PUBLICATIONS AND SCIENTIFIC PRESENTATIONS

HANNA USCKA-WEHLOU, PH.D.

I. LIST OF PUBLICATIONS:

07-1 Uscka-Wehlou, Hanna, 2007. Digital lines with irrational slopes. *Theoretical Computer Science* **377**, 157–169.

08-1 Uscka-Wehlou, Hanna, 2008. Continued Fractions and Digital Lines with Irrational Slopes. In D. Coeurjolly et al. (Eds.): DGCI 2008, LNCS **4992**, pp. 93–104.

08-2 Uscka-Wehlou, Hanna, 2008. A run-hierarchical description of upper mechanical words with irrational slopes using continued fractions; 15 pp. In *Proceedings of 12th Mons Theoretical Computer Science Days (Mons, Belgium), 27–30 August 2008.* http://www.jmit.ulg.ac.be/jm2008/index-en.html. Preprint: http://wehlou.com/hania/files/uu/mons08rev.pdf.

09-1 Uscka-Wehlou, Hanna, 2009. Run-hierarchical structure of digital lines with irrational slopes in terms of continued fractions and the Gauss map. *Pattern Recognition* **42**, 2247–2254.

09-2 Uscka-Wehlou, Hanna, 2009. Two equivalence relations on digital lines with irrational slopes. A continued fraction approach to upper mechanical words. *Theoretical Computer Science* **410** (38–40), 3655–3669.

09-3 Uscka-Wehlou, Hanna, 2009. Sturmian words with balanced construction; 12 pp. In *Proceedings of Words 2009, the 7th International Conference on Words (Salerno, Italy), 14–18 September 2009.* http://words2009.dia.unisa.it/accepted.html. Preprint: http://wehlou.com/hania/files/uu/words2009.pdf.

09-4 Uscka-Wehlou, Hanna, 2009. *Digital Lines, Sturmian Words, and Continued Fractions.* Ph.D. Thesis, 152 pp. In *Uppsala Dissertations in Mathematics* **65**. ISBN: 978-91-506-2090-0.

10-1 Uscka-Wehlou, Hanna, 2010. Continued fractions, Fibonacci numbers, and some classes of irrational numbers. To appear in *Acta Mathematica Academiae Paedagogicae Nyíregyháziensis* **26(1)**.

Date: January 21, 2010.

IIa. PUBLICATION DETAILS:

1. ***Digital lines with irrational slopes*** (a journal paper)

Abstract. How to construct a digitization of a straight line and be able to recognize a straight line in a set of pixels are very important topics in computer graphics. The aim of the present paper is to give a mathematically exact and consistent description of digital straight lines according to Rosenfeld's definition. The digitizations of the lines with slopes $0 < a < 1$, where a is irrational, are considered. We formulate a definition of digitization runs, formulate and prove theorems containing necessary and sufficient conditions for digital straightness. The proof was successfully constructed using only methods of elementary mathematics. The developed and proved theory can be used in the research into the theory of digital lines, their symmetries, translations etc.

Number of pages. 13

Status. Published in *Theoret. Comput. Sci.* (**377** (2007) 157–169).

2. ***Continued Fractions and Digital Lines with Irrational Slopes*** (a peer-refereed conference paper)

Abstract. This paper expands on previous work on relationships between digital lines and continued fractions (CF). The main result is a parsimonious description of the construction of the digital line based only on the elements of the CF representing its slope and containing only simple integer computations. The description reflects the hierarchy of digitization runs, which raises the possibility of dividing digital lines into equivalence classes depending on the CF expansions of their slopes. Our work is confined to irrational slopes since, to our knowledge, there exists no such description for these, in contrast to rational slopes which have been extensively examined. The description is exact and does not use approximations by rationals. Examples of lines with irrational slopes and with very simple digitization patterns are presented. These include both slopes with periodic and non-periodic CF expansions, i.e. both quadratic surds and other irrationals.

Number of pages. 12

Status. Published (DGCI 2008, LNCS **4992**, pp. 93–104, 2008).

3. ***A Run-hierarchical Description of Upper Mechanical Words with Irrational Slopes Using Continued Fractions*** (a refereed conference paper)

Abstract. The main result is a run-hierarchical description (by contin-

ued fractions) of upper mechanical words with slope $a \in]0,1[\setminus \mathbf{Q}$ and intercept 0. We compare this description with two classical methods of forming of such words. In order to be able to perform the comparison, we present a quantitative analysis of our method. We use the denominators of the convergents of the continued fraction expansion of the slope to compute the length of the prefixes obtained by our method. Due to the special treatment which is given to the elements equal to 1, our method gives in some cases longer prefixes than the two other methods. Our method reflects the hierarchy of runs, by analogy to digital lines, which can give a new understanding of the construction of upper mechanical words.

Number of pages. 15

Status. Published (Proceedings of JM 2008, Mons, Belgium).

4. *Run-hierarchical structure of digital lines with irrational slopes in terms of continued fractions and the Gauss map*
(a journal paper; an extended version of the conference paper 2.)

Abstract. We study relations between digital lines and continued fractions. The main result is a parsimonious description of the construction of the digital line based only on the elements of the continued fraction representing its slope and containing only simple integer computations. The description reflects the hierarchy of digitization runs, which raises the possibility of dividing digital lines into equivalence classes depending on the continued fraction expansions of their slopes. Our work is confined to irrational slopes since, to our knowledge, there exists no run-hierarchical and continued fraction based description for these, in contrast to rational slopes which have been extensively examined. The description is exact (it does not use approximations by rationals). Examples of lines with irrational slopes and with very simple digitization patterns are presented. These include both slopes with periodic and non-periodic continued fraction expansions, i.e. both quadratic surds and other irrationals. We also derive the connection between the Gauss map and the digitization parameters introduced by the author in 2007.

Number of pages. 8

Status. Published in *Pattern Recognition* (**42** (2009) 2247–2254).

5. *Two Equivalence Relations on Digital Lines with Irrational Slopes. A Continued Fraction Approach to Upper Mechanical Words* (a journal paper)

Abstract. We examine the influence of the elements of the continued fraction (CF) expansion of $a \in]0,1[\setminus \mathbf{Q}$ on the construction of runs

in the digitization of the positive half line $y = ax$ or, equivalently, on the run-hierarchical structure of the upper mechanical word with slope a and intercept 0. Special attention is given to the CF elements equal to 1. We define two complementary equivalence relations on the set of slopes, based on their CF expansions. A new description of digital lines is presented; we show how to define a straight line or upper mechanical word by two sequences of positive integers, fulfilling some extra conditions. These equivalence relations and this new description enable us to analyze the construction of digital lines and upper mechanical words.

Number of pages. 15

Status. Published in *Theoret. Comput. Sci.* (**410** (2009) 3655–3669).

6. *Sturmian words with balanced construction* (a refereed conference paper)

Abstract. In this paper we define Sturmian words with balanced construction. We formulate a fixed-point theorem for Sturmian words and analyze the set of all fixed points. The inspiration for this work came from the Kolakoski word and the general idea of self-reading sequences by Păun and Salomaa. The basis for this article is the author's earlier research on the influence of the continued fraction elements in the expansion of $a \in]0, 1[\setminus \mathbf{Q}$ on the construction of runs for the upper mechanical word with slope a and intercept 0.

Number of pages. 12

Status. Published in *Proceedings of Words 2009.*

7. *Continued Fractions, Fibonacci Numbers, and Some Classes of Irrational Numbers* (to be published in an electronical journal)

Abstract. In this paper we define an equivalence relation on the set of positive irrational numbers less than 1. The relation is defined by means of continued fractions. Equivalence classes under this relation are determined by the places of some elements equal to 1 (called *essential 1's*) in the continued fraction expansion of numbers. Analysis of suprema of all the equivalence classes leads to a solution which involves Fibonacci numbers and constitutes the main result of this paper. The problem has its origin in the author's research on the construction of digital lines and upper and lower mechanical and characteristic words according to the hierarchy of runs.

Number of pages. 14

Status. Acc. *Acta Mathematica Academiae Paedagogicae Nyíregyháziensis.*

IIb. PH.D. THESIS (defended 25 September 2009):

Digital Lines, Sturmian Words, and Continued Fractions.

152 pp. In *Uppsala Dissertations in Mathematics* **65** (2009)

ISBN: 978-91-506-2090-0.

Abstract. In this thesis we present and solve selected problems arising from digital geometry and combinatorics on words. We consider digital straight lines and, equivalently, upper mechanical words with slope $a \in]0,1[\setminus \mathbf{Q}$ and intercept 0. We formulate a continued fraction (CF) based description of their run-hierarchical structure.

Paper I gives a theoretical basis for the CF-description of digital lines. We define for each irrational positive slope less than 1 a sequence of digitization parameters which fully specifies the run-hierarchical construction.

In Paper II we use the digitization parameters in order to get a description of runs using only integers. We show that the CF-elements of the slopes contain the complete information about the run-hierarchical structure of the line. The index jump function introduced by the author indicates for each $k \in \mathbf{N}^+$ the index of the CF-element which determines the shape of the digitization runs on level k.

In Paper III we present the results for upper mechanical words and compare our CF-based formula with two well-known methods, one of which was formulated by Johann III Bernoulli and proven by Markov, while the second one is known as the standard sequences method. Due to the special treatment of some CF-elements equal to 1 (*essential* 1's in Paper IV), our method is currently the only one which reflects the run-hierarchical structure of upper mechanical words by analogy to digital lines.

In Paper IV we define two equivalence relations on the set of all digital lines with slopes $a \in]0,1[\setminus \mathbf{Q}$. One of them groups into classes all the lines with the same run length on all digitization levels, the second one groups the lines according to the run construction in terms of long and short runs on all levels. We analyse the equivalence classes with respect to minimal and maximal elements. In Paper V we take another look at the equivalence relation defined by run construction, this time independently of the context, which makes the results more general.

In Paper VI we define a *run-construction encoding operator*, by analogy with the well-known *run-length encoding operator*. We formulate and present a proof of a fixed-point theorem for Sturmian words. We

show that in each equivalence class under the relation based on run length on all digitization levels (as defined in Paper IV), there exists exactly one fixed point of the run-construction encoding operator.

IIc. REPORTS AND TEACHING MATERIALS:

1. ***Complexe getallen (Complex numbers)*** (a *Mathematica* notebook, in Dutch)
Abstract. An interactive electronical textbook for students from VUB (The Flemish University of Brussels, Belgium), which I designed and programmed while working for the project Exploot led by Professor Ivan Cnop. The notebook contains the definition of complex numbers, visualises the operations of adding and multiplying complex numbers, and offers a number of experiments with complex functions.
Link. http://we.vub.ac.be/exploot/lesmateriaal/welcome_NS. html
Status. Teaching material for the students from VUB from 2001.

2. ***Complexe veeltermen (Complex polynomials)*** (a *Mathematica* notebook, in Dutch)
Abstract. An interactive electronical textbook for students from VUB (The Flemish University of Brussels, Belgium), which I designed and programmed while working for the project Exploot led by Professor Ivan Cnop. The notebook is about spirographs. It contains a number of experiments with complex polynomials.
Link. http://we.vub.ac.be/exploot/lesmateriaal/welcome_NS. html
Status. Teaching material for the students from VUB from 2001.

3. ***Complexe getallen en de meetkunde (Complex numbers and geometry)*** (a *Mathematica* notebook, in Dutch)
Abstract. An interactive electronical textbook for students from VUB (The Flemish University of Brussels, Belgium), which I designed and programmed while working for the project Exploot led by Professor Ivan Cnop. The notebook shows the geometrical interpretation of complex numbers, their addition and multiplication and shows how to use complex numbers for solving of geometrical problems.
Link. http://wehlou.com/hania/index.htm.
Status. Published on my personal home page.

4. ***Digital lines with irrational slopes*** (an internal report)
Abstract. The same as in paper 1. in the list IIa, but published as an

internal report.
Number of pages. 17
Status. Scientific report (U.U.D.M. Report 2005:20).

5. ***Digital lines*** (an internal report)
Abstract. This paper is a short presentation of the first year of my research as a PhD student. I was working with Rosenfeld's digitization of straight lines. I introduced a modification of Rosenfeld's digitization and digitization parameters determining it. I showed how Euclid's algorithm for m and n describes the construction of the digital line $y = \frac{m}{n}$, where $m, n \in \mathbf{N}^+$, $m < n$ and $\gcd(m, n) = 1$. I also formulated a theorem giving an exact description of all the translations of digital lines with rational slopes.
Number of pages. 5
Status. Presentation of my research for FMB Open House.

6. ***Groups*** (teaching material in Swedish)
Abstract. Theoretical introduction to groups and related topics: definitions, examples, exercises.
Link. http://wehlou.com/hania/files/uu/SKGroups.pdf
Status. Teaching material for the students participating in Sonja Kovalevskydagarna organized by Uppsala University in November 2008.

III. SCIENTIFIC PRESENTATIONS:

1. Event. The "33rd session of the Seminaire de Mathematique Superieures (SMS) on Topological Methods in Differential Equations and Inclusions"
Place. Montreal, Canada.
Time. July 11–22, 1994.
Abstract. No paper presented.

2. Event. Presentation *Digital lines*
Place. The Centre for Image Analysis, Uppsala.
Time. December 15, 2003.
Abstract (the original version I sent to CBA on December 8, 2003).
I am going to talk about my research in digital geometry. I am, for the moment, most interested in digital lines, so my presentation next Monday will be related to what Erik Melin presented this Monday. I

am mainly interested in construction *of a digital line, I mean - pixel by pixel,* block *by* block, sequence *by* sequence *(it will all become clear next Monday) and how to properly recognize a digital line in a set of pixels. I have mainly worked with Rosenfeld digitization, but most (maybe all) of my results are also valid for the Khalimsky-Melin case. Like Erik, I did not use any complicated mathematical tools, only some very elementary geometry and some extraordinarily elementary number theory (only Euclid's algorithm and some simple conclusions following from it). I hope you will enjoy it and that we will have an interesting discussion afterwards.*

3. Event. Presentation *Digital lines* during the FMB-FMD Open House Conference
Place. Uppsala University.
Time. November 15–16, 2004.
Abstract. As internal report 5 in the list IIc above.

4. Event. Presentation *Theory of digital lines*
Place. The Centre for Image Analysis, Uppsala.
Time. January 24, 2005.
Abstract (the original version I sent to CBA on January 21, 2005).
I will present the results of my research about digital lines with irrational slopes. It will be the formal description of such lines. I will present the theory needed for the formulation of necessary and sufficient conditions to be a digital line with irrational slope. There is no advanced mathematics involved, but quite a lot of definitions are needed for the final formulation of the conditions.

5. Event. The 14th International Conference on DISCRETE GEOMETRY for COMPUTER IMAGERY, DGCI 2008
Place. Lyon, France.
Time. April 16–18, 2008.
Abstract. As Paper 2 in the list IIa above (a poster and a short oral presentation).

5. Event. The 12th Mons Theoret. Comp. Science Days, JM 2008
Place. Mons, Belgium.
Time. August 27–30, 2008.
Abstract. As Paper 3 in the list IIa above (oral presentation).

6. Event. Sonja Kovalevskydagarna 2008

Place. Uppsala University.

Time. November 7–8, 2008.

Abstract. I was the leader of a workshop called "Groups" during "Sonja Kovalevskydagarna" at Uppsala University. I wrote the notes (in Swedish) for the participants of this workshop. The notes can be found under http://wehlou.com/hania/files/uu/SKGroups.pdf. I worked also in the problem-solving group (preparing the problems, being there during the problem-solving time, and correcting the solutions afterwards).

7. Event. The 7th International Conference on Words, Words 2009

Place. Salerno, Italy.

Time. September 14–18, 2009.

Abstract. As Paper 6 in the list IIa above (oral presentation).

8. Event. Presentation of my Ph.D. thesis

Place. Uppsala University.

Time. September 25, 2009.

Abstract. Above (in IIb).

9. Event. Presentation *Klasy liczb niewymiernych zdefiniowane przez ich rozwinięcia na ułamki łańcuchowe* (in Polish) [Classes of irrational numbers defined by their continued fraction expansions].

Place. Copernicus University, Toruń (Poland).

Time. October 27, 2009.

Abstract. As Paper 7 in the list IIa above.

10. Event. Presentation *Some combinatorial problems related to digital straight lines with irrational slopes and to balanced aperiodic words.*

Place. KTH, Stockholm (the combinatorics seminar).

Time. December 2, 2009.

Abstract. There is a very close relationship between digital straight lines and binary words with the property of balance. During my Ph.D. studies I described, by means of continued fractions, the run hierarchical structure of digital straight lines (equivalently, of upper mechanical

words) with irrational positive slopes less than one. I defined two equivalence relations on the set of slopes. Both relations are defined on the continued fraction elements of the slopes and they have their interpretation in the run hierarchical structure. This causes some questions with combinatorial nature, for example, about the description of the continued fraction expansion of slopes in each equivalence class. Another combinatorial piece is the fixed point theorem for the lines and words. The fixed points are the lines (words) with self-balanced construction in terms of long and short runs on all digitization levels.

11. Planned event. A scientific visit in the research group *Formal Languages and Concurrency* (Professor Edward Ochmański).
Place. Copernicus University, Toruń (Poland).
Time. February 22–25, 2010.
Abstract. Discussions about the project *Mathematical methods in modeling and analysis of concurrent systems.*