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Sturmian words with balanced construction

Hanna Uscka-Wehlou

$$\begin{tikzcd}
]0,1[\setminus \mathbf{Q} \arrow[r, "s'"] \arrow[dr, "1\gamma"] & \mathcal{UM}_0 \arrow[d, "\Delta_c"] \\
& \{0,1\}^\omega \supset \mathcal{UM}_0
\end{tikzcd}$$

William G. Kolakoski

Run-length encoding operator / fixed point

Herbert Freeman

Chain codes / uniformity / balance

Azriel Rosenfeld

Hierarchy of runs

The run-length encoding operator

$$\Delta_l \colon \{1,2\}^\omega \to \mathbf{N}^\omega$$

$$w = \begin{cases} 1^{k_1}2^{k_2}1^{k_3}2^{k_4}\cdots, & \text{if } w \in 1\cdot\{1,2\}^\omega \\ 2^{k_1}1^{k_2}2^{k_3}1^{k_4}\cdots, & \text{if } w \in 2\cdot\{1,2\}^\omega \end{cases}$$

$$\Delta_l(w) = k_1k_2k_3\cdots$$

The Kolakoski word

$$w = 2\ 2\ 1\ 1\ 2\ 1\ 2\ 2\ 1\ 2\ 2\ 1\ 1\ 2\ \ldots$$

$$\Delta_l(w) = 2\ 2\ 1\ 1\ 2\ 1\ 2\ 2\ 1\ 2\ 2\ 1\ 1\ 2\ \ldots$$

Digital lines...

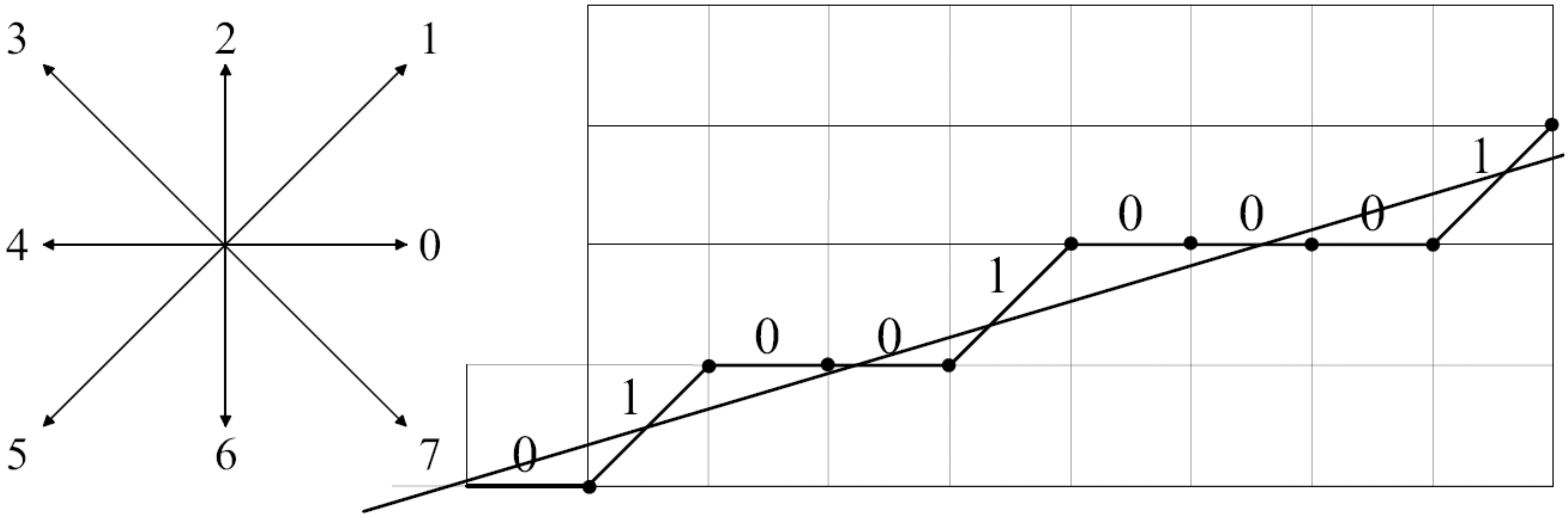
$$D_{R'}(y = ax,\ x > 0) = \{(k, \lceil ak \rceil);\quad k \in \mathbf{N}^+\}$$

...and binary words

$$s'(a)\colon \mathbf{N} \to \{0, 1\}$$

$$\forall\, n \in \mathbf{N} \qquad s'_n(a) = \lceil a(n+1) \rceil - \lceil an \rceil$$

The Freeman code

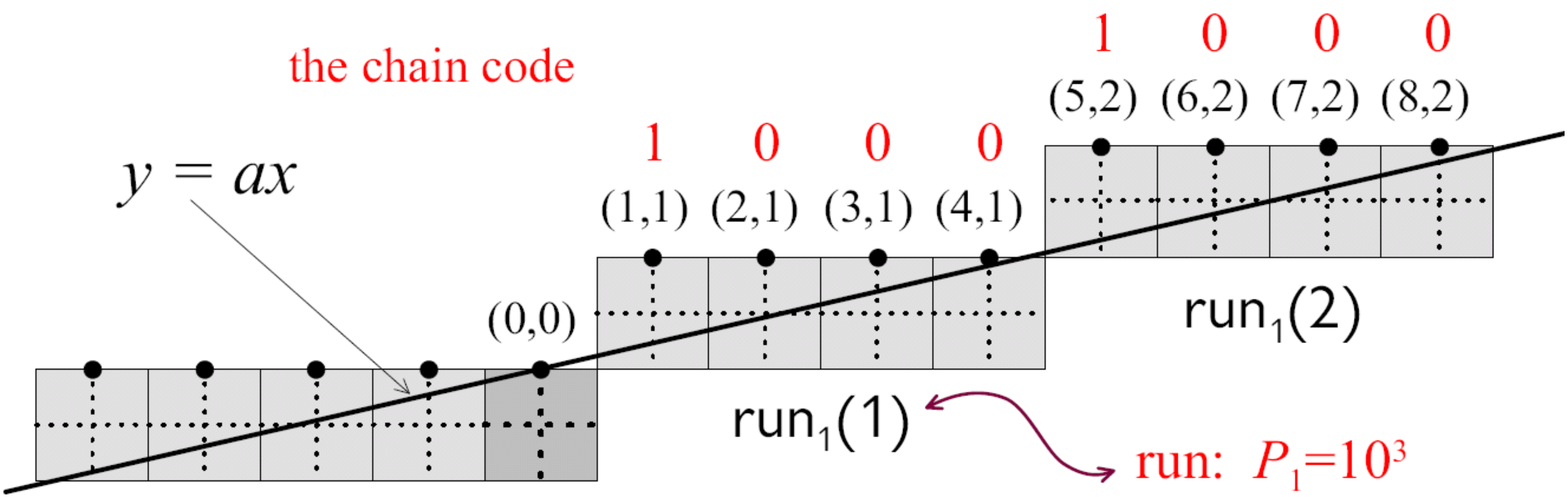


the chain code : ... 010010001 ...

Freeman's notion of balance

(F3) successive occurrences of the element occurring singly are as uniformly spaced as possible.

Rosenfeld and his runs



Rosenfeld and his runs

Two run lengths on level 1: runs$_1$ $S_1=10^m$ and $L_1=10^{m+1}$

Two run lengths on level 2: runs$_2$ $S_2=S_1L_1^{\,k}$ and $L_2=S_1L_1^{\,k+1}$

or $S_2=S_1^{\,k}L_1$ and $L_2=S_1^{\,k+1}L_1$

Two run lengths on level n: runs$_n$ $S_n=S_{n-1}L_{n-1}^{\,l}$ and $L_n=S_{n-1}L_{n-1}^{\,l+1}$

or $S_n=S_{n-1}^{\,l}L_{n-1}$ and $L_n=S_{n-1}^{\,l+1}L_{n-1}$

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Three questions. About:

the run length on level n

the main run on level n-1

the first run on level n-1

The index jump function

$$a = [0; a_1, a_2, a_3, a_4, a_5, a_6, a_7, \ldots]$$

$$i_a : \mathbf{N}^+ \to \mathbf{N}^+$$

$$i_a(1) = 1,\ i_a(2) = 2, \text{ for } n \geq 2:$$

$$i_a(n+1) = i_a(n) + 1 + \delta_1(a_{i_a(n)})$$

$$\begin{array}{lcccccccccccccl}
a = [0; & \overset{b_1}{1}, & \overset{b_2}{a_2}, & \overbrace{\underline{1},\ 1}^{b_3}, & \overset{b_4}{a_5}, & \overbrace{\underline{1},\ 1}^{b_5}, & \overset{b_6}{a_8}, & \overset{b_7}{a_9}, & \overbrace{\underline{1},\ a_{11}}^{b_8}, & \overset{b_9}{a_{12}}, & \overbrace{\underline{1},\ 1}^{b_{10}}, & \overbrace{\underline{1},\ a_{16}}^{b_{11}}, & \overset{b_{12}}{a_{17}}, & \ldots] \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \ldots \\
(i_a(k))_{k \in \mathbf{N}^+} = (& 1, & 2, & 3, & 5, & 6, & 8, & 9, & 10, & 12, & 13, & 15, & 17, & \ldots)
\end{array}$$

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$$(i_a(k))_{k \in \mathbf{N}^+} = (1, 2, 3, 5, 6, 8, 9, 10, 12, 13, 15, 17, \ldots)$$

Sturmian words with balanced construction

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The slope of the line (upper mech. word) is $a = [\,0\,; a_1, a_2, a_3, \ldots\,]$

How $i_a(k)$ and $a_{i_a(k)}$ describe the form of run_k:

$S_1 = 10^{a_1-1}$, $L_1 = 10^{a_1}$, and, for $k \geq 2$:

$$a_{i_a(k)} \geq 2 \quad \Rightarrow \quad S_k = S_{k-1}^{a_{i_a(k)}-1} L_{k-1} \ (\text{or } L_{k-1} S_{k-1}^{a_{i_a(k)}-1}),$$

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$$i_a(k) \ \text{ is even} \quad \Rightarrow \quad P_{k-1} = S_{k-1},$$
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The slope of the line (upper mech. word) is $a = [\,0\,; a_1, a_2, a_3, \ldots\,]$

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slopes: $a = [0; 1, 2, 1, 1, 3, 1, 1, a_8, a_9, \ldots]$

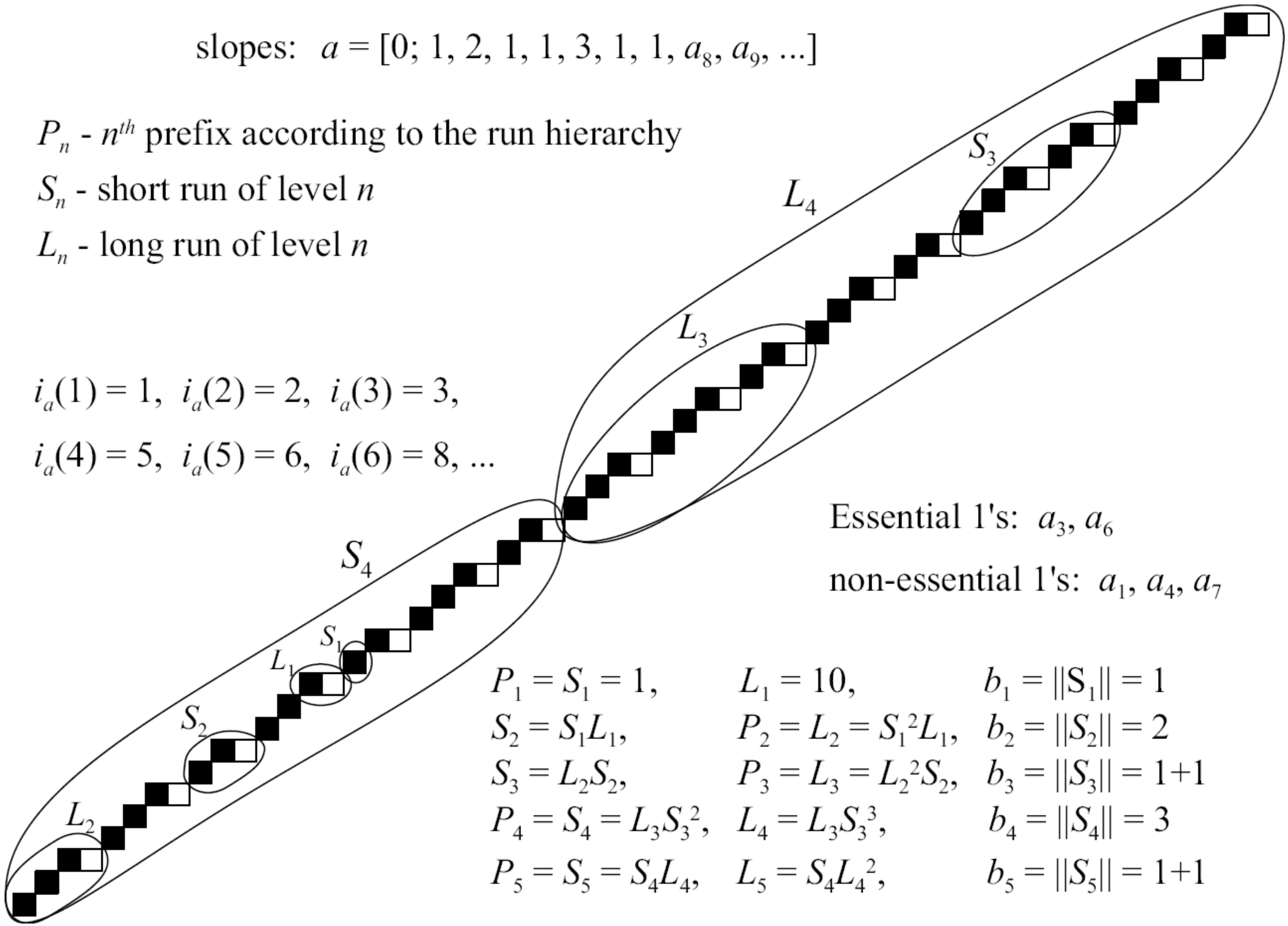
P_n - n^{th} prefix according to the run hierarchy

S_n - short run of level n

L_n - long run of level n

$i_a(1) = 1,\ \ i_a(2) = 2,\ \ i_a(3) = 3,$

$i_a(4) = 5,\ \ i_a(5) = 6,\ \ i_a(6) = 8, \ldots$



Essential 1's: a_3, a_6

non-essential 1's: a_1, a_4, a_7

$P_1 = S_1 = 1,$	$L_1 = 10,$	$b_1 = \lVert S_1 \rVert = 1$
$S_2 = S_1 L_1,$	$P_2 = L_2 = S_1^{\,2} L_1,$	$b_2 = \lVert S_2 \rVert = 2$
$S_3 = L_2 S_2,$	$P_3 = L_3 = L_2^{\,2} S_2,$	$b_3 = \lVert S_3 \rVert = 1+1$
$P_4 = S_4 = L_3 S_3^{\,2},$	$L_4 = L_3 S_3^{\,3},$	$b_4 = \lVert S_4 \rVert = 3$
$P_5 = S_5 = S_4 L_4,$	$L_5 = S_4 L_4^{\,2},$	$b_5 = \lVert S_5 \rVert = 1+1$

Calculations for the latest example

k	$a_{i_a(k+1)} = 1?$	main_k	$i_a(k+1)$	b_k	prefix P_k of $s'(a)$
1	$a_{i_a(2)} = a_2 = 2 \geq 2$	S_1	even	1	$S_1 = 1$
2	$a_{i_a(3)} = a_3 = 1$	L_2	odd	2	$L_2 = S_1^2 L_1 = 1110$
3	$a_{i_a(4)} = a_5 = 3 \geq 2$	S_3	odd	2	$L_3 = L_2^2 S_2$
4	$a_{i_a(5)} = a_6 = 1$	L_4	even	3	$S_4 = L_3 S_3^2$
5	$a_{i_a(6)} = a_8 \geq 2$	S_5	even	2	$S_5 = S_4 L_4$
6	$a_{i_a(7)} = a_9 \geq 2$	S_6	odd	a_8	$L_6 = S_5^{a_8} L_5$

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4	$a_{i_a(5)} = a_6 = 1$	L_4	even	3	$S_4 = L_3 S_3^2$
5	$a_{i_a(6)} = a_8 \geq 2$	S_5	even	2	$S_5 = S_4 L_4$
6	$a_{i_a(7)} = a_9 \geq 2$	S_6	odd	a_8	$L_6 = S_5^{a_8} L_5$

Calculations for the latest example

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2	$a_{i_a(3)} = a_3 = 1$	L_2	odd	2	$L_2 = S_1^2 L_1 = 1110$
3	$a_{i_a(4)} = a_5 = 3 \geq 2$	S_3	odd	2	$L_3 = L_2^2 S_2$
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Two equivalence relations on the set of slopes

1. based on run length on all levels for $s'(a)$:

$$a \in [(b_1, b_2, b_3, \dots)]_{\sim_{\mathrm{len}}} \quad \Leftrightarrow \quad \forall \; k \in \mathbf{N}^+ \;\; \|S_k\| = b_k$$

2. based on run construction on all levels for $s'(a)$:

$$a \sim_{\mathrm{con}} a' \quad \Leftrightarrow \quad i_a \equiv i_{a'}$$

Sturmian words with balanced construction

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The constructional word $\gamma(a) \in \{0,1\}^\omega$

Let $a = [0; a_1, a_2, \ldots]$. For $n \in \mathbf{N}^+$:

$$\gamma_n(a) = i_a(n+2) - i_a(n+1) - 1$$

$$\gamma_n(a) = \delta_1(a_{i_a(n+1)})$$

$$\gamma_n(a) = \begin{cases} 0, & S_n \text{ is the most frequent run on level } n \text{ for } s'(a) \\ 1, & L_n \text{ is the most frequent run on level } n \text{ for } s'(a). \end{cases}$$

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Definition The *run-construction encoding operator* $\Delta_c : \mathcal{UM}_0 \longrightarrow \{0,1\}^\omega$ is defined as $\Delta_c = (1\gamma)\circ(s')^{-1}$.

$$\begin{tikzcd}]0,1[\setminus \mathbf{Q} \arrow[r, "s'"] \arrow[dr, "1\gamma"'] & \mathcal{UM}_0 \arrow[d, "\Delta_c"] \\ & \{0,1\}^\omega \supset \mathcal{UM}_0 \end{tikzcd}$$

where $\mathcal{UM}_0$ denotes the set of all upper mechanical words with irrational slope $0 < a < 1$ and with intercept 0.

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Let $a \in \,]0,1[\, \setminus \mathbf{Q}$. The word $s'(a) = 1c(a)$ has

balanced construction if

$$\exists \, \alpha \in \mathbf{R} \quad \gamma(a) = c(\alpha)$$

Sturmian-balanced construction if

$$\exists \, \alpha \in \,]0,1[\, \setminus \mathbf{Q} \quad \gamma(a) = c(\alpha)$$

self-balanced construction

$$1\gamma(a) = \Delta_c(1c(a)) = 1c(a)$$

A fixed-point theorem: exactly 1 fixed point in each equivalence class

Let $(b_n)_{n\in\mathbf{N}^+}$ be such that $b_1 \in \mathbf{N}^+$ and $b_n \in \mathbf{N}^+ \setminus \{1\}$ for all $n \geq 2$. Then

$$\exists^1_{a\in]0,1[\setminus\mathbf{Q}}$$

$$a \in [(b_n)_{n\in\mathbf{N}^+}]_{\sim_{\mathrm{len}}} \ \wedge \ s'(a) = \Delta_c(s'(a)).$$

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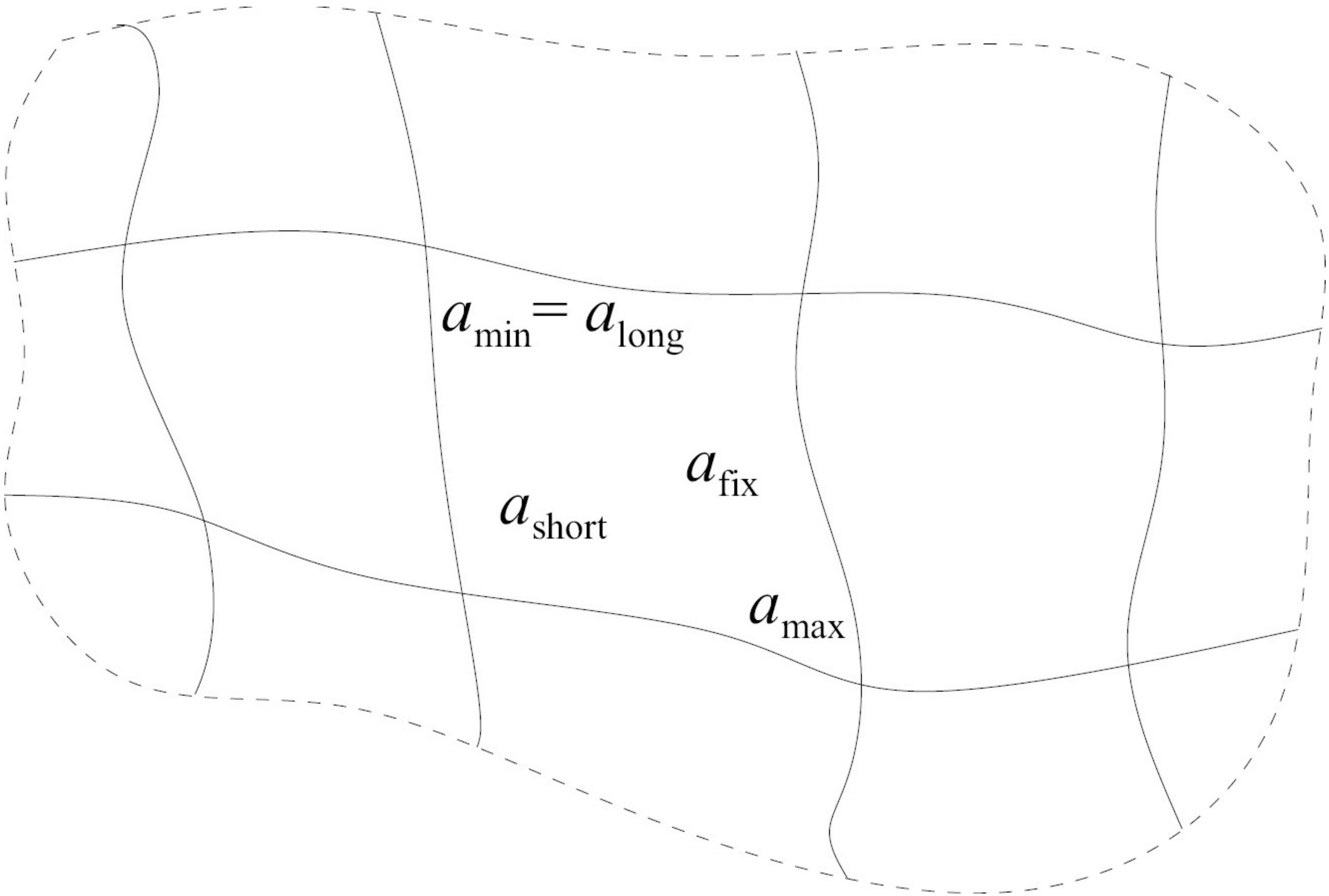

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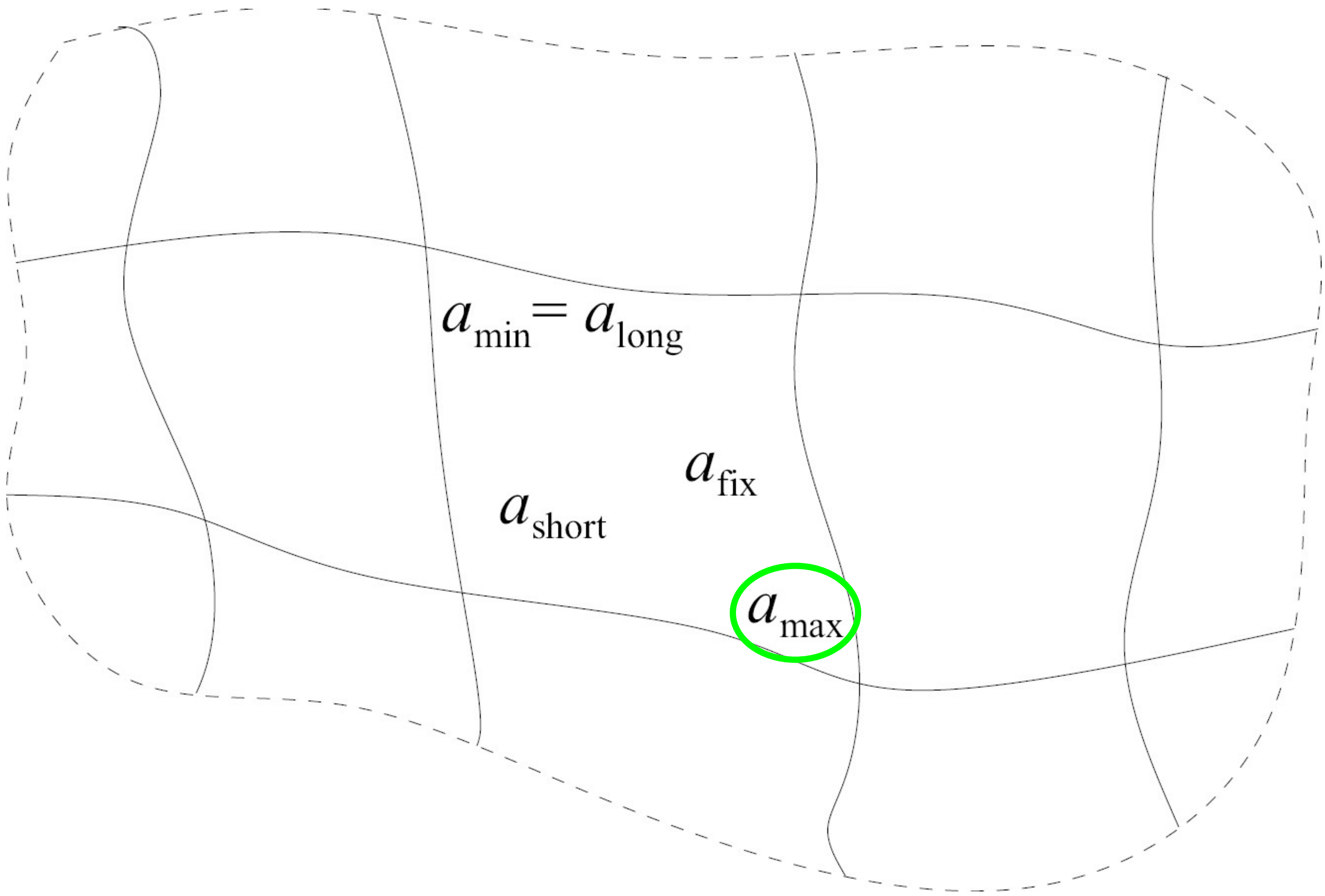

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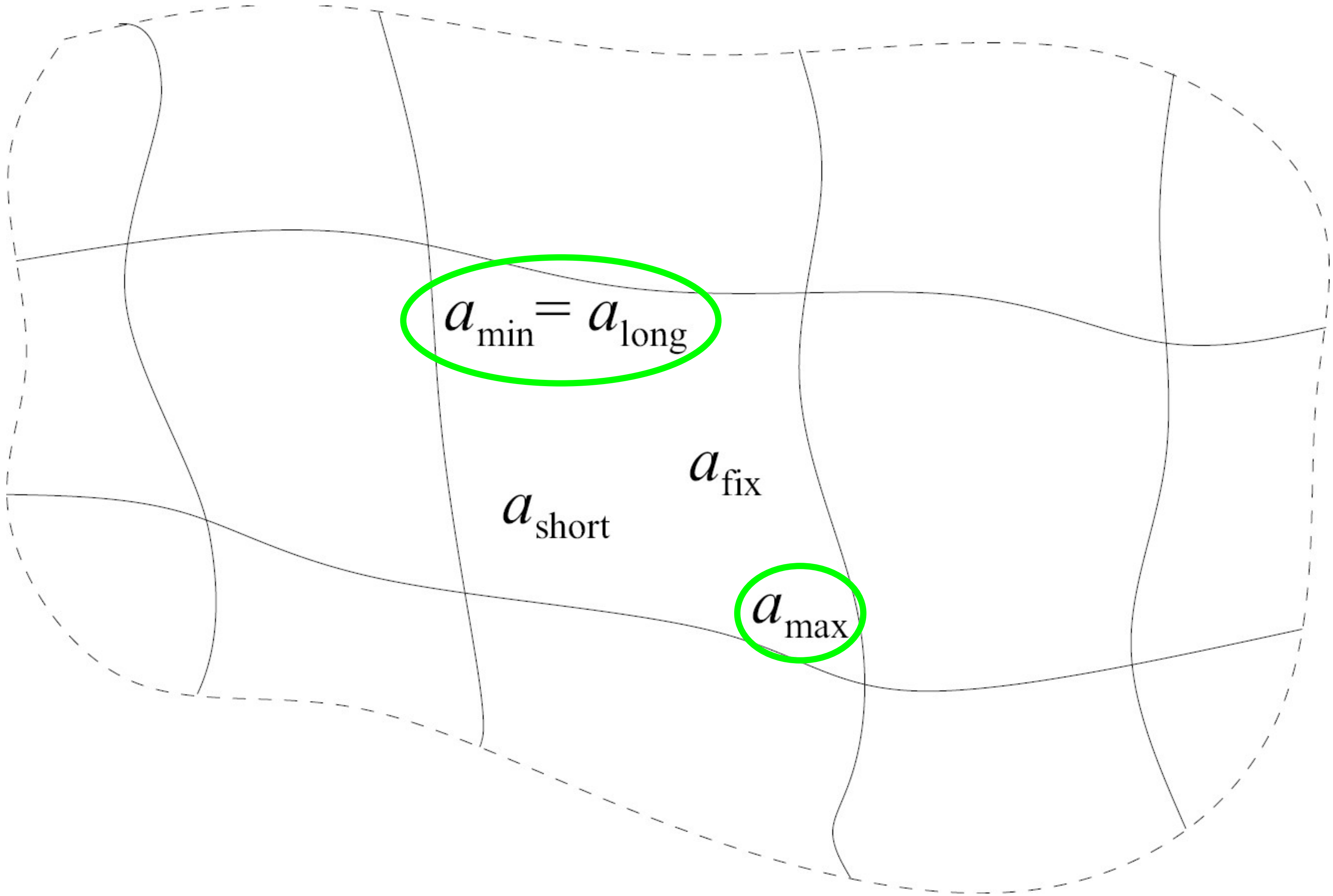

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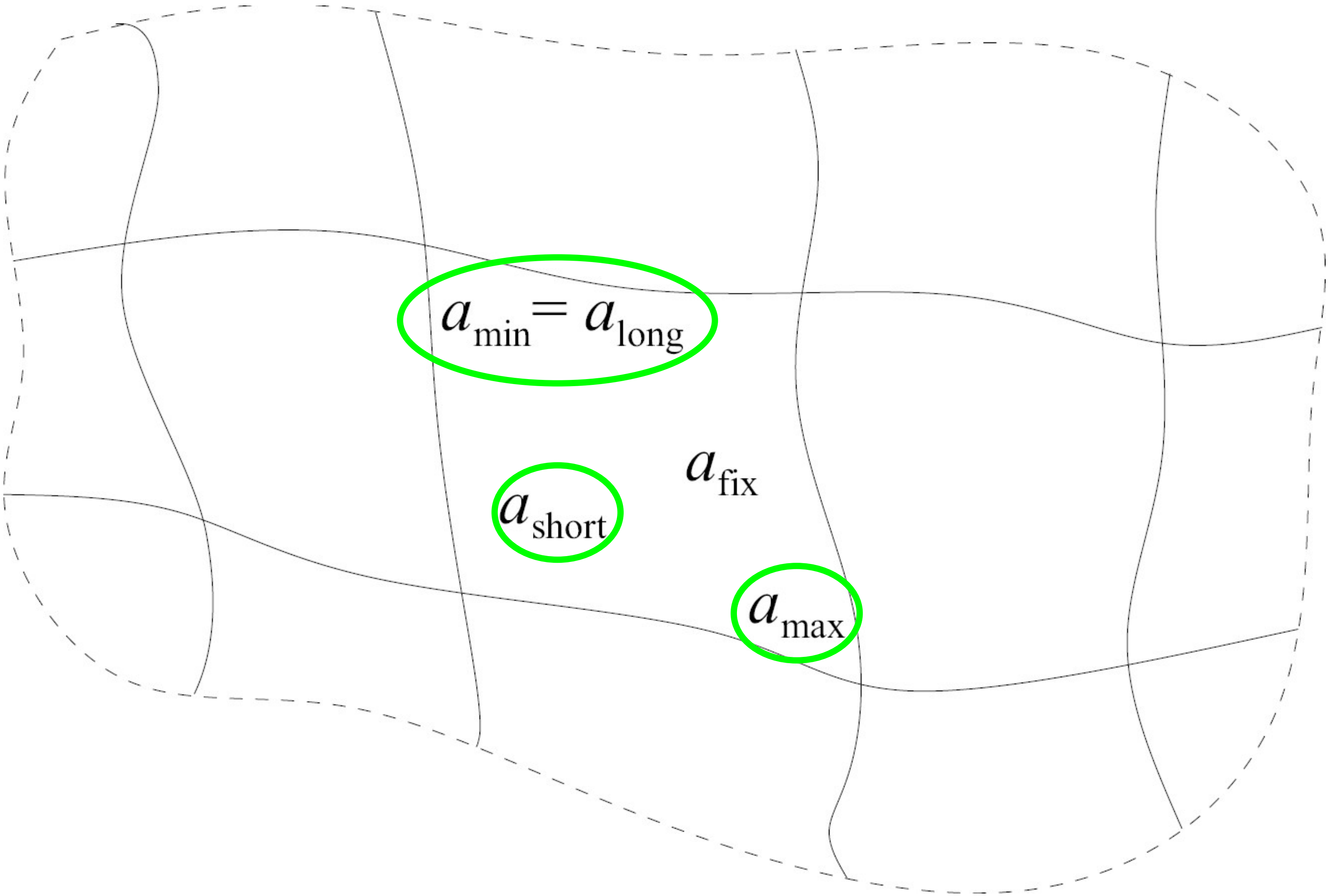

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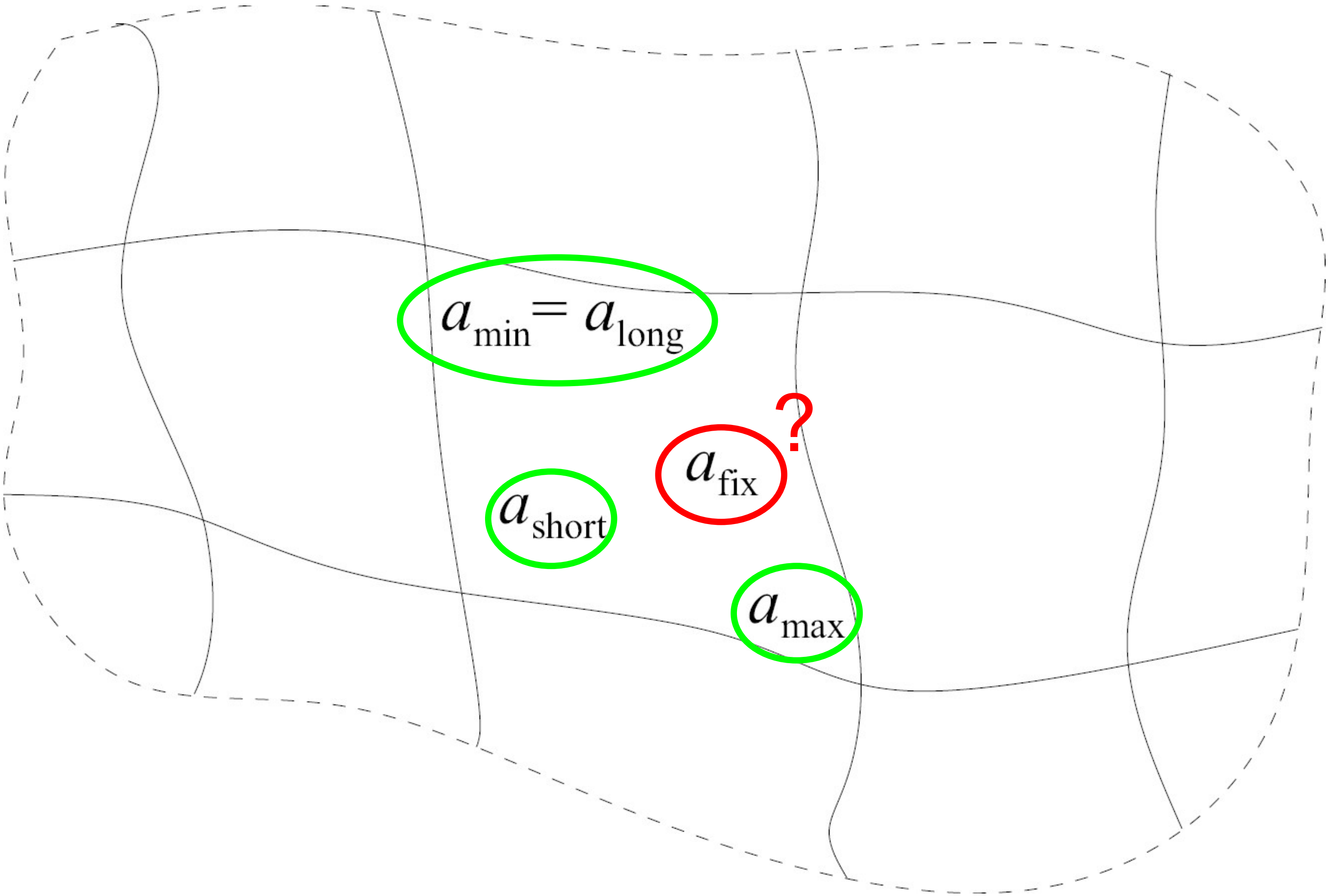

Sturmian words with balanced construction

Theorem Let $\mathrm{Fix}(\Delta_c) \subset \mathcal{UM}_0$ denote the set of all fixed points of Δ_c. Then:

1. $\mathrm{Fix}(\Delta_c) \subset s'(]0, \frac{2}{3}[\setminus \mathbf{Q})$; numbers 0 and $\frac{2}{3}$ are accumulation points of $(s')^{-1}(\mathrm{Fix}(\Delta_c))$.

2. $\mathrm{card}(\mathrm{Fix}(\Delta_c))$ is equal to that of the continuum.

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Sturmian words with balanced construction

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Ph.D. Thesis (to be defended in 8 days!):

Digital lines, Sturmian words, and continued fractions