**Continued Fractions & Digital Lines**

Hanna Uscka-Wehlou

Sweden, Uppsala University, Department of Mathematics

We present a new, continued fraction based, recursive description of digital half lines $y = ax, x > 0$ for irrational $a$ from the interval $]0,1[$. The description reflects the hierarchy of runs.

**Continued fractions.** Positive integers $a_1, a_2, ...$ are the continued fraction elements of $a$, if

$$a = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}} \text{ def. } [0; a_1, a_2, a_3, \ldots].$$

**The denominators of the convergents** of the continued fraction expansion of $a$: $q_0 = 1$, $q_1 = a_1$, and, for $n \geq 2$, $q_n = a_n q_{n-1} + q_{n-2}$

**The index jump function**

For the slope $a = [0; a_1, a_2, a_3, \ldots]$, the index jump function is

$$\delta_i(x) = \begin{cases} 1, & x = 1 \\ 0, & x \neq 1. \end{cases}$$

An example how it works for the lines like on the picture.

The last row shows the first 12 elements of the sequence of the values of the index jump function.

**Run hierarchic description** in terms of long and short runs on all the digitization levels. Expressed by the index jump function. Illustrated on the picture.

Sequences $(b_i)$ of length specification define a quantitative equivalence relation LENGTH.

The places of essential 1’s define a qualitative equivalence relation CONSTRUCTION. FIBONACCI numbers and THE GOLDEN SECTION

**Two kinds of run length**

1. $|S_n|, |L_n|$ - the number of pixels contained in $S_n, L_n$ respectively.

2. $||S_n||, ||L_n||$ - the number of runs contained in $S_n, L_n$.

**The pixel-wise length of runs**

Expressed by means of the denominators of the convergents and the index jump function. The number of pixels in the line segment $y = ax, x > 0$, achieved in the $n^{th}$ step:

$$|S_n| = q_{i_{a(n-1)+1}}, |L_n| = q_{i_{a(n+1)+1}} + q_{i_{a(n-1)+2}}$$

**The cardinality of runs**

Calculated according to the following formula:

$$||S_n|| = b_n + a_{i_{a(n)}} + \delta_1(a_{i_{a(n)}}) \cdot a_{i_{a(n)+1}}$$

1. Continued Fractions and Digital Lines with Irrational Slopes (the DGCI 2008 paper)
2. A New Description of Upper Mechanical Words with Irrational Slopes Using Continued Fractions (submitted in March 2008)