Exercise 7.

Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable on the entire $\mathbb{R}$. Consider the function $|f| : \mathbb{R} \to \mathbb{R}$ defined by $|f|(x) := |f(x)|$ for all $x \in \mathbb{R}$. The following two statements describe differentiability of $|f|$:

C1 Function $|f|$ is differentiable at each $x_0 \in \mathbb{R}$ such that $f(x_0) \neq 0$. Moreover, in such cases $|f|'(x_0) = f'(x_0)$ if $f(x_0) > 0$ and $|f|'(x_0) = -f'(x_0)$ if $f(x_0) < 0$.

C2 In $x_0 \in \mathbb{R}$ such that $f(x_0) = 0$ we distinguish two cases:

C2.1 if $f'(x_0) = 0$, then $|f|$ is differentiable at x_0 and $|f|'(x_0) = 0$,

C2.2 if $f'(x_0) \neq 0$, then $|f|$ is not differentiable at x_0.

This means that the only problem with differentiability of $|f|$ is in such arguments x_0 that at the same time $f(x_0) = 0$ and $f'(x_0) \neq 0$.

Note: This theorem is not a part of a typical Calculus curriculum. The illustration given in the video is **not** a proof, it is just an illustration, but quite a convincing one, I believe. A proof of the first part (**C1**) of the theorem will be given in Video 24, a proof of **C2.1** will be given in Video 25, and a proof of **C2.2** will be given in Section 6, as we need to develop some theory first.

22 Yet another way of defining differentiability at a point.

Theorem: Function f is differentiable at $x_0 \in D_f \cap D_f'$ **if and only if** there exists such a number A that:

(1) for all $x \in D_f$ we have $f(x) - f(x_0) = A(x - x_0) + r(x, x_0)$,

(2) the remainder $r(x, x_0)$ satisfies the condition $\lim\limits_{x \to x_0} \frac{r(x, x_0)}{x - x_0} = 0$,

i.e., the remainder $r(x, x_0)$ is an infinitesimal quantity of *a higher order* than $x - x_0$ (tends to zero faster than $x - x_0$ when x tends to x_0).

The theorem can be reformulated using $h = x - x_0$:

Theorem: Function f is differentiable at $x_0 \in D_f \cap D_f'$ **if and only if** there exists such a number A that:

(1) for all h (such that $x_0 + h \in D_f$) we have $f(x_0 + h) - f(x_0) = Ah + \rho(h)h$,

(2) the remainder $\rho(h)h$ satisfies the condition $\lim\limits_{h \to 0} \rho(h) = 0$,

i.e., the remainder $\rho(h)h$ is an infinitesimal quantity of *a higher order* than h (tends to zero faster than h when h tends to 0).

Extra material: notes with a proof of the theorem.

23 Each differentiable function is continuous, but is the converse true?

Theorem *A necessary (but not sufficient) condition for differentiability*: If $f : D_f \to \mathbb{R}$ is differentiable at $x_0 \in D_f \cap D_f'$ then it is also continuous at x_0.

The converse is not true, though: if f is continuous at x_0 then it doesn't need to be differentiable at x_0.

Example: The floor function is not differentiable at any $x_0 \in \mathbb{Z}$.

Extra material: notes with a proof of the theorem.

24 **Optional**: Proof of the part **C1** from the theorem in Video 21.

Extra material: notes with a proof of the theorem.

25 **Optional**: Proof of the part **C2.1** from the theorem in Video 21.

Extra material: notes with a proof of the theorem.

26 Is the absolute value always a bad news for global differentiability? Problem 1.

Problem 1: Use the definition of derivatives to show that $f(x) = x \cdot |x|$ is differentiable at zero. Plot $y = f(x)$ and $y = f'(x)$ in the same coordinate system.

Extra material: notes with solved Problem 1.

27 Derivatives of piecewise functions, Problem 2.

Problem 2: (**Precalculus 1**, Video 94) Given the function:

$$f(x) = \begin{cases} -x\,, & -2 \leqslant x < -1 \\ x^2\,, & -1 \leqslant x \leqslant 1 \\ x\,, & 1 < x \leqslant 2 \end{cases}$$

Examine differentiability of this function.

Extra material: notes with solved Problem 2.

28 Recognising derivatives, Problem 3.

Problem 3: In **Calculus 1, part 1 of 2** (V157) we computed the following limit:

$$\lim_{x \to 1} \frac{\frac{1}{x+1} - \frac{1}{2}}{x - 1}.$$

Actually, this result helps us determine the derivative of some function at some point. Identify this function and the point, and plot this function together with the tangent line to the graph at this point.

Extra material: notes with solved Problem 3.

29 Recognising derivatives, Problem 4.

Problem 4: In **Calculus 1, part 1 of 2** (V157) we computed the following limit:

$$\lim_{x\to 0} \frac{1-\sqrt{1+x}}{x}.$$

Actually, this result helps us determine the derivative of some function at some point. Identify this function and the point, and plot this function together with the tangent line to the graph at this point.

Extra material: notes with solved Problem 4.

30 Recognising derivatives, Problem 5.

Problem 5: In **Calculus 1, part 1 of 2** (V159 and V161) we computed the following standard limits in zero:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1, \qquad \lim_{x\to 0} \frac{e^x - 1}{x} = 1.$$

Actually, these results help us determine the derivative of some functions at some point. Identify these functions and the point, and plot these functions together with the tangent lines to the graphs at this point.

31 Computing derivatives from the definition, Problem 6.

Problem 6: Using the definition of derivative, compute $f'(1)$ for $f(x) = x + (x-1)\arcsin\sqrt{\frac{x}{x+1}}$.

Extra material: notes with solved Problem 6.

32 One of my favourite problems, Problem 7.

Problem 7: Consider the part of the graph of $f(x) = \frac{1}{x}$ in the first quadrant. Choose any point on this graph and draw the tangent line to the graph through this point. This tangent line forms a right triangle together with the positive x and y axes. Show that the area of such triangle does not depend of the choice of point on the curve.

Extra material: notes with solved Problem 7.

33 Higher order derivatives, definition and notation.

34 Geometric interpretation of differentials.

35 What is linearization and why it is good for you.

Example: Find linearization of $f(x) = \frac{1}{x}$ about $x_0 = \frac{1}{2}$.

Extra material: notes with solved Example.

36 Linearization works locally, Problem 8.

Problem 8: Use the linearization for $f(x) = \sqrt{x}$ about $x_0 = 4$ to find an approximate value for $\sqrt{4.01}$.

Show that the linearization about $x_0 = 4$ doesn't give any good approximation of $\sqrt{9}$, $\sqrt{16}$, $\sqrt{25}$.

Extra material: notes with solved Problem 8.

S3 Deriving the derivatives of elementary functions

You will learn: how to derive the formulas for derivatives of basic elementary functions: the constant function, monic monomials, roots, trigonometric and inverse trigonometric functions, exponential functions, logarithmic functions, and some power functions (more to come in the next section); how to prove and apply the Sum Rule, the Scaling Rule, the Product Rule, and the Quotient Rule for derivatives, and how to use these rules for differentiating plenty of new elementary functions formed from the basic ones; differentiability of continuous piecewise functions defined with help of the elementary ones.

Read along with this section: **Calculus book**: Chapters 3.2 *The Derivative as a Function* and 3.3 *Differentiation Rules*, pages: from 273 (3.2.1) to 312 (3.3E.6). Chapter 3.5 *Derivatives of Trigonometric Functions*, pages: from 325 (3.5.1) to 337 (3.5E.4).

37 Our plan.

38 The derivative of monic monomials (power functions 1), method 1.

If $n \in \mathbb{N}^+$ and $f(x) = x^n$, then $f'(x) = nx^{n-1}$; proof using the formula for the difference of nth powers.

Extra material: notes with the derivation of the formula.

39 The derivative of monic monomials (power functions 1), method 2.

If $n \in \mathbb{N}^+$ and $f(x) = x^n$, then $f'(x) = nx^{n-1}$; proof using the Binomial Theorem. (There will be method 3! In Video 57, using induction and the Product Rule for derivatives.)

40 The derivative of roots (power functions 2), method 1.

If $n \in \mathbb{N}^+$, $n \geqslant 2$ and $f(x) = \sqrt[n]{x} = x^{1/n}$ (for $x \geqslant 0$ if n is even), then $f'(x) = \frac{1}{n}x^{1/n-1}$ (for $x \neq 0$); a proof using the formula for the difference of nth powers. (Methods 2 and 3 will be given in Sections 5 and 11.)

Extra material: notes with the derivation of the formula.

41 The derivative of power functions 3, method 1.

If $k \in \mathbb{Z}^-$ and $f(x) = x^k$ (for $x \neq 0$), then $f'(x) = kx^{k-1}$; a proof using the formula for the difference of nth powers. (Method 2 will be given in Video 59.)

Extra material: notes with the derivation of the formula.

42 The derivative of sine, method 1.

Show that $(\sin x)' = \cos x$. Use the formula for the difference of the sines of two arguments.

Extra material: notes with the derivation of the formula.

43 The derivative of cosine, method 1.

Show that $(\cos x)' = -\sin x$. Use the formula for the difference of the cosines of two arguments.

Extra material: notes with the derivation of the formula.

44 The derivative of sine and cosine, method 2.

Show that $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$. Use the sum identities for the sine and for the cosine.

45 The derivative of sine inverse, method 1.

Show that $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$ for $x \in (-1, 1)$. (Methods 2 and 3 will be given in Sections 5 and 11.)

Extra material: notes with the derivation of the formula.

46 The derivative of cosine inverse.

Show that $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$ for $x \in (-1, 1)$.

Extra material: notes with the derivation of the formula.

47 The derivative of the exponential function.

Show that $(e^x)' = e^x$.

48 The derivative of the natural logarithm, method 1.

Show that $(\ln x)' = \frac{1}{x}$ for $x > 0$. (Methods 2 and 3 will be given in Sections 5 and 11.)

Motivate graphically that $g'(x) = \frac{1}{x}$ for $g(x) = \ln|x|$ (for $x \neq 0$).

Extra material: notes with the derivation of the formula.

49 The derivative of logarithms with any base.

Let $a > 0$ and $a \neq 1$. Show that $(\log_a x)' = \frac{1}{x \ln a}$ for $x > 0$.

Extra material: notes with the derivation of the formula.

50 Rules of differentiation: the main theorem.

Theorem: Let f and g be two functions that are differentiable at x_0 (where x_0, together with some open and at least one-sided neighbourhood, belongs to the domains of both functions). Let c be a constant. Then:

* $(f+g)'(x_0) = f'(x_0) + g'(x_0)$ (**the Sum Rule**)
* $(cf)'(x_0) = cf'(x_0)$ (**the Scaling Rule**)
* $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$ (**the Product Rule**)
* $\dfrac{d}{dx}\left(\dfrac{f}{g}\right)(x_0) = \dfrac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}$ if $g(x_0) \neq 0$ (**the Quotient Rule**)
* $\dfrac{d}{dx}\left(\dfrac{1}{g}\right)(x_0) = -\dfrac{g'(x_0)}{g(x_0)^2}$ if $g(x_0) \neq 0$ (**the Reciprocal Rule**).

Some examples how to apply these rules: Compute $f'(x)$ using the rules above:

* $f(x) = \sqrt[5]{x} + \sin x + e^x$ (**the Sum Rule**)
* $f(x) = 2e^x - \ln x - 4\cos x$ (**the Scaling Rule** combined with **the Sum Rule**)
* $f(x) = x^4 \cdot \sin x$ (**the Product Rule**)
* $f(x) = \dfrac{\sin x}{e^x}$ (**the Quotient Rule**)
* $f(x) = \frac{1}{\sqrt[3]{x}}$ (**the Reciprocal Rule**).

Extra material: notes from the iPad with solved examples.

51 An illustration for the Sum Rule.

Given two differentiable functions f and g with the following properties:

$$f(2) = -6, \quad f'(2) = -1, \quad g(2) = 2, \quad g'(2) = 3.$$

Let $s(x) = f(x) + g(x)$. Compute $s'(2)$. Make an illustration showing how the three functions (f, g, s) can behave around the argument $x_0 = 2$.

52 Some illustrations for the Product Rule.

0) An abstract illustration for the rule $(fg)' = f'g + fg'$,
a) $-x^2 + 4x - 3 = (x-1)(-x+3)$,
b) $x^2 - 4x + 3 = (x-1)(x-3)$,
c) $-x^2 - x + 6 = (-x-3)(x-2)$,
d) $x^2 + x - 6 = (x+3)(x-2)$.

53 Three ways of writing a proof of the Sum Rule.

Three ways of writing the proof of the Sum Rule:

* using limit of the difference quotient with $x \to x_0$,
* using limit of the difference quotient with $h \to 0$,
* working with the Δ notation.

54 Three ways of writing a proof of the Scaling Rule.

Three ways of writing the proof of the Scaling Rule:

* using limit of the difference quotient with $x \to x_0$,
* using limit of the difference quotient with $h \to 0$,
* working with the Δ notation.

55 Linearity of the differential operator, and its consequences.

The differential operator $D : C^\infty[\mathbb{R}] \to C^\infty[\mathbb{R}]$ is linear, i.e. $D(\alpha f + \beta g) = \alpha D(f) + \beta D(g)$ for any $f, g \in C^\infty[\mathbb{R}]$. As a consequence, we can deduce that for each $n \in \mathbb{N}^+$ we have:

$$D(\sum_{i=1}^{n} \alpha_i f_i) = \sum_{i=1}^{n} \alpha_i D(f_i) \qquad \text{for any} \quad \alpha_i \in \mathbb{R} \quad \text{and} \quad f_i \in C^\infty[\mathbb{R}] \quad \text{for} \quad i = 1, 2, \ldots, n.$$

Extra material: notes from the iPad.

56 A proof of the Product Rule.

A proof using limit of the difference quotient with $h \to 0$. (As an exercise, write down the two other notations.)

57 Derivatives of polynomials are polynomials.

Using the Product Rule, show by induction that $p_n'(x) = nx^{n-1}$ for $p_n(x) = x^n$ where $n \in \mathbb{N}^+$. Use the first two rules from the theorem in Video 50 to show that the derivative of a polynomial is a polynomial of degree one less than the given polynomial.

Extra material: notes from the iPad.

58 A proof of the Quotient (and Reciprocal) Rule.

A proof using the Δ notation. (As an exercise, write down the two other notations.)

59 The derivative of power functions 3, method 2.

If $k \in \mathbb{Z}^-$ and $f(x) = x^k$, then $f'(x) = kx^{k-1}$; a proof using the Reciprocal Rule and $(x^n)' = nx^{n-1}$ for $n \in \mathbb{N}^+$.

Extra material: notes with the derivation of the formula.

60 Derivatives of rational functions are rational functions.

Example: Differentiate $f(x) = \dfrac{x}{x^2+1}$.

Extra material: notes with solved Example.

61 A generalization of the Product Rule.

Product Formula for Derivatives: We know (V50, V56) that the following formula holds for differentiable functions: $(fg)' = f'g + fg'$. Generalise this formula for more than two functions.

Extra material: notes from the iPad.

62 The derivative of tangent.

Use the Quotient Rule and the formulas for the derivatives of the sine and the cosine, to derive the formula for the derivative of the tangent function.

Extra material: notes with the derivation of the formula.

63 The derivative of arctangent, method 1.

Show that $(\arctan x)' = \frac{1}{1+x^2}$. (Methods 2 and 3 will be given in Sections 5 and 11.)

Extra material: notes with the derivation of the formula.

64 Some practice in differentiation, Exercise 1.

Exercise 1: Differentiate $f(x) = a^2 + 5a^3 \cdot x^2 - x^3$.

Extra material: notes with solved Exercise 1.

65 Some practice in differentiation, Exercise 2.

Exercise 2: Differentiate $f(x) = x + \sqrt{x} + \sqrt[3]{x}$.

Extra material: notes with solved Exercise 2.

66 Some practice in differentiation, Exercise 3.

Exercise 3: Differentiate $f(x) = \dfrac{1}{x} + \dfrac{2}{x^2} + \dfrac{3}{x^3}$.

Extra material: notes with solved Exercise 3.

67 Some practice in differentiation, Exercise 4.

Exercise 4: Calculate the derivatives given that $f(2) = 2$ and $f'(2) = 3$:

a) $\left.\dfrac{d}{dx}\left(\dfrac{x^2}{f(x)}\right)\right|_{x=2}$

b) $\left.\dfrac{d}{dx}\left(x^2 f(x)\right)\right|_{x=2}$

Extra material: notes with solved Exercise 4.

68 Some practice in differentiation, Exercise 5.

Exercise 5: Differentiate $f(x) = \dfrac{x^2+x+1}{x^3}$.

Extra material: notes with solved Exercise 5.

69 Some practice in differentiation, Exercise 6.

Exercise 6: Let $y = uv$ be the product of the functions u and v. Find $y'(9)$ if $u(9) = 2$, $u'(9) = -5$, $v(9) = 1$, and $v'(9) = 3$.

Extra material: notes with solved Exercise 6.

70 Some practice in differentiation, Exercise 7.

Exercise 7: Differentiate $f(x) = x^2 \sin x$ and $g(x) = x^2 \sin x \cdot \arctan x$.

Extra material: notes with solved Exercise 7.

71 Some practice in differentiation, Exercise 8.

Exercise 8: Differentiate $f(x) = \dfrac{1+x-x^2}{1-x+x^2}$.

Extra material: notes with solved Exercise 8.

72 Some practice in differentiation, Exercise 9.

Exercise 9: Compute the derivative of $f(x) = (x^2+1)(x^3+4)$ with two methods: with help of the Product Rule, and with help of the Sum Rule combined with the Scaling Rule.

Extra material: notes with solved Exercise 9.

73 Some practice in differentiation, Exercise 10.

Exercise 10: Compute the derivative of $f(x) = \dfrac{x}{x^2+1}$ using the definition; compute then the derivative using the theorem and compare the results.

Extra material: notes with solved Exercise 10.

74 Differentiability of piecewise functions, Exercise 11.

Exercise 11: We have shown in **Calculus 1, part 1 of 2** (V191) that the following function is continuous:

$$f(x) = \begin{cases} 2-x^2\,, & x \geqslant 2 \\ 4-3x\,, & x < 2 \end{cases}$$

Discuss its differentiability.

Extra material: notes with solved Exercise 11.

75 The derivative of the sine of a scaled argument.

Compute the derivative of $f(x) = \sin 3x$. Present some arguments showing that this cannot be $g(x) = \cos 3x$. Later, in Section 4, we will come back to this problem and see that it has a really simple solution.

Extra material: notes with the derivation of the formula.

76 Differentiability of piecewise functions, Exercise 12.

Exercise 12: We have shown in **Calculus 1, part 1 of 2** (V193) that the following function is continuous:

$$f(x) = \begin{cases} \dfrac{\sin 3x}{x}\,, & x < 0 \\ -\frac{4}{\pi}x + 3\,, & 0 \leqslant x \leqslant \frac{\pi}{2} \\ \sin x\,, & x > \frac{\pi}{2} \end{cases}$$

Discuss its differentiability.

Extra material: notes with solved Exercise 12.

77 Multiple zeros of polynomials and the round shapes of the graphs.

Show that if a polynomial $p(x)$ has a multiple zero x_0 (i.e., contains in its factorization a factor $(x - x_0)^k$ where $k \geqslant 2$ is a natural number), then x_0 is also a zero of the derivative $p'(x)$, and its multiplicity is one lower. This means that:

* the graph of a polynomial with a multiple zero x_0 cuts (if the multiplicity k is odd) or touches (if k is even) the x-axis at x_0 in a *rounded* way; we look at the following example: $p(x) = (x+2)^3x^3(x-1)^2$.
* if all the zeros of $p(x)$ have multiplicity of 2 or higher, then the new function $y = |p|(x)$ defined as $|p|(x) := |p(x)|$ for all $x \in \mathbb{R}$ is differentiable on the entire real axis (this is motivated by the Theorem from Video 21; Part **C2.1** shows the only sensitive part (the zeros of the original function $p(x)$).

Extra material: notes from the iPad.

78 A really cool problem about polynomials, Problem 1.

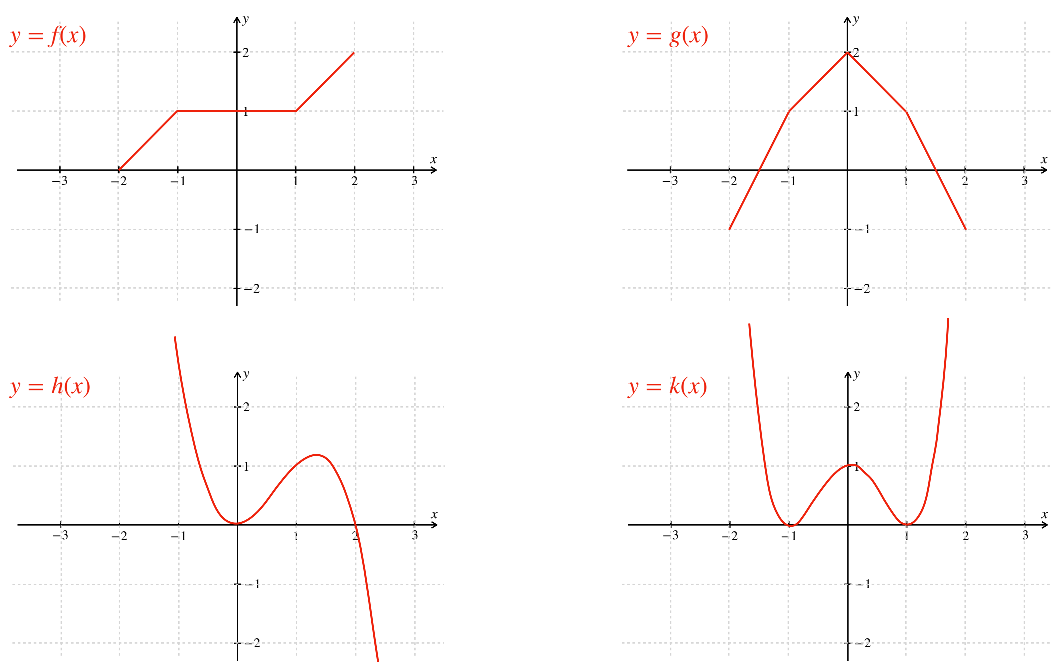
Problem 1: Let $p(x)$ be a non-negative 4th degree polynomial function in one real variable with $p(0) = 1$ and $p(1) = p(-1) = 0$. Compute $p(2)$.

Extra material: notes with solved Problem 1.

79 The one with a picture, Problem 2.

Problem 2: Discuss differentiability of the four functions in the picture. Sketch the derivative functions in their domains. Try to figure out the formulas of the functions and compute the derivatives.

Extra material: notes with solved Problem 2.



80 Finding the tangent line, Problem 3.

Problem 3: Find equations of any lines that pass through the point $(-1,0)$ and are tangent to the curve $y=\dfrac{x-1}{x+1}$. Explain how to draw the curve using graph transformations of some well known curve.

Extra material: notes with solved Problem 3.

81 Where to find more exercises for practice; some hints and tricks.

How to analyze the construction of functions in order to apply the differentiation rules correctly: a diagram for Problem 4 and Problem 6 from the article.

Extra material: notes from the iPad.

Extra material: an article with more solved problems on computing derivatives. In Problems 2–8, compute the first derivative to function f (sometimes you get to do more than that, then it is stated separately):

- **Extra problem 1**: Given two differentiable functions f and g with the following properties:

$$\begin{aligned} f(2) &= -6 \\ f'(2) &= -1 \\ g(2) &= 2 \\ g'(2) &= 3. \end{aligned}$$

  Let $s(x)=f(x)+g(x),\ p(x)=f(x)g(x),\ q(x)=\frac{f(x)}{g(x)}$. Compute $s'(2),\ p'(2),\ q'(2)$.

- **Extra problem 2**: $f(x)=x\cdot 2^x\cdot\sin x$.
- **Extra problem 3**: $f(x)=\dfrac{x}{1+\ln x}$. Determine the domain of f.
- **Extra problem 4**: $f(x)=\dfrac{\arcsin x+1}{2x-1}$. Determine the domain of f.
- **Extra problem 5**: $f(x)=\dfrac{\arcsin x}{x+1}$. Determine the domain of f.
- **Extra problem 6**: $f(x)=\dfrac{e^x\cdot\arcsin x}{x+2}$. Determine the domain of f.
- **Extra problem 7**: $f(x)=\dfrac{xe^x}{\ln x+1},\quad x>0,\ \ x\neq e^{-1}$.
- **Extra problem 8**: $f(x)=\dfrac{x}{x+e^x}$. Compute also the second derivative.

S4 The Chain Rule and related rates

You will learn: how to compute derivatives of composite functions using the Chain Rule; some illustrations and a proof of the Chain Rule; derivations of the formulas for the derivatives of a more general variant of power functions, and of exponential functions with the basis different than e; how to solve some types of problems concerning related rates (the ones that can be solved with help of the Chain Rule).

Read along with this section: **Calculus book**: Chapter 3.4 *Derivatives as Rates of Change*, pages: from 313 (3.4.1) to 324 (3.4E.6); Chapter 3.6 *The Chain Rule*, pages: from 338 (3.6.1) to 351 (3.6E.4); Chapter 4.1 *Related Rates*, pages: from 395 (4.1.1) to 408 (4.1E.6).

82 About this section; some reading recommendations.

83 Repetition from Precalculus 1: compositions of functions.

84 Transformations of graphs that involve scalings of the argument.

Some examples discussed in the Precalculus series:

a) Back to the problem from Video 75: if $f(x) = \sin 3x$ then $f'(x) = 3\cos 3x$; a computation that will resemble the proof of the general rule (see next video).
b) *Precalculus 3: Trigonometry*: $\sin 2x$ and $\sin \frac{1}{2}x$.
c) *Precalculus 1: Basic notions*: $g(x) = f(2x)$ gives $g'(x) = 2f'(2x)$; $g(x) = f\left(\frac{1}{2}x\right)$ gives $g'(x) = \frac{1}{2}f'\left(\frac{1}{2}x\right)$.

85 The Chain Rule: the theorem, an example, and a proof.

Example: Differentiate the following functions:

$$\text{a) } h(x) = \cos 5x, \quad \text{b) } h(x) = \ln 3x, \quad \text{c) } h(x) = \arctan 4x.$$

Extra material: notes with solved Example.

86 A generalization of The Chain Rule and some related topics.

* Some examples of notation
* Generalization of The Chain Rule to a composition of any number of differentiable functions
* An explanation for the name of the rule
* Example: differentiate $k(x) = \sqrt{\cos(5x^3 + 3x^2 + 1)}$.

Extra material: notes with solved Example.

87 Back to Video 87 in Precalculus 1.

Example: Analyze the following functions: which of them are composite functions, which not:

$$\text{a) } f(x) = x^3 + \sqrt{x} \quad \text{b) } f(x) = \sqrt{x^2+4} \quad \text{c) } f(x) = x^5 \sin x \quad \text{d) } f(x) = \frac{3x-1}{x^2+4} \quad \text{e) } f(x) = e^{\sqrt{x^4+x^2}}.$$

Differentiate all of them.

Extra material: notes with solved Example.

88 Back to Video 88 from Precalculus 1: the order of functions in a composition is important.

Example: Analyze the compositions $f_1(x) = \ln(\sin x^2)$, $f_2(x) = \ln(\sin^2 x)$, and $f_3 = \sin(\ln x^2)$ and compute the derivatives of the three functions.

Extra material: notes with solved Example.

89 The derivative of power functions 3, method 3.

If $k \in \mathbb{Z}^-$ and $f(x) = x^k$, then $f'(x) = kx^{k-1}$; prove using the Chain Rule and the fact that $(x^n)' = nx^{n-1}$ for $n \in \mathbb{N}^+$ (proven in V14 and later) and that $(x^{-1})' = -x^{-2}$ (proven in V32).

Extra material: notes with the derivation of the formula.

90 The derivative of power functions 4.

If $q \in \mathbb{Q}$ and $f(x) = x^q$, then $f'(x) = qx^{q-1}$; prove using the Chain Rule and the derivatives $(x^k)' = kx^{k-1}$ for $k \in \mathbb{Z}$ (V41, V59, V89) and $(\sqrt[n]{x})' = \frac{1}{n}x^{1/n-1}$ for $n \in \mathbb{N}^+ \setminus \{1\}$ (V40).

Extra material: notes with the derivation of the formula.

91 Some useful formulas from Precalculus 4.

92 The derivative of power functions 5.

If $\alpha \in \mathbb{R}$ and $f(x) = x^\alpha$, then $f'(x) = \alpha x^{\alpha-1}$; prove using the Chain Rule and some formulas from *Precalculus 4: Exponentials and logarithms*.

Extra material: notes with the derivation of the formula.

93 The derivative of exponential functions.

If $a > 0$, $a \neq 1$, and $f(x) = a^x$, then $f'(x) = \ln a \cdot a^x$.

Extra material: notes with the derivation of the formula.

94 Neither exponential nor power functions.

Given two differentiable functions f and g such that f attains only strictly positive values, i.e., $f(x) > 0$ for all x, and $f'(x)$ and $g'(x)$ exist for each x. Motivate that $h(x) = f(x)^{g(x)}$ is differentiable, and derive the formula for the derivative of h. Use the same method for computing the derivative of $h(x) = x^{1/x}$ for $x > 0$.

95 Back to some details from Videos 48 and 77.

In this video we go back to some old videos and motivate some stuff with a proper computation:

* Motivate with a computation that $g'(x) = \frac{1}{x}$ for $g(x) = \ln|x|$ (for $x \neq 0$); see Video 48.
* Motivate with a computation that if f is a differentiable function on $\mathbb{R}$ and $x_0 \in \mathbb{R}$ is any constant, then $h(x) = f(x - x_0)$ is also differentiable on $\mathbb{R}$ and $h'(x) = f'(x - a)$. In particular: $((x - x_0)^k)' = k(x - x_0)^{k-1}$ for $k \in \mathbb{N}^+ \setminus \{1\}$ (see Video 77).

96 The Chain Rule, an example.

Example: Functions f and g are differentiable and such that

$$f(0) = 1, \quad g(0) = 5, \quad f(1) = 1, \quad g(1) = 2, \quad f'(0) = 2, \quad g'(0) = -1, \quad f'(1) = 4, \quad g'(1) = 3.$$

Explain why $g \circ f$ is differentiable, and compute the value of its derivative at $x = 0$.

Extra material: notes with solved Example.

97 How to handle differentiation in easy and complicated cases.

How to analyze the construction of functions in order to apply the differentiation rules correctly: a diagram for Problems 9, 28, 30, 34 from the article attached to Video 116.

Extra material: notes from the iPad.

98 Some practice in differentiation (ChR), Exercise 1.

Exercise 1: Differentiate $f(x) = (5 + 2x)^{10} \cdot (3 - 4x)^{20}$.

Extra material: notes with solved Exercise 1.

99 Some practice in differentiation (ChR), Exercise 2.

Exercise 2: Differentiate $f(x) = x\sqrt{1 + x^2}$.

Extra material: notes with solved Exercise 2.

100 Some practice in differentiation (ChR), Exercise 3.

Exercise 3: Differentiate $f(x) = \sqrt[3]{\dfrac{1 + x^3}{1 - x^3}}$.

Extra material: notes with solved Exercise 3.

101 Some practice in differentiation (ChR), Exercise 4.

Exercise 4: Differentiate $f(x) = \sin(\cos^2 x) \cdot \cos(\sin^2 x)$.

Extra material: notes with solved Exercise 4.

102 Some practice in differentiation (ChR), Exercise 5.

Exercise 5: Differentiate $f(x) = \arcsin(\sin x)$.

Extra material: notes with solved Exercise 5.

103 Some practice in differentiation (ChR), Exercise 6.

Exercise 6: Differentiate $f(x) = \tan x - \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x$.

Extra material: notes with solved Exercise 6.

104 Some practice in differentiation (ChR), Exercise 7.

Exercise 7: Differentiate $f(x) = \sqrt{x + \sqrt{x + \sqrt{x}}}$.

Extra material: notes with solved Exercise 7.

105 Some practice in differentiation (ChR), Exercise 8.

Exercise 8: Compute the derivative of $f(x) = (x + ((3x)^5 - 2)^{-1/2})^{-6}$.

Extra material: notes with solved Exercise 8.

106 Some practice in differentiation (ChR), Exercise 9.

Exercise 9: Differentiate $f(x) = \arctan\left(\dfrac{1+x}{1-x}\right)$. (Note: Interesting result! Comes back in Section 6.)

Extra material: notes with solved Exercise 9.

107 Some practice in differentiation (ChR), Exercise 10.

Exercise 10: Differentiate $f(x) = 2\arctan x + \arcsin \frac{2x}{1+x^2}$ for $|x| \geqslant 1$. (Note: Interesting result! See Section 6.)

Extra material: notes with solved Exercise 10.

108 Derivatives of hyperbolic functions, Exercise 11.

Exercise 11: The following functions are called *hyperbolic cosine* and *hyperbolic sine*:

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}.$$

Compute their derivatives. *Hyperbolic tangent* is defined as the quotient of hyperbolic sine by hyperbolic cosine. Compute its derivative, using the identity $\cosh^2 x - \sinh^2 x = 1$ proven in *Precalculus 4: Exponentials and logarithms* (V221). Enjoy observing the similarities with derivatives of the regular trigonometric functions.

Extra material: notes with solved Exercise 11.

109 Derivatives of inverse hyperbolic functions, Exercise 12.

Exercise 12: In Video 169 (*Precalculus 4: Exponentials and logarithms*) we showed that $y = \ln(x + \sqrt{x^2+1})$ is the inverse function to the hyperbolic sine. Compute the derivative to this function; compare the result to the derivative of arcsine. (Note: the function inverse to the hyperbolic sine is called **arsinh**.)

Extra material: notes with solved Exercise 12.

110 Related rates and The Chain Rule.

111 Related Rates, Problem 1.

Problem 1: Air is being pumped into a spherical balloon. The volume of the balloon is increasing at a rate of $20\,\text{cm}^3/\text{s}$ when the radius is $30\,\text{cm}$. How fast is the radius increasing at that time?

Extra material: notes with solved Problem 1.

112 Related Rates, Problem 2.

Problem 2: When the diameter of a ball of ice is $6\,\text{cm}$, it is decreasing at a rate of $0.5\,\text{cm/h}$ due to melting of the ice. How fast is the volume of the ice ball decreasing at that time?

Extra material: notes with solved Problem 2.

113 Related Rates, Problem 3.

Problem 3: How fast is the area of the rectangle changing if one side is $10\,\text{cm}$ long and is increasing at a rate of $2\,\text{cm/s}$ and the other side is $8\,\text{cm}$ long and is decreasing at a rate of $3\,\text{cm/s}$?

Extra material: notes with solved Problem 3.

114 Related Rates, Problem 4.

Problem 4: A leaky water tank is in the shape of an inverted right circular cone with depth $5\,\text{m}$ and top radius $2\,\text{m}$. When the water in the tank is $4\,\text{m}$ deep, it is leaking out at a rate of $\frac{1}{12}\,\text{m}^3/\text{min}$. How fast is the water level in the tank dropping at that time?

Extra material: notes with solved Problem 4.

115 Related Rates, Problem 5.

Problem 5: A right circular cone is placed on its vertex as in the picture. The radius of its base is $R = 6\,\text{m}$ and the height is $H = 8\,\mathrm{m}$. The cone is being filled with water at a rate of $0.1\,\mathrm{m}^3/\mathrm{min}$. How fast is the water level increasing when the water level is $h = 4\,\mathrm{m}$?

Extra material: notes with solved Problem 5.

116 More practice: in the article and in the book.

Extra material: an article with more solved problems on computing derivatives. In Problems 3–37, compute the first derivative to function f (sometimes you get to do more than that, then it is stated separately):

- ⋆ **Extra problem 1**: $f(x) = \sqrt{5x^2 + 11x + 6}$. Determine directly from the definition the derivative $f'(0)$. Verify then the obtained result with help of the computational rules for the derivatives. What is the name of the rule and to what type of functions does it apply?

- ⋆ **Extra problem 2**: Given two differentiable functions f and g with the following properties:

$$f(4) = 5, \quad f'(4) = -1, \quad g(1) = 4, \quad g'(1) = 3.$$

  Let $h(x) = x \cdot f(g(x))$. Compute $h'(1)$.

- ⋆ **Extra problem 3**: $f(x) = \cos^3(x^2 + x + 1)$.

- ⋆ **Extra problem 4**: $f(x) = \arcsin\sqrt{x-1}$. Determine the domain of f.

- ⋆ **Extra problem 5**: $f(x) = \arcsin(\ln x)$. Determine the domain of f.

- ⋆ **Extra problem 6**: $f(x) = e^{\frac{1}{1+x}}, \quad x \neq -1$.

- ⋆ **Extra problem 7**: $f(x) = \sin(e^x + 1)$. Compute also the second derivative.

- ⋆ **Extra problem 8**: $f(x) = \ln(\tan x)$ in the intervals where $\tan x > 0$. Formulate the answer without using the function tan.

- ⋆ **Extra problem 9**: $f(x) = \cos\left(\dfrac{e^x+1}{x+1}\right), \quad x \neq -1$.

- ⋆ **Extra problem 10**: $f(x) = \arctan\left(\dfrac{1+x}{1-x}\right), \quad x \neq 1$.

- ⋆ **Extra problem 11**: $f(x) = \dfrac{e^{2x}}{2x+1}, \quad x \neq -\frac{1}{2}$. Compute also the second derivative.

- ⋆ **Extra problem 12**: $f(x) = \tan(3x^3 - x)$. Find all the *critical points* of f, i.e., verify at which points the derivative is equal to zero.

- ⋆ **Extra problem 13**: $f(x) = \ln\left(\dfrac{x}{x+1}\right)$, $x \in (-\infty, -1) \cup (0, \infty)$. Compute also the second derivative.

- ⋆ **Extra problem 14**: $f(x) = (2x+1)^3(5x+4)^6$. Give the answer is a factored form; determine the zeros of both f and its derivative.

- ⋆ **Extra problem 15**: $f(x) = x^6\sqrt{5x^2 - x}$. Determine the domain of both f and its derivative.

- ⋆ **Extra problem 16**: $f(x) = \dfrac{x^2-2}{(x+1)^4}, \quad x \neq -1$.

- ⋆ **Extra problem 17**: $f(x) = \dfrac{\arctan(x+1)}{x+2}, \quad x \neq -2$.

- ⋆ **Extra problem 18**: $f(x) = \ln(x + \sqrt{1+x^2})$.

- ⋆ **Extra problem 19**: $f(x) = 2\arctan x - \ln(1+x^2)$.

- ⋆ **Extra problem 20**: $f(x) = x \cdot \arctan x - \ln(\sqrt{1+x^2})$.

- ⋆ **Extra problem 21**: $f(x) = \ln(\arcsin(e^x))$. Determine the domain of f.
- ⋆ **Extra problem 22**: $f(x) = \arctan e^x \cdot \cos x$.
- ⋆ **Extra problem 23**: $f(x) = x \cdot \sin x^2 \cdot \arctan e^x$.
- ⋆ **Extra problem 24**: $f(x) = e^x \cdot \arccos x \cdot \cos x^2$.
- ⋆ **Extra problem 25**: $f(x) = \arcsin x^2 \cdot \sqrt{x} \cdot \ln x, \quad x \in (0, 1]$.
- ⋆ **Extra problem 26**: $f(x) = \sin e^x \cdot \arctan \sqrt{x} \cdot x, \quad x \geqslant 0$.
- ⋆ **Extra problem 27**: $f(x) = \sin 3x \cdot e^{\cos x} \cdot \arcsin x$.
- ⋆ **Extra problem 28**: $f(x) = x \cdot e^{\sin x} \cdot \arctan(\ln x), \quad x > 0$.
- ⋆ **Extra problem 29**: $f(x) = \sin(x+1) \cdot \cos\left(x^2 e^x\right) \cdot \ln x, \quad x > 0$.
- ⋆ **Extra problem 30**: $f(x) = \arctan\left(\dfrac{\sin x}{e^x + 1}\right)$.
- ⋆ **Extra problem 31**: $f(x) = \tan(e^{-2x}) \cdot \sin(x^{\frac{2}{3}})$.
- ⋆ **Extra problem 32**: $f(x) = \arcsin\left(x \cdot e^{\tan x}\right)$.
- ⋆ **Extra problem 33**: $f(x) = \tan x^2 \cdot e^{\arctan x}$.
- ⋆ **Extra problem 34**: $f(x) = \cos(\sin x \cdot \ln x), \quad x > 0$.
- ⋆ **Extra problem 35**: $f(x) = \arcsin(x \cos x^2)$.
- ⋆ **Extra problem 36**: $f(x) = \arctan x \cdot \sin(\ln x + e^x), \quad x > 0$.
- ⋆ **Extra problem 37**: $f(x) = \ln(\arccos x + e^{-\sin x})$.

S5 Derivatives of inverse functions

You will learn: the formula for the derivative of an inverse function to a differentiable invertible function defined on an interval (with a very nice geometrical/trigonometrical intuition behind it); we will revisit some formulas that have been derived earlier in the course and we will show how they can be motivated with help of the new theorem, but you will also see some other examples of application of this theorem.

Read along with this section: **Calculus book**: Chapter 3.7 *Derivatives of Inverse Functions*, pages: from 352 (3.7.1) to 365 (3.7E.6).

117 Derivative of functions inverse to differentiable functions, an intuition.

118 Derivative of functions inverse to differentiable functions, the theorem.

119 Derivative of functions inverse to differentiable functions, Example 1.

Example 1: Illustrate the theorem from V118 using the function $f(x) = \dfrac{x+2}{x}$ by computing its inverse and its derivative (with help of the Quotient Rule), and then computing the derivative of the inverse using the theorem.

Extra material: notes with solved Example 1.

120 The derivative of an inverse, Example 2.

Example 2: Function $y = f(x)$ is defined, strictly increasing, and differentiable on the entire $\mathbb{R}^+$. Further: $f(1) = 2$, $f(2) = 10$, $f'(1) = 3$, and $f'(2) = 5$. Is the inverse function f^{-1} differentiable at 2? If the answer is positive, compute $(f^{-1})'(2)$.

Extra material: notes with solved Example 2.

121 The derivative of roots (power functions 2), method 2.

If $n \in \mathbb{N}^+$, $n \geqslant 2$ and $f^{-1}(x) = \sqrt[n]{x} = x^{1/n}$ (for $x \geqslant 0$ if n is even), then $(f^{-1})'(x) = \frac{1}{n}x^{1/n-1}$ (for $x > 0$); a proof using the formula for the derivatives of inverse functions and the fact that $f'(x) = nx^{n-1}$ for $f(x) = x^n$. (Method 1 was given in Video 40, Method 3 will be given in Section 11.)

Extra material: notes with the derivation of the formula for the derivative.

122 The derivative of the natural logarithm, method 2.

Show using the theorem from V118 that the derivative of $f^{-1}(x) = \ln x$ is $(f^{-1})'(x) = \frac{1}{x}$.
Use the fact that $f'(x) = e^x$ for $f(x) = e^x$. (Method 1 was given in V48.)

Extra material: notes with the derivation of the formula for the derivative.

123 The derivative of sine inverse, method 2.

Show using the theorem from V118 that the derivative of $f^{-1}(x) = \arcsin x$ is $(f^{-1})'(x) = \frac{1}{\sqrt{1-x^2}}$.
Use the fact that $f'(x) = \cos x$ for $f(x) = \sin x$. (Method 1 was given in V45.)

Extra material: notes with the derivation of the formula for the derivative.

124 The derivative of arctangent, method 2.

Show using the theorem from V118 that the derivative of $f^{-1}(x) = \arctan x$ is $(f^{-1})'(x) = \frac{1}{1+x^2}$.
Use the fact that $f'(x) = 1 + \tan^2 x$ for $f(x) = \tan x$. (Method 1 was given in V63.)

Extra material: notes with the derivation of the formula for the derivative.

125 **Optional**: Intercept equations of straight line.

Fact: Let $a, b \neq 0$. Lines with equations $\dfrac{x}{a} + \dfrac{y}{b} = 1$ and $\dfrac{x}{b} + \dfrac{y}{a} = 1$ are symmetric about the diagonal $y = x$. If $a + b \neq 0$, these lines intersect at the point $\left(\dfrac{ab}{a+b}, \dfrac{ab}{a+b}\right)$. What happens when $a + b = 0$?

Three examples: $\frac{x}{1} + \frac{y}{4} = 1$ and $\frac{x}{4} + \frac{y}{1} = 1$, $\frac{x}{-3} + \frac{y}{1} = 1$ and $\frac{x}{1} + \frac{y}{-3} = 1$, $\frac{x}{-4} + \frac{y}{4} = 1$ and $\frac{x}{4} + \frac{y}{-4} = 1$.

Extra material: notes with a proof of the fact.

126 **Optional**: A theoretical statement about tangent lines

Statement: Given a differentiable and invertible on $\mathbb{R}$ function f, and $x_0 \in \mathbb{R}$ such that $f'(x_0) \neq 0$.

* The tangent line to $y = f(x)$ through the point $(x_0, f(x_0))$ has the slope-intercept equation

$$y = f'(x_0)x + f(x_0) - f'(x_0)x_0.$$

(Formula derived in V13.) If this tangent line doesn't pass the origin, then its intercept equation is

$$\frac{x}{x_0 - \frac{f(x_0)}{f'(x_0)}} + \frac{y}{f(x_0) - f'(x_0)x_0} = 1,$$

otherwise, if the line does pass the origin, its equation is $y = f'(x_0)x$.

* The tangent line to $y = f^{-1}(x)$ through the point $(f(x_0), x_0)$ has the slope-intercept equation

$$y = \frac{1}{f'(x_0)}x + x_0 - \frac{f(x_0)}{f'(x_0)}.$$

If this tangent line doesn't pass the origin (and this happens iff the tangent line discussed above doesn't pass the origin), then its intercept equation is

$$\frac{x}{f(x_0) - f'(x_0)x_0} + \frac{y}{x_0 - \frac{f(x_0)}{f'(x_0)}} = 1,$$

otherwise, if the line does pass the origin, its equation is $y = \frac{1}{f'(x_0)}x$.

Extra material: notes with the proof of the statement.

127 The derivative of an inverse, Problem 1.

Problem 1: Show that $f(x) = x^3 + x$ is 1–1 on $\mathbb{R}$, and, noting that $f(2) = 10$, find $(f^{-1})'(10)$. Write equations of tangent lines to both curves at points $(2, 10)$ and $(10, 2)$ respectively.

Extra material: notes with solved Problem 1.

128 The derivative of an inverse, Problem 2.

Problem 2: Function $f(x) = x^5 + x + 1$ has a differentiable inverse g. Compute $g'(35)$.

Extra material: notes with solved Problem 2.

129 The derivative of an inverse, Problem 3.

Problem 3: Consider $f : [0, 4] \to \mathbb{R}$ defined as $f(x) = x^3 - 6x^2 + 7$.

- Show that f is invertible.
- Determine the domain and the range for both f and f^{-1}.
- Compute $(f^{-1})'(2)$.

Now we prove the first part just on the level of intuition, using the fact (true, but not formally proven yet) that if f has a negative derivative on the entire interval $(0, 4)$ then the slopes of the tangent lines to the curve (or: the rates of change) are negative at each point of the interval, which means that f is strictly decreasing on the entire interval, and thus an injection; this method will be formalised in Section 6.

Extra material: notes with solved Problem 3.

S6 Mean value theorems and other important theorems

You will learn: various theorems that play an important role for further applications: Mean Value Theorems (Lagrange, Cauchy), Darboux property, Rolle's Theorem, Fermat's Theorem; you will learn their formulations, proofs, intuitive/geometrical interpretations, examples of applications, importance of various assumptions; you will learn some new terms like CP (*critical point*, a.k.a. *stationary point*) and *singular point*; the definitions of *local/relative maximum/minimum* and *global/absolute maximum/minimum* will be repeated from *Precalculus* 1, so that we can use them in the context of Calculus (they will be discussed in a more practical way in Sections 7, 17, and 18).

Read along with this section: **Calculus book**: Chapter 4.3 *Maxima and Minima*, pages: from 422 (4.3.1) to 437 (4.3E.6); Chapter 4.4 *The Mean Value Theorem*, pages: from 438 (4.4.1) to 448 (4.4E.3).

130 Lots of theory that can be illustrated in a very intuitive way.

Extra material: An article with some important theorems and their proofs:

a) Lemma: About positive and negative derivatives. [V133]
b) Theorem 1 (**Fermat**): Necessary condition for extremums at interior points. [V135]
c) Theorem 2 (**Rolle**): About existence of a stationary point. [V137]
d) Theorem 3 (**Lagrange**): The Mean Value Theorem. [V140]
e) Theorem 4 (**Cauchy**): Extended Mean Value Theorem. [V144]
f) Theorem 5 (**Darboux**): Darboux property for derivatives (the image of the derivative of a function on an interval is an interval). [V145]
g) Theorem 6: About monotonicity of functions. [V147]
h) Theorem 7: Functions with derivative equal to zero. [V151]

131 The concepts of maximum and minimum are not reserved for continuous functions.

Some questions about functions $f : \mathbb{R} \to \mathbb{R}$:

Q1 Do all continuous functions have a maximum/minimum?
Q2 Do all continuous functions on $[a, b]$ have a maximum/minimum on this interval?
Q3 Do all continuous functions on (a, b) have a maximum/minimum on this interval?
Q4 If function f has a maximum/minimum on $[a, b]$, does f need to be continuous?

132 The concept of monotonicity is not reserved for continuous functions.

Some questions about functions $f : \mathbb{R} \to \mathbb{R}$:

- Q1 Is each strictly monotone function (on some interval) continuous?
- Q2 Is each continuous and strictly monotone function (on some interval) differentiable?
- Q3 Is each strictly monotone function invertible?
- Q4 Does invertible function need to be strictly monotone?
- Q5 Does each continuous invertible function on an interval need to be strictly monotone?

133 Lemma about positive and negative derivatives.

Lemma: Let $X \subset \mathbb{R}$ be an interval and $f : X \to \mathbb{R}$ be a function differentiable at $x_0 \in X$. Then

* if $f'(x_0) > 0$, then there exists such a number $\delta > 0$ that
$$f(x) < f(x_0) \quad \text{for each} \quad x \in X \cap (x_0 - \delta, x_0)$$
and
$$f(x) > f(x_0) \quad \text{for each} \quad x \in X \cap (x_0, x_0 + \delta),$$
* if $f'(x_0) < 0$, then there exists such a number $\delta > 0$ that
$$f(x) > f(x_0) \quad \text{for each} \quad x \in X \cap (x_0 - \delta, x_0)$$
and
$$f(x) < f(x_0) \quad \text{for each} \quad x \in X \cap (x_0, x_0 + \delta).$$

134 Absolute values and cusps: proof of part C2.2 from V21.

Extra material: notes from the iPad.

135 Fermat's Theorem: Necessary condition for extremums at interior points.

Theorem 1 (Fermat): *Necessary condition for extremums at interior points.*

Let $X \subset \mathbb{R}$ be an interval. If $x_0 \in X$ is an interior point of X (i.e., is included in X together with some open neighbourhood, i.e., is not an endpoint of X) and $f : X \to \mathbb{R}$ is differentiable at x_0 and f attains its largest (smallest) value in X at x_0, then $f'(x_0) = 0$. (Two proofs: from the definition of derivative, from the Lemma.)

Examples showing why the condition $f'(x_0) = 0$ is necessary but *not* sufficient for an extremum at an interior point of the domain, and why being an interior point of the domain is important: $f(x) = x^{2n+1}$ on $[-1, 1]$.

136 Critical (stationary) points and singular points; plateaus.

Some Examples:

- E1 A function can have its minimum/maximum at a singular point: $f(x) = |x|$, $g(x) = -|x|$.
- E2 A critical point doesn't need to be an extremum point: $f(x) = x^3$ and $g(x) = -x^3$ have a *plateau* at $x_0 = 0$.
- E3 A critical point can be a global minimum/maximum.
- E4 A critical point that is a local extremum point doesn't need to be a global minimum/maximum.

137 Rolle's Theorem: About existence of a stationary point.

Theorem 2 (Rolle): *About existence of a stationary point.*

Let $a, b \in \mathbb{R}$ be such that $a < b$. If $f : [a, b] \to \mathbb{R}$ is:

* continuous,
* such that $f'(x)$ exists for all $x \in (a, b)$,
* such that $f(a) = f(b)$,

then there exists $c \in (a, b)$ such that $f'(c) = 0$.

138 Rolle's Theorem, Example 1.

Example 1: Examine the number of zeros of the derivative to $f(x) = (x - 1)(x - 2)(x - 3)(x - 4)(x - 5)$.

Extra material: notes with solved Example 1.

139 Rolle's Theorem, Example 2.

Example 2: Let $g(x) = 1 + x^{1990}(x-1)^{2000}$. Show that the derivative $g'(x)$ has a zero in the interval $(0, 1)$. Solve using Rolle's Theorem, and verify your result by computing the derivative and analysing its zeros.

Extra material: notes with solved Example 2.

140 The Mean Value Theorem (Lagrange).

Theorem 3 (Lagrange): *The Mean Value Theorem.*

Let $a, b \in \mathbb{R}$ be such that $a < b$. If $f : [a, b] \to \mathbb{R}$ is:

* continuous,
* such that $f'(x)$ exists for all $x \in (a, b)$,

then there exists $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

141 Lagrange's Theorem, Example 1.

Example 1: Illustrate Lagrange's Theorem by finding such a $c \in (1, 2)$ that the tangent line to the graph of $f(x) = \frac{1}{x}$ through $(c, f(c))$ is parallel to the secant through the points $(1, 1)$ and $(2, \frac{1}{2})$ on the hyperbola.

Extra material: notes with solved Example 1.

142 Lagrange's Theorem, Example 2.

Example 2: Use Lagrange's Theorem to show that $\sin x < x$ for all $x > 0$.

Extra material: notes with solved Example 2.

143 Lagrange's Theorem, Example 3.

Example 3: Show that $\sqrt{1 + x} < 1 + \frac{x}{2}$ for $x > 0$ and for $-1 \leqslant x < 0$.

Extra material: notes with solved Example 3.

144 Extended Mean Value Theorem (Cauchy).

Theorem 4 (Cauchy): *Extended Mean Value Theorem.*

Let $a, b \in \mathbb{R}$ be such that $a < b$. If two functions $f, g : [a, b] \to \mathbb{R}$ are:

* continuous,
* such that $f'(x)$ and $g'(x)$ exist for all $x \in (a, b)$,
* $g'(x) \neq 0$ for all $x \in (a, b)$,

then there exists $c \in (a, b)$ such that $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}$.

Fun fact about the three theorems (Rolle's, Lagrange's, and Cauchy's): how they are related.

145 Darboux property for derivatives.

Theorem 5: *Darboux property for derivatives.*

Let $a, b \in \mathbb{R}$ be such that $a < b$. If function $f : [a, b] \to \mathbb{R}$ is differentiable at all points of $[a, b]$, then for each number d between the numbers $f'(a)$ and $f'(b)$ there exists such an $c \in [a, b]$ that $f'(c) = d$.

146 Darboux property for derivatives, some examples.

Examples 1 and 2: The following functions cannot be derivatives for any functions: $f(x) = \text{sgn}(x)$, $g(x) = [x]$.

Example 3: Is $p(x) = x^5 - 3x^4 + 7x^2 + 7x - 3$ the derivative to some function? If yes, try to determine it.

Extra material: notes with solved Example 3.

147 About monotonicity of differentiable functions.

Theorem 6: *About monotonicity of functions.*

Let $a, b \in \mathbb{R}$ be such that $a < b$. Function $f : [a, b] \to \mathbb{R}$ is differentiable. If $f'(x) \geqslant 0$ at all points of $[a, b]$ (or: $f'(x) \leqslant 0$ at all points of $[a, b]$) then f is non-decreasing (or: non-increasing) on the entire interval.

Example showing that constant sign on the entire *interval* is important: $f(x) = \frac{1}{x}$ is decreasing on intervals, but not decreasing on $\mathbb{R}$.

148 About monotonicity of differentiable functions, some examples.

Knowing how to differentiate the following functions, we can confirm their monotonicity:

E1 Function $f(x) = e^x$ is strictly increasing on $\mathbb{R}$.

E2 Function $f(x) = \ln x$ is strictly increasing on $\mathbb{R}^+$.

E3 Function $f(x) = \arctan x$ is strictly increasing on $\mathbb{R}$.

E4 Function $f(x) = \arcsin x$ is strictly increasing on $(-1, 1)$.

E5 Function $f(x) = x^\alpha$ is strictly increasing on $\mathbb{R}^+$ if $\alpha > 0$ and strictly decreasing on $\mathbb{R}^+$ if $\alpha < 0$.

149 Monotonicity of exponential functions.

Example: Confirm with help of derivatives that $f(x) = a^x$ with $a > 1$ is strictly increasing on $\mathbb{R}$, and $g(x) = b^x$ with $0 < b < 1$ is strictly decreasing on $\mathbb{R}$.

150 Back to the derivatives of inverse functions, Problem 4.

Problems 1, 2, and 3 from Videos 127, 128, and 129: constant sign of the derivative on an interval guarantees invertibility of the function on this interval.

Problem 4: Show that $f(x) = \dfrac{\sin x}{x}$ is decreasing on the interval $(0, \pi)$. Motivate that this function is invertible and differentiable. Compute $(f^{-1})'(\frac{2}{\pi})$.

Extra material: notes with solved Problem 4.

151 Functions with derivative equal to zero.

Theorem 7: *Functions with derivative equal to zero.*

Let $a, b \in \mathbb{R}$ be such that $a < b$. Function $f : [a, b] \to \mathbb{R}$ is differentiable. If $f'(x) = 0$ at all points of $[a, b]$ then f is constant on the entire interval.

152 Corollary about functions that have the same derivative on an interval.

Corollary: Let $a, b \in \mathbb{R}$ be such that $a < b$. Functions $f, g : [a, b] \to \mathbb{R}$ are differentiable. If $f'(x) = g'(x)$ at all points of $[a, b]$ then $f = g + c$ for some constant $c \in \mathbb{R}$ on the entire interval.

Example: Back to Example b from Video 85: $f(x) = \ln 3x$.

153 Back to Video 106: explaining some subtleties in the Corollary in V152.

In Exercise 9 (V106) we showed that function $f(x) = \arctan\left(\dfrac{1+x}{1-x}\right)$ has the same derivative as $g(x) = \arctan x$ (in its entire domain). These functions don't differ by a constant, though. Explain why it doesn't contradict the Corollary from Video 152; make a picture.

Extra material: notes from the iPad.

154 Back to Video 107: explaining some subtleties in the Corollary in V152.

In Exercise 10 (V107) we showed that

$$f(x) = 2\arctan x + \arcsin\frac{2x}{1+x^2}$$

has derivative 0 for $|x| \geqslant 1$. What kind of conclusions can you make about this function using this information?

155 **Optional**: Back to Video 107; a trigonometry-based solution.

Simplify the formula for $f(x) = 2\arctan x + \arcsin\frac{2x}{1+x^2}$ using only trigonometry. Prove that

$$2\arctan x + \arcsin\frac{2x}{1+x^2} = \begin{cases} -\pi\,, & x \leqslant -1 \\ 4\arctan x\,, & -1 \leqslant x \leqslant 1 \\ \pi\,, & x \geqslant 1 \end{cases}$$

Extra material: notes from the iPad.

S7 Applications: monotonicity and optimisation

You will learn: how to apply the results from the previous section in more practical settings like examining monotonicity of differentiable functions and optimising (mainly continuous) functions; The First Derivative Test and The Second Derivative Test for classifications of CP (critical points) of differentiable functions.

Read along with this section: **Calculus book**: Chapter 4.3 *Maxima and Minima*, pages: from 422 (4.3.1) to 437 (4.3E.6); Chapter 4.4 *The Mean Value Theorem*, pages: from 438 (4.4.1) to 448 (4.4E.3); Chapter 4.5 *Derivatives and the Shape of a Graph* (without the discussion of concavity; we come back to this in Section 8), pages: from 449 (4.5.1) to 468 (4.5E.9).

156 You know everything you need to know about monotonicity and optimisation.

Examples from Video 155 in *Precalculus* 2: Plot the polynomials:

a) $p(x) = a_1x + a_0, \;\; a_1 \neq 0$
b) $p(x) = x^2 + x - 2$
c) $p(x) = x^3 - 2x^2 - 5x + 6$
d) $p(x) = x^4 + x^3 - 11x^2 - 9x + 18$.

157 Polynomials: some examples from Precalculus 2 revisited.

Problem 1 from Video 156: Sketch the graph of $p(x) = x^3 - 4x^2 + 4x$.

Problem 2 from Video 157: Sketch the graph of $p(x) = x^3 - 3x - 2$.

Problem 3 from Video 158: Sketch the graph of $p(x) = x^4 - 2x^2 - 3$.

158 How to find the vertex of a parabola if you hate completing the square.

Use derivatives to show that the vertex of the parabola $y = ax^2 + bx + c$ (where $a \neq 0$) is at

$$\left(-\frac{b}{2a}, -\frac{\Delta}{4a}\right),$$

where $\Delta = b^2 - 4ac$.

Extra material: notes from the iPad.

159 A rational function, Problem 1 from V220 in Calc1p1.

Problem 1: Determine the local and global extremums, and intervals of monotonicity for $f(x) = \dfrac{x^2 + 2x + 5}{x + 1}$.

Extra material: notes with solved Problem 1.

160 The one with two square roots, Problem 2 from V221 in Calc1p1.

Problem 2: Determine local and global extremums, and intervals of monotonicity for $f(x) = \sqrt{x^2 + x} - \sqrt{x^2 + 1}$.

Extra material: notes with solved Problem 2.

161 The one with arctangent, Problem 3 from V222 in Calc1p1.

Problem 3: Determine local and global extremums, and intervals of monotonicity for $f(x) = \arctan\left(\dfrac{x}{x+1}\right)$.

Extra material: notes with solved Problem 3.

162 Where the derivative doesn't help much, Problem 4 from V223 in Calc1p1.

Problem 4: Compute the derivative of $f(x) = \dfrac{\arcsin x}{\tan(x+1)}$.

Extra material: notes with solved Problem 4.

163 The one with a one-sided vertical asymptote, Problem 5 from V224 in Calc1p1.

Problem 5: Determine local and globbal extremums, and intervals of monotonicity for $f(x) = e^{1/(1+x)}$.

164 The First Derivative Test.

Example: Find and classify the critical points of $f(x) = x^4 - 2x^3 + 1$.

Extra material: notes with solved Example.

165 The Second Derivative Test.

Example: Find and classify the critical points of $f(x) = x^2 e^{-x}$. Solve using both tests.

Extra material: notes with solved Example.

166 Comparison between two tests: advantages and disadvantages.

167 Optimisation of continuous functions on compact and non-compact domains.

168 Optimisation, Problem 1.

Problem 1: Determine local and global extremums for $f(x) = |x^2 - x - 2|$ on $[-3, 3]$.

Extra material: notes with solved Problem 1.

169 Optimisation, Problem 2.

Problem 2: back to Problem 10 from V88 in *Precalculus* 4: Determine the range of $f(x) = -9^x - 2\cdot 3^x + 8,\ x \in \mathbb{R}$.

Extra material: notes with solved Problem 2.

170 Optimisation, Problem 3.

Problem 3: back to Problem 11 from V89 in *Precalculus* 4: Determine the range of $f(x) = 49^x - 7^x - 6,\ x \in \mathbb{R}$.

Extra material: notes with solved Problem 3.

171 Optimisation, Problem 4.

Problem 4: back to Problem 13 from V90 in *Precalculus* 4: Determine the range of

$$f(x) = \left(\frac{1}{2}\right)^{x^2 - 4x},\ x \in [1,\ 4.5].$$

Extra material: notes with solved Problem 4.

172 Optimisation, Problem 5.

Problem 5: Determine the largest and the smallest value of $f(x) = x \cdot e^{x - x^2}$ on $[-1, \frac{1}{2}]$.

Extra material: notes with solved Problem 5.

173 Comparing numbers, Problem 6.

Problem 6: back to Problems 11 and 12 from Videos 219 and 220 in *Precalculus* 4:

- Which is greater: π^e or e^π?
- Which is greater: 2022^{2023} or 2023^{2022}?

We have prepared some theory for this in Video 94 in the present course.

174 Comparing numbers, Problem 7.

Problem 7: back to Problem 23 from Video 147 in *Precalculus* 4.

Theorem: If $n \in \mathbb{N}$ and $n \geqslant 3$, then $\log_{n-1} n > \log_n (n+1)$.

Extra material: notes with solved Problem 7.

S8 Convexity and second derivatives

You will learn: how to determine with help of the second derivative whether a function is concave of convex on an interval; inflection points and how they look on graphs of functions; the concept of convexity is a general concept, but here we will only apply it to twice differentiable functions.

Read along with this section: **Calculus book**: Chapter 4.5 *Derivatives and the Shape of a Graph*, pages: from 449 (4.5.1) to 468 (4.5E.9).

175 Convexity, concavity, inflection points.

176 Convexity, concavity, inflection points: many examples.

Some examples:

* $f(x) = x^{2n}$ is convex on $\mathbb{R}$ and $g(x) = -x^{2n}$ is concave on $\mathbb{R}$ (for each $n \in \mathbb{N}^+$).
* $f(x) = x^{2n+1}$ (where $n \in \mathbb{N}^+$) is convex on $\mathbb{R}^+$ and concave on $\mathbb{R}^-$; $x_0 = 0$ is its inflection point. (The inflection point is at the same time a **critical point**.)
* $f(x) = e^x$ is convex on $\mathbb{R}$.
* $f(x) = \ln x$ is concave on $\mathbb{R}^+$.
* $f(x) = \sin x$ is convex on $((2k-1)\pi, 2k\pi)$ and concave on $(2k\pi, (2k+1)\pi)$ for all $k \in \mathbb{Z}$. (The inflection points are **not** critical points, and no critical points are inflection points.)
* Several examples of power functions: $f(x) = x^{4/3}$ is convex on $\mathbb{R}$. Its derivative $f'(x) = \frac{4}{3}x^{1/3}$ is convex on $\mathbb{R}^-$ and concave on $\mathbb{R}^+$ (the inflection point is at a **singular** point); $h(x) = x^{2/3}$ is concave on $\mathbb{R}^-$ and $\mathbb{R}^+$.

177 Convexity, Problem 1.

Problem 1: Examine convexity and possible inflection points for $f(x) = x^2e^{-x}$ from Video 165.

Extra material: notes with solved Problem 1.

178 Convexity, Problem 2.

Problem 2: Examine convexity and possible inflection points for $f(x) = 49^x - 7^x - 6$ from Video 170.

Extra material: notes with solved Problem 2.

179 Convexity, Problem 3.

Problem 3: Examine convexity and possible inflection points for $f(x) = \arctan\left(\dfrac{x}{x+1}\right)$ from Video 161.

Extra material: notes with solved Problem 3.

S9 l'Hôpital's rule with applications

You will learn: use l'Hôpital's rule for computing the limits of indeterminate forms; you get a very detailed proof in an article attached to the first video in this section.

Read along with this section: **Calculus book**: Chapter 4.8 *l'Hôpital's Rule*, pages: from 518 (4.8.1) to 536 (4.8E.4).

180 l'Hôpital's rule, the theorem with a proof (in an article).

Theorem (l'Hôpital's rule): Let X be a subset of $\mathbb{R}$ having one of the following forms[2]:

$$X = (a, x_0), \quad \text{where} \quad -\infty \leqslant a < x_0 \leqslant +\infty$$

or

$$X = (x_0, a), \quad \text{where} \quad -\infty \leqslant x_0 < a \leqslant +\infty$$

or

$$X = (a, x_0) \cup (x_0, b), \quad \text{where} \quad -\infty \leqslant a < x_0 < b \leqslant +\infty$$

and let $f, g : X \to \mathbb{R}$ be two differentiable functions such that $g'(x) \neq 0$ for all $x \in X$. Moreover, one of the requirements holds:

[2] Basically, X is either an (open) interval with x_0 as one of the endpoints, or a union of two (open) intervals (almost) meeting at x_0. This means that x_0 does **not** belong to X, but it is an accumulation point of X, so you can discuss (at least one-sided) limits (with x tending to x_0) of functions defined on X.

H1 $\lim_{x \to x_0} g(x) = \lim_{x \to x_0} f(x) = 0$

H2 $\lim_{x \to x_0} |g(x)| = +\infty$.

Under the assumptions above, if

$$\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = \alpha,$$

where $\alpha \in \mathbb{R} \cup \{-\infty, +\infty\}$, then also

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \alpha.$$

Extra material: an article with a (very detailed) proof of the rule.

181 l'Hôpital's rule, Exercise 1.

Exercise 1: Compute $\lim_{x \to 1} \frac{x^n - 1}{x^m - 1}$ (where $m \neq 0$), $\lim_{x \to 0} \frac{x^2 - 2 + 2\cos x}{x^4}$.

Extra material: notes with solved Exercise 1.

182 l'Hôpital's rule, Exercise 2.

Exercise 2: Compute $\lim_{x \to \infty} \frac{x^5}{e^x}$, $\lim_{x \to \infty} \frac{\ln x}{x^3}$.

Extra material: notes with solved Exercise 2.

183 l'Hôpital's rule, Exercise 3.

Exercise 3: Compute $\lim_{x \to 0^+} x \ln x$.

Extra material: notes with solved Exercise 3.

184 l'Hôpital's rule, Exercise 4.

Exercise 4: Compute $\lim_{x \to 1} \left(\frac{1}{x-1} - \frac{1}{\ln x} \right)$.

Extra material: notes with solved Exercise 4.

185 l'Hôpital's rule, Exercise 5.

Exercise 5: Compute $\lim_{x \to 0^+} x^{\sin x}$.

Extra material: notes with solved Exercise 5.

186 l'Hôpital's rule, Exercise 6.

Exercise 6: Compute $\lim_{x \to 0^+} \left(\frac{1}{x} \right)^{\sin x}$.

Extra material: notes with solved Exercise 6.

187 l'Hôpital's rule, Exercise 7.

Exercise 7: Compute $\lim_{x \to 1} x^{\frac{1}{1-x}}$.

Extra material: notes with solved Exercise 7.

188 l'Hôpital's rule, Exercise 8.

Exercise 8: Compute $\lim_{x \to 0} \left(\frac{1}{x} - \cot x \right)$.

Extra material: notes with solved Exercise 8.

189 l'Hôpital's rule, Exercise 9.

Exercise 9: $\lim_{x \to 0^+} \frac{\ln \sin 2x}{\ln \sin x}$.

Extra material: notes with solved Exercise 9.

S10 Higher order derivatives and an intro to Taylor's formula

You will learn: about classes of real-valued functions of a single real variable: C^0, C^1, $\ldots$, C^∞ and some prominent members of these classes; the importance of Taylor/Maclaurin polynomials and their shape for the exponential function, for the sine and for the cosine; you only get a glimpse into these topics, as they are usually a part of Calculus 2.

Read along with this section: **Calculus book**: Chapter 3.5 *Derivatives of Trigonometric Functions*, pages: from 325 (3.5.1) to 337 (3.5E.4); Chapter 10.3 *Taylor and Maclaurin Series*, pages: from 1226 (10.3.1) to 1246 (10.3E.7). (Large parts of Chapter 10.3 will be too hard to deal with now, though. You can wait with the study of these topics when we arrive at *Calculus 2, part 2 of 2: Sequences and series.*)

190 Why we want to approximate functions with polynomials.

191 Monic monomials closely to zero.

192 Remember higher-order derivatives? Smooth and less smooth functions.

- $f(x) = |x|$ is C^0, but not differentiable, $g(x) = x \cdot |x|$ is C^1, but not twice differentiable.
- $f(x) = x^{8/3}$ is C^2 but not C^3: $f'(x) = \frac{8}{3}x^{5/3}$, $f''(x) = \frac{40}{9}x^{2/3}$, $f'''(x) = \frac{80}{27}x^{-1/3}$, $x \neq 0$.
- $f(x) = x^{4/3}$ is C^1 but not C^2: $f'(x) = \frac{4}{3}x^{1/3}$, $f''(x) = \frac{4}{9}x^{-2/3}$, $x \neq 0$.

193 How to construct a polynomial that has the same derivatives at some point as a given function.

Extra material: notes from the iPad showing that $p_n(x_0) = f(x_0),\ p_n'(x_0) = f'(x_0),\ \ldots,\ p_n^{(n)}(x_0) = f^{(n)}(x_0)$.

194 Maclaurin polynomial for the exponential function.

Exercise 1: Derive Maclaurins polynomials for $f(x) = e^x$ and show that $\lim\limits_{n\to\infty} R_n(x) = 0$ for each $x \in \mathbb{R}$.

Extra material: notes with the derivation.

195 Maclaurin polynomial for the sine.

Exercise 2: Derive Maclaurins polynomials for $f(x) = \sin x$ and show that $\lim\limits_{n\to\infty} R_n(x) = 0$ for each $x \in \mathbb{R}$.

Extra material: notes with the derivation.

196 Maclaurin polynomial for the cosine.

Exercise 3: Derive Maclaurins polynomials for $f(x) = \cos x$ and show that $\lim\limits_{n\to\infty} R_n(x) = 0$ for each $x \in \mathbb{R}$.

Extra material: notes with the derivation.

197 Taylor polynomial and the Second (and more!) Derivative Test.

198 Approximation, an example.

Example: Find the Taylor polynomial $p_2(x)$ for $f(x) = \sqrt{x}$ about $x_0 = 4$ and use it to approximate $\sqrt{4.01}$. Compare your result to the one obtained in Video 36.

Extra material: notes with solved Example.

199 Limit, an example.

Back to V76: compute the limit $\lim\limits_{x\to 0} \dfrac{3x \cdot \cos 3x - \sin 3x}{x^2}$. Use both Maclaurin polynomials (without a proper knowledge of the theory) and l'Hôpital's rule (fully proven).

Extra material: notes from the iPad.

200 Order relation on the classes of functions introduced in V192.

Use the following functions to show that the inclusion between classes is strict:

$$f(x) = \begin{cases} \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}, \qquad g(x) = \begin{cases} x\sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}, \qquad h(x) = \begin{cases} x^2\sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

S11 Implicit differentiation

You will learn: how to find the derivative $y'(x)$ from an implicit relation $F(x, y) = 0$ by combining various rules for differentiation; you will get some examples of curves described by implicit relations, but their study is *not* included in this course (it is usually studied in *Algebraic Geometry*, *Differential Geometry* or *Geometry and Topology*; the topic is also partially covered in *Calculus 3 (Multivariable Calculus), part* 1 *of* 2: Implicit Function Theorem).

Read along with this section: **Calculus book**: Chapter 3.8 *Implicit Differentiation*, pages: from 366 (3.8.1) to 376 (3.8E.4).

201 Explicit versus implicit, Example 0.

Example 0: Describe the curve $x^2 + y^2 = r^2$ (where $r > 0$) as explicit functions $y = y(x)$ and $x = x(y)$ wherever possible. Compute their derivatives in both explicit and implicit way.

202 Implicit differentiation and how it works.

203 One more example easy to handle both ways, Example 1.

Example 1: Given a curve defined by $x^3 - x = y^2$. Find the derivative $y'(x)$ everywhere it is possible and write an equation for the tangent line through the point $(-\frac{1}{2}, \frac{\sqrt{6}}{4})$. Verify your result by computing $y = y(x)$ in an explicit way and by making a picture in software of your choice. (This problem comes back in Section 18.)

Extra material: notes with solved Example 1.

204 Derivative of the logarithm, method 3.

Derive the derivative of $f(x) = \ln x$ knowing that $(e^x)' = e^x$. Use implicit differentiation and the cancellation identity $e^{\ln x} = x$.

Extra material: notes with the derivation of the formula.

205 The derivative of roots (power functions 2), method 3.

If $n \in \mathbb{N}^+$, $n \geqslant 2$ and $f(x) = \sqrt[n]{x} = x^{1/n}$ (for $x \geqslant 0$ if n is even), then $f'(x) = \frac{1}{n}x^{1/n-1}$ (for $x > 0$); a proof using implicit differentiation and the fact that $(x^n)' = nx^{n-1}$.

Extra material: notes with the derivation of the formula.

206 Derivative of arctangent, method 3.

Derive the derivative of $f(x) = \arctan x$. Use implicit differentiation and $f(\tan x) = x$ for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Extra material: notes with the derivation of the formula.

207 Derivative of arcsine, method 3.

Compute the derivative of $f(x) = \arcsin x$. Use implicit differentiation and $f(\sin x) = x$ for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Extra material: notes with the derivation of the formula.

208 The one with a heart.

Problem: Given the curve $(x^2 + y^2 - 1)^3 - x^2y^3 = 0$. Find an equation for the tangent line through $(-1, 1)$.

Extra material: notes with the solution to Problem.

209 Plenty of problems to solve; we walk through two of them.

A walkthrough: Problems 23 and 24 from the attached article.

Extra material: an article with more solved problems on implicit differentiation. In all the problems formulated below, determine $\frac{dy}{dx}$ ($y'(x)$; or: $f'(x)$ if the condition is formulated in terms of $y = f(x)$) if:

- ⋆ **Extra problem 1**: $x + y = \ln y$.
- ⋆ **Extra problem 2**: $e^{x+y} = x$.
- ⋆ **Extra problem 3**: $x + y = e^y$.
- ⋆ **Extra problem 4**: $x^2 + y^2 = \tan y$.
- ⋆ **Extra problem 5**: $xy = e^y$. Find also the second derivative $y''(x)$.
- ⋆ **Extra problem 6**: $y^2 + y = e^{xy}$.

- ⋆ **Extra problem 7**: $x + xy = \ln y$.
- ⋆ **Extra problem 8**: $\sin(x + y) = x$.
- ⋆ **Extra problem 9**: $ye^y = e^x - x$.
- ⋆ **Extra problem 10**: $xy^2 = \ln y$.
- ⋆ **Extra problem 11**: $x \cos y = y \sin x$.
- ⋆ **Extra problem 12**: $x^2 + f(x)^2 = \sin(f(x))$.
- ⋆ **Extra problem 13**: $x^2 + x + xy + e^y = 1$.
- ⋆ **Extra problem 14**: $x^3 + 2y^3 + xy = \cos y$.
- ⋆ **Extra problem 15**: $\sin x \cdot f(x) + \sqrt{1 + f(x)} = x$.
- ⋆ **Extra problem 16**: $e^{xy} + \sin x + y^2 = 2$.
- ⋆ **Extra problem 17**: $e^{f(x)} + \ln(f(x)) = x$.
- ⋆ **Extra problem 18**: $(1 + 2x) \cdot f(x) + \cos(f(x)) = e^x$. Assuming that $f(0) = 0$, compute $f'(0)$.
- ⋆ **Extra problem 19**: $f(x)^2 + 2f(x)^3 - xf(x) = x + 1$. Assuming that $f(1) = 1$, compute $f'(1)$.
- ⋆ **Extra problem 20**: $e^{f(x)} + xf(x) + x^3 = 2$. Assuming that $f(1) = 0$, compute $f'(1)$.
- ⋆ **Extra problem 21**: $x^2 + xy + y^2 = 3$. Determine the equation to the tangent line to the curve in $(1, 1)$.
- ⋆ **Extra problem 22**: $x^2 + xy + 2y^3 = 4$. Determine the equation to the tangent line to the curve in $(-2, 1)$.
- ⋆ **Extra problem 23**: $x^3 + 4xy - 3y^{\frac{4}{3}} = 2x$.
- ⋆ **Extra problem 24**: Determine $\frac{dx}{dy}$ if $x^3 + 4xy - 3y^{\frac{4}{3}} = 2x$. **(Attention! For once we will think about a function $x = x(y)$, i.e., y is an independent variable, and x is a function of y.)**

S12 Logarithmic differentiation

You will learn: how to perform logarithmic differentiation and in what type of cases it is practical to apply.

Read along with this section: **Calculus book**: Chapter 3.9 *Derivatives of Exponential and Logarithmic Functions* (at the very end of the chapter), pages: from 377 (3.9.1) to 389 (3.9E.4).

210 Logarithmic differentiation, Problem 1.

Problem 1: Differentiate $f(x) = \dfrac{(x+1)(x+2)(x+3)}{x+4}$.

Extra material: notes with solved Problem 1.

211 Logarithmic differentiation, Problem 2.

Problem 2: Differentiate $f(x) = \sqrt{(x+1)(x^2+1)(x^3+1)}$.

Extra material: notes with solved Problem 2.

212 Logarithmic differentiation, Problem 3.

Problem 3: Differentiate $f(x) = \prod_{n=1}^{100} (x^2 + n)$.

Extra material: notes with solved Problem 3.

213 Logarithmic differentiation, Problem 4.

Problem 4: Differentiate $f(x) = \dfrac{x^5}{(1-10x)\sqrt{x^2+2}}$.

Extra material: notes with solved Problem 4.

214 Back to the powers, Problem 5.

Problem 5: Differentiate $f(x) = (1-3x)^{\cos x}$.

Extra material: notes with solved Problem 5.

S13 Very briefly about partial derivatives

You will learn: how to compute partial derivatives to multivariable functions (just an introduction).

Read along with this section: **Calculus book**: Chapter 14.3 *Partial Derivatives*, page 1716 (14.3.1).

215 Functions of several variables.

216 Partial derivatives, Exercise 1.

Example 1: Compute all the first-order partial derivatives for $f(x, y, z) = x^3y^4z^5$.

Example 2: Compute all the first-order partial derivatives for $f(x, y) = e^{x+y} \sin(xy)$.

Exercise 1: Find all the first-order partial derivatives of the function $f(x, y, z) = x^{y \ln z}$.

Extra material: notes with solved Exercise 1.

217 Higher-order partial derivatives, Exercise 2.

Exercise 2: Compute all the second-order partial derivatives for $f(x, y, z) = x^3y^4z^5$.

Extra material: notes with solved Exercise 2.

S14 Very briefly about antiderivatives

You will learn: about the wonderful applicability of integrals and about the main integration techniques.

Read along with this section: **Calculus book**: Chapter 4.10 *Antiderivatives*, pages: from 550 (4.10.1) to 564 (4.10E.4).

218 Reverting differentiation.

219 A word about main integration techniques.

220 It is more useful than you think.

S15 A very brief introduction to the topic of ODE

You will learn: some very basic stuff about ordinary differential equations.

Read along with this section: **Calculus book**: Chapter 6.8 *Exponential Growth and Decay*, pages: from 845 (6.8.1) to 856 (6.8E.3). Differential equations will be covered in my upcoming course about ODE; in the **Calculus book** the topic is covered in Chapters 8 and 17 (but the text is outside the scope of the present course).

221 Various types of equations.

222 Differential equations mentioned in Precalculus 4.

223 Solving versus verifying solutions to ODE.

Example: Verify that functions $y(x) = C_1e^{x/3} + C_2e^{-x/2} + \frac{1}{4}e^{3x}$, where C_1, $C_2 \in \mathbb{R}$ are any real numbers, satisfy the equation $6y'' + y' - y = 14e^{3x}$.

Extra material: notes with solved Example.

S16 More advanced concepts built upon the concept of derivative

You will learn: about some more advanced concepts based on the concept of derivative: partial derivative, gradient, jacobian, hessian, derivative of vector-valued functions, divergence, rotation (curl).

224 Plenty of derivative-like creatures.

225 Multivariate Taylor polynomials.

226 Mean Value Theorem used for functions of several variables.

S17 Problem solving: optimisation

You will learn: how to solve optimisation problems (practice to Section 7).

Read along with this section: **Calculus book**: Chapter 4.3 *Maxima and Minima*, pages: from 422 (4.3.1) to 437 (4.3E.6); Chapter 4.7 *Applied Optimization Problems*, pages: from 502 (4.7.1) to 517 (4.7E.5).

227 Optimisation: a practical section.

Extra material: an article with more solved problems on optimisation. In all the problems, determine all the critical points and describe their type (local max, local min, none of the above) for the function f; in some cases, when the interval I is given, determine the largest or the smallest (or both, depending on the problem) value of f on I.

- ⋆ **Extra problem 1**: $f(x) = 4x^3 - 3x$.
- ⋆ **Extra problem 2**: $f(x) = -2x^3 - 3x^2 + 12x$, on the interval $I = [-3, 3]$.
- ⋆ **Extra problem 3**: $f(x) = x^3 - 3x^2 + 3$, on the interval $I = [1, 3]$.
- ⋆ **Extra problem 4**: $f(x) = \dfrac{x}{1 + x^2}$, on the interval $I = [0, 10]$.
- ⋆ **Extra problem 5**: $f(x) = e^x(x^2 + 3x - 3)$, on the interval $I = [-10, 10]$.
- ⋆ **Extra problem 6**: $f(x) = e^x(x^2 - x - 1)$, on the interval $I = [0, 3]$.
- ⋆ **Extra problem 7**: $f(x) = e^{2x}(2x^2 - 2x - 7)$.
- ⋆ **Extra problem 8**: $f(x) = e^{2x}\left(2x^2 - 2x - 17\right)$, on the interval $I = [0, 10]$.
- ⋆ **Extra problem 9**: $f(x) = e^{-2x}(2x^2 + 6x - 3)$.
- ⋆ **Extra problem 10**: $f(x) = e^x \sin x$, on the interval $I = [-\pi, \pi]$.
- ⋆ **Extra problem 11**: $f(x) = x - 2\arctan x$, on the interval $I = [0, 10]$.
- ⋆ **Extra problem 12**: $f(x) = \arctan(4x - 4x^2)$, on the interval $I = [0, 1]$.

228 Optimisation, Problem 1.

Problem 1: The sum of two positive numbers is equal to 7. What is the maximal possible value of their product?

Extra material: notes with solved Problem 1.

229 Optimisation, Problem 2.

Problem 2: The sum of two non-negative numbers is equal to n. What is the least possible value of the sum of their squares?

Extra material: notes with solved Problem 2.

230 Optimisation, Problem 3.

Problem 3: A rectangular animal enclosure is to be constructed having one side along an existing long wall and the other three sides fenced. If 100m of fence are available, what is the largest possible area of the enclosure?

Extra material: notes with solved Problem 3.

231 Optimisation, Problem 4.

Problem 4: Given a circle with radius R. Determine the side lengths of the rectangle with maximal possible area that can be inscribed in this circle.

Extra material: notes with solved Problem 4.

232 Optimisation, Problem 5.

Problem 5: Given a sphere with radius R. Determine the minimal volume of a right circular cone circumscribed around this sphere.

Extra material: notes with solved Problem 5.

233 Optimisation, Problem 6.

Problem 6: Find the shortest possible distance from the origin to a point on the curve given by $x^2y^4 = 1$.

Extra material: notes with solved Problem 6.

234 Proving inequalities, Problem 7.

Problem 7: Use the theory of monotonicity to show that $\sin x \leqslant x$ for all $x \in [0, \frac{\pi}{2}]$.

Extra material: notes with solved Problem 7.

235 Proving inequalities, Problem 8.

Problem 8: Use the theory of monotonicity to show that $x - \frac{x^3}{3!} \leqslant \sin x$ for all $x \in [0, \frac{\pi}{2}]$.

Extra material: notes with solved Problem 8.

S18 Problem solving: plotting functions

You will learn: how to make the table of (sign) variations for the function and its derivatives; you get a lot of practice in plotting functions (topic covered partly in *Calculus 1, part 1 of 2: Limits and continuity*, and completed in Sections 6–8 of the present course).

Read along with this section: **Calculus book**: Chapter 4.3 *Maxima and Minima*, pages: from 422 (4.3.1) to 437 (4.3E.6); Chapter 4.5 *Derivatives and the Shape of a Graph*, pages: from 449 (4.5.1) to 468 (4.5E.9); Chapter 4.6 *Limits at Infinity and Asymptotes*, pages: from 469 (4.6.1) to 501 (4.6E.8). [The topics from Chapter 4.6 were covered in *Calculus 1, part 1 of 2: Limits and continuity*, but you can read the chapter from the book as repetition.]

236 How to make a table of (sign) variations.

237 A brief repetition about asymptotes.

Example 1: (from V219 in *Calculus 1, part 1 of 2*): $f(x) = \frac{3x}{10} + \arctan x$.

Example 2: $f(x) = x + \sin x$.

Example 3: $f(x) = x + \frac{1}{2}\sin x$.

Example 4: $f(x) = x + \dfrac{\sin x}{x}, \quad g(x) = x + \dfrac{10\sin x}{x} - 3$.

Extra material: an article with more solved problems on plotting functions.

- ⋆ **Extra problem 1**: Plot $f(x) = e^{x^2-2x}$.
- ⋆ **Extra problem 2**: Plot $f(x) = e^x(-x^2 + 5x - 7)$.
- ⋆ **Extra problem 3**: Plot $f(x) = e^x(x^2 - 4x + 4)$.
- ⋆ **Extra problem 4**: Plot $f(x) = \dfrac{(x-1)^3}{(x+1)^2}$.

238 Plotting functions, Problem 1.

Problem 1: Back to the upper part of the curve from Video 203; plot $f(x) = \sqrt{x^3 - x}$.

Extra material: notes with solved Problem 1.

239 Plotting functions, Problem 2.

Problem 2: Plot $f(x) = 3x^4 - 16x^3 + 24x^2$.

Extra material: notes with solved Problem 2.

240 Plotting functions, Problem 3.

Problem 3: Plot

$$f(x) = \frac{x^3}{(x-1)^2}.$$

Extra material: notes with solved Problem 3.

241 Plotting functions, Problem 4.

Problem 4: Find all the CPs for:

$$f(x) = \frac{e^{x+\frac{1}{x}}}{x}$$

and discuss monotonicity of f.

Extra material: notes with solved Problem 4.

242 Plotting functions, Problem 5.

Problem 5: Plot

$$f(x) = \arcsin \sqrt{x-1}.$$

Extra material: notes with solved Problem 5.

243 Plotting functions, Problem 6.

Problem 6: Plot

$$f(x) = \frac{x^2}{2} + 2x - \ln|x+2|.$$

Extra material: notes with solved Problem 6.

244 Plotting functions, Problem 7.

Problem 7: Plot

$$f(x) = (x+2)^{2/3} - (x-2)^{2/3}.$$

Extra material: notes with solved Problem 7.

245 Wrap-up Calculus 1.

S19 Extras

You will learn: about all the courses we offer, and where to find discount coupons. You will also get a glimpse into our plans for future courses, with approximate (very hypothetical!) release dates.

B Bonus lecture.

Extra material 1: a pdf with all the links to our courses, and coupon codes.

Extra material 2: a pdf with an advice about optimal order of studying our courses.

Extra material 3: a pdf with information about course books, and how to get more practice.