Some combinatorial problems related to digital straight lines with irrational slopes and to balanced aperiodic words
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Hanna Uscka-Wehlou

Some combinatorial problems related to digital straight lines with irrational slopes and to balanced aperiodic words

KTH, 2 December 2009
Items:

Mechanical words and digital lines

A short introduction to continued fractions

Some combinatorics on continued fraction elements

Questions and problems
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Mechanical words and digital lines

A short introduction to continued fractions

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Questions and problems
Words and lines
1.1 Words
Finite words

\( A \) - alphabet (a set of symbols)

\( A^* \) - the set of finite words over \( A \)

\( (A^*, +) \) - is a monoid:
- concatenation \((+\) is associative \( (u+v)+w=u+(v+w) \)
  
  \[
  101010+1111=1010101111
  \]
- the empty word \( \varepsilon \) is the neutral element

\( (A^*, +) \) is called the free monoid on the set \( A \).

- no inverse operation, no commutativity
Infinite words

\( A \) - alphabet (a set of symbols)

\( A^\omega \) - the set of right infinite words over \( A \)

For example, if \( A=\{1,2\} \), then the words are:

\[
\begin{align*}
  w &: \mathbb{N}^+ \rightarrow \{1, 2\} \\
  w &= w(1)w(2)w(3) \cdots \in \{1, 2\}^\omega
\end{align*}
\]
The word $w$ is called a factor of a word $u$ if there exist words $x, y$ such that $u = x + w + y$.

1222 is a factor of 000122211113213110101001

10101 is a factor of 10101010101010101

ABCDA is a factor of CBBBDACADBCAABCDA

10101 is a factor of 10101
Sturmian words are infinite words which have exactly $m+1$ different factors of length $m$ for every natural $m$. 
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$m=1 \quad \rightarrow \quad$ two letters (binary words)
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$m=1 \quad \rightarrow \quad$ two letters (binary words)

101010010100101001010010100101001010010100...
Sturmian words are infinite words which have exactly $m+1$ different factors of length $m$ for every natural $m$.

$m=1$ → two letters (binary words)

10101001010100101001010010101001010010100 ...

$m=4$

1010, 0101, 0010, 1001, 0100.
Sturmian words are infinite words which have exactly \( m+1 \) different factors of length \( m \) for every natural \( m \).

\[ m=1 \quad \rightarrow \quad \text{two letters (binary words)} \]

1001010010101001010010101001010010101001010010101001010010100 \ldots

\[ m=4 \]

1010, 0101, 0010, 1001, 0100.
Sturmian words are infinite words which have exactly \( m+1 \) different factors of length \( m \) for every natural \( m \).

\[
m=1 \quad \rightarrow \quad \text{two letters (binary words)}
\]

10101001010010101001010010101001010010100 \ldots

\[
m=4
\]

1010, 0101, \text{0010, 1001, 0100}.\]
Balanced words (binary)

\( n \) - the length of the word

\( m \) - any positive natural number less than \( n \)

Each \( m \)-letter long factor of this word can contain either \( k \) or \( k+1 \) 1's

An example:

\( n = 41 \)

\( m = 16 \)

\[
10101001010010101001010010101001010010100
\]

\( k = 6 \)

\( 7 \)

\( 6 \)
Balanced words give straight lines
Upper and lower mechanical, characteristic words

\[ s'(a), s(a): \mathbb{N} \rightarrow \{0, 1\} \]

\[ \forall n \in \mathbb{N} \quad s'_n(a) = \left[ a(n + 1) \right] - \left[ an \right], \]

\[ s_n(a) = \left[ a(n + 1) \right] - \left[ an \right] \]

\[ c(a): \mathbb{N}^{+} \rightarrow \{0, 1\} \]

\[ \forall n \in \mathbb{N}^{+} \quad c_n(a) = \left[ a(n + 1) \right] - \left[ an \right] \]
Theorem Let $s$ be an infinite word. The following are equivalent:

- $s$ is Sturmian;
- $s$ is balanced and aperiodic;
- $s$ is irrational (lower or upper) mechanical.
1.2

Lines
Digital geometry – R'-digitization
Digital geometry - $R'$-digitization
The arithmetical expression of the $R'$-digitization of the line $y = ax$ for irrational positive $a$ less than 1:

$$D_{R'}(y = ax) = \{(k, \lfloor ak \rfloor) ; \ k \in \mathbb{Z}\}$$
The R'-digital line $y = ax$ with irrational slope

$a = [ \, 0 ; a_1, a_2, \ldots \, ]$

Points on the line:
- $(0,0)$
- $(1, \lceil a \rceil)$
- $(2, \lceil 2a \rceil)$
- $(3, \lceil 3a \rceil)$
- $(4, \lceil 4a \rceil)$
- $(5, \lceil 5a \rceil)$
- $(6, \lceil 6a \rceil)$
- $(7, \lceil 7a \rceil)$

Graphical representation of the line $y = ax$.
Digital geometry – straight lines and mechanical words

The $\mathbb{R}'$-digital line $y = ax$ with slope $a = [0; a_1, a_2, ...]$ and the corresponding upper mechanical word $s'(a)$:

\[
s'(a) = 10010010010001... \]

\[
\begin{align*}
\lfloor 4a \rfloor - \lfloor 3a \rfloor &= 1 \\
\lfloor 3a \rfloor - \lfloor 2a \rfloor &= 0 \\
\lfloor 2a \rfloor - \lfloor a \rfloor &= 0 \\
1 - 0 &= 1
\end{align*}
\]

\[
\begin{align*}
\lfloor 5a \rfloor - \lfloor 4a \rfloor &= 0 \\
\lfloor 6a \rfloor - \lfloor 5a \rfloor &= 1 \\
\lfloor 7a \rfloor - \lfloor 6a \rfloor &= 0 \\
\lfloor 8a \rfloor - \lfloor 7a \rfloor &= 1 \\
\lfloor 9a \rfloor - \lfloor 8a \rfloor &= 0 \\
\lfloor 10a \rfloor - \lfloor 9a \rfloor &= 1
\end{align*}
\]
Digital geometry - the concept of run

$y = ax$

the chain code

run$_1$(1)

run$_1$(2)

run: $P_1 = 10^3$
Digital geometry – the concept of run

\( P_n \) - the \( n^{th} \) prefix according to the run hierarchy

\( S_n \) - short run of level \( n \)

\( L_n \) - long run of level \( n \)

\[
\begin{align*}
P_1 &= S_1 = 1, & L_1 &= 10 \\
S_2 &= S_1 L_1, & P_2 &= L_2 = S_1^2 L_1 \\
S_3 &= L_2 S_2, & P_3 &= L_3 = L_2^2 S_2 \\
P_4 &= S_4 = L_3 S_3^2, & L_4 &= L_3 S_3^3 \\
P_5 &= S_5 = S_4 L_4, & L_5 &= S_4 L_4^2
\end{align*}
\]

\[
\frac{41}{57} = [0; 1, 2, 1, 1, 3, 1, 1]
\]

\[
\begin{align*}
P_5 &= S_5 = S_4 L_4 = (L_3 S_3^2)(L_3 S_3^3) = (L_2^2 S_2)(L_2 S_2)^2(L_2^2 S_2)(L_2 S_2)^3 \\
&= (S_1^2 L_1)^2 S_1 L_1 (S_1^2 L_1 S_1 L_1)^2 (S_1^2 L_1)^2 S_1 L_1 (S_1^2 L_1 S_1 L_1)^3 \\
&= (1110)^2 110 (1110110)^2 (1110)^2 110 (1110110)^3
\end{align*}
\]
Digital geometry - the concept of run

Two run lengths on level 1: runs\(_1\) \(S_1=10^m\) and \(L_1=10^{m+1}\)

\[
\begin{array}{cccccccccc}
& & S_1 & & L_1 & & & & & \\
& & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\]

Two run lengths on level 2: runs\(_2\) \(S_2=S_1L_1^k\) and \(L_2=S_1L_1^{k+1}\)

or \(S_2=S_1^kL_1\) and \(L_2=S_1^{k+1}L_1\)

Two run lengths on level \(n\): runs\(_n\) \(S_n=S_{n-1}L_{n-1}^l\) and \(L_n=S_{n-1}L_{n-1}^{l+1}\)

or \(S_n=S_{n-1}^lL_{n-1}\) and \(L_n=S_{n-1}^{l+1}L_{n-1}\)

or \(S_n=L_{n-1}S_{n-1}^l\) and \(L_n=L_{n-1}S_{n-1}^{l+1}\)

or \(S_n=L_{n-1}^lS_{n-1}\) and \(L_n=L_{n-1}^{l+1}S_{n-1}\)
Hierarchy of runs - runs on level $k+1$

$$L_k S^m_k \quad S^m_k L_k \quad L^m_k S_k \quad S_k L^m_k$$
Hierarchy of runs

Three questions. About:

the run length on level $k+1$

the main run on level $k$

the first run on level $k$
Continued fractions
Continued fractions - notation

\[ a = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}} = [0; a_1, a_2, a_3, \ldots] \]
Continued fractions - the CF-elements

\[ \frac{1}{a} = \left[ \frac{1}{a} \right] + \frac{\frac{1}{a}}{1} = \left[ \frac{1}{a} \right] + \frac{1}{\left[ \frac{1}{\frac{1}{a}} \right]} + \frac{1}{\left[ \frac{1}{\frac{1}{\frac{1}{a}}} \right]} \]

\[ a_1 \quad a_2 \]
Continued fractions - a definition

\[ a = [a_0; a_1, a_2, a_3, \ldots] \]

\[ \alpha_0 = a; \quad \text{for} \quad n \geq 0 : \]

\[ a_n = \lfloor \alpha_n \rfloor, \quad \alpha_{n+1} = \frac{1}{\alpha_n - a_n} = \frac{1}{\text{frac}(\alpha_n)} \]
Continued fractions – an example

$$\frac{13}{41} = \frac{1}{\frac{41}{13}} = \frac{1}{3 + \frac{2}{13}} = \frac{1}{3 + \frac{1}{\frac{13}{2}}}$$

$$\frac{1}{3 + \frac{1}{6 + \frac{1}{2}}} = [0; 3, 6, 2].$$
Continued fractions and decimal expansions

\[ 0.1111 \cdots = \frac{1}{9} = [0; 9] \]

\[ [0; 1, 1, 1, \ldots] = \frac{\sqrt{5} - 1}{2} = 0.6180339887\ldots \]
The CF-expansion of $\alpha$ is periodic

$\alpha$ is a quadratic surd
... is an algebraic number of the second degree, i.e.:

is irrational and is a root of some equation

\[ a_2 x^2 + a_1 x + a_0 = 0 \]

with integer coefficients.

\[ \frac{\sqrt{5} - 1}{2} \] is a root of \[ x^2 + x - 1 = 0 \]
Periodicity of continued fractions

\[ [0; 1, 1, 1, 1, \ldots] = ? \]

\[ x = \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \cdots}}} \]
Periodicity of continued fractions

\[ [0; 1, 1, 1, 1, \ldots] = ? \]

\[ x = \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \cdots}}} \]
Periodicity of continued fractions

\[ [0; 1, 1, 1, 1, \ldots] = \, ? \]

\[ x = \frac{1}{\frac{1}{1 + \frac{1}{\frac{1}{1 + \ldots}}} + x} = \frac{1}{1 + x} \]
Periodicity of continued fractions

\[ [0; 1, 1, 1, 1, \ldots] = ? \]

\[
x = \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \cdots}}} = \frac{1}{1 + x}
\]

\[ x = \frac{1}{1 + x} \iff x^2 + x - 1 = 0 \]
Periodicity of continued fractions

\[ [0; 1, 1, 1, 1, \ldots] = \frac{\sqrt{5} - 1}{2} \]

\[ x = \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \cdots}}} = \frac{1}{1 + x} \]

\[ x = \frac{1}{1 + x} \iff x^2 + x - 1 = 0 \]
The CF-expansion of $\alpha$ is periodic

Euler 1737

$\alpha$ is a quadratic surd
The CF-expansion of $\alpha$ is periodic

$\alpha$ is a quadratic surd

Lagrange 1770

Euler 1737
# Continued fractions and decimal expansions

<table>
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<tr>
<td><strong>finite</strong></td>
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<td>rational</td>
</tr>
<tr>
<td><strong>infinite</strong></td>
<td>irrational</td>
<td>rational</td>
</tr>
<tr>
<td>periodic</td>
<td>irrational (quadratic surd)</td>
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</tr>
<tr>
<td>aperiodic</td>
<td>irrational (no quadratic surd)</td>
<td>irrational</td>
</tr>
</tbody>
</table>
Continued fractions – periodic patterns
(Euler 1737)

\[ e - 2 = \left[ 0; 1, 2, 1, 1, 4, 1, 1, 6, 1, \ldots, 1, 2k, 1, \ldots \right] \]
Continued fractions - periodic patterns
(Euler 1737)

\[ e - 2 = [0; 1, 2, 1, 1, 4, 1, 1, 6, 1, \ldots , 1, 2k, 1, \ldots ] \]
Continued fractions – periodic patterns
(Euler 1737)

\[ e - 2 = [0; \underbrace{1, 2, 1, 1, 4, 1, 1, 6, 1, \ldots}, \underbrace{1, 2k, 1, \ldots}] \]

\[ = [0; \overbrace{1, 2k, 1}^k_{k=1}]^{\infty} \]
Continued fractions - periodic patterns
(Euler 1737)

\[ e - 2 = \left[ 0; 1, 2, 1, 1, 4, 1, 1, 6, 1, \ldots, 1, 2k, 1, \ldots \right] \]

\[ = \left[ 0; \overline{1, 2k, 1} \right]_{k=1}^{\infty} \]

for \( k \geq 2 \)

\[ \sqrt[n]{e} - 1 = \left[ 0; \overline{(2k - 1)n - 1, 1, 1} \right]_{k=1}^{\infty} \]
for $k \geq 2$

$$\tan(1/k) = [0; k-1, 1, (2n + 1)k - 2]_{n=1}^{\infty}$$
Continued fractions – periodic patterns

(Lambert 1761)

for $k \geq 2$

$$\tan(1/k) = [0; k-1, 1, (2n + 1)k - 2]_{n=1}^{\infty}$$

$$\tan(1/2) = [0; 1, 1, 4, 1, 8, 1, 12, 1, 16, \ldots]$$
Combinatorics on CFs
Important issues

Two equivalence relations on the set of slopes

A new fixed point theorem for words
Important issues

Two equivalence relations on the set of slopes

A new fixed point theorem for words

A new CF-description (essential 1's, run hierarchy)
Two equivalence relations on the set of slopes

A new CF-description (essential 1's, run hierarchy)
An informal introduction to the equivalence relations on CFs

\[ a = [0; 1, a_2, 1, 1, a_5, 1, 1, a_8, a_9, 1, a_{11}, a_{12}, 1, 1, 1, a_{16}, a_{17}, \ldots] \]
An informal introduction to the equivalence relations on CFs

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\[ (b_k)_{k=1}^{\infty} = (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b_{11}, b_{12}, \ldots) \]
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\[ a = [0; 1, a_2, 1, 1, a_5, 1, 1, a_8, a_9, 1, a_{11}, a_{12}, 1, 1, 1, a_{16}, a_{17}, \ldots] \]

\[ (b_k)_{k=1}^\infty = (1, \ldots) \]

\[ (b_k)_{k=1}^\infty = (b_1, \ldots) \]
An informal introduction to the equivalence relations on CFs

\[
a = [0; 1, a_2, 1, 1, a_5, 1, 1, a_8, a_9, 1, a_{11}, a_{12}, 1, 1, 1, a_{16}, a_{17}, \ldots]
\]

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(b_k)_{k=1}^\infty = (1, a_2, \ldots)
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(b_k)_{k=1}^\infty = (b_1, b_2, \ldots)
\]
An informal introduction to the equivalence relations on CFs

\[ a = [0; 1, a_2, 1, 1, a_5, 1, 1, a_8, a_9, 1, a_{11}, a_{12}, 1, 1, 1, a_{16}, a_{17}, \ldots] \]

\[ (b_k)_{k=1}^{\infty} = (1, a_2, 1 + 1) \]

\[ (b_k)_{k=1}^{\infty} = (b_1, b_2, b_3, \ldots) \]
An informal introduction to the equivalence relations on CFs

\[ a = [0; 1, a_2, 1, 1, a_5, 1, 1, a_8, a_9, 1, a_{11}, a_{12}, 1, 1, 1, a_{16}, a_{17}, \ldots] \]

\[ (b_k)_{k=1}^\infty = (1, a_2, 1 \_+ 1, a_5) \]

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\[ (b_k)_{k=1}^\infty = (b_1, b_2, b_3, b_4, b_5, b_6, \ldots) \]
An informal introduction to the equivalence relations on CFs

\[ a = \left[ 0; 1, a_2, 1, 1, a_5, 1, 1, a_8, a_9, 1, a_{11}, a_{12}, 1, 1, 1, a_{16}, a_{17}, \ldots \right] \]

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\[ (b_k)_{k=1}^{\infty} = (b_1, b_2, b_3, b_4, b_5, b_6, b_7, \ldots) \]
An informal introduction to the equivalence relations on CFs

\[ a = [0; 1, a_2, 1, 1, a_5, 1, 1, a_8, a_9, 1, a_{11}, a_{12}, 1, 1, 1, a_{16}, a_{17}, \ldots] \]

\[ (b_k)_{k=1}^\infty = (1, a_2, 1 + 1, a_5, 1 + 1, a_8, a_9, 1 + a_{11}, \ldots) \]

\[ (b_k)_{k=1}^\infty = (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, \ldots) \]
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\[ a = \left[ 0; 1, a_2, 1, 1, a_5, 1, 1, a_8, a_9, 1, a_{11}, a_{12}, 1, 1, 1, a_{16}, a_{17}, \ldots \right] \]

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An informal introduction to the equivalence relations on CFs

\[
a = [0; 1, a_2, 1, 1, a_5, 1, 1, a_8, a_9, 1, a_{11}, a_{12}, 1, 1, 1, a_{16}, a_{17}, \ldots]
\]

\[
(b_k)_{k=1}^\infty = (1, a_2, 1 \oplus 1, a_5, 1 \oplus 1, a_8, a_9, 1 \oplus a_{11}, a_{12}, 1 \oplus 1, \ldots)
\]

\[
(b_k)_{k=1}^\infty = (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, \ldots)
\]
An informal introduction to the equivalence relations on CFs

\[
\alpha = [0; 1, a_2, 1, 1, a_5, 1, 1, a_8, a_9, a_{11}, a_{12}, 1, 1, 1, a_{16}, a_{17}, \ldots]
\]

\[
(b_k)_{k=1}^\infty = (1, a_2, 1 + 1, a_5, 1 + 1, a_8, a_9, 1 + a_{11}, a_{12}, 1 + 1, 1 + a_{16},
\]

\[
(b_k)_{k=1}^\infty = (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b_{11},
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An informal introduction to the equivalence relations on CFs

\[ a = [0; 1, a_2, 1, 1, a_5, 1, 1, a_8, a_9, 1, a_{11}, a_{12}, 1, 1, 1, a_{16}, a_{17}, \ldots] \]

\[ (b_k)_{k=1}^\infty = (1, a_2, 1 + 1, a_5, 1 + 1, a_8, a_9, 1 + a_{11}, a_{12}, 1 + 1, 1 + a_{16}, a_{17}, \ldots) \]

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\[ (b_k)_{k=1}^{\infty} = (1, a_2, 1 + 1, a_5, 1 + 1, a_8, a_9, 1 + a_{11}, a_{12}, 1 + 1, 1 + a_{16}, a_{17}, \ldots) \]

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\[ a' = \left[ 0; 1, 1, a_2 - 1, 2, a_5, 1, 1, 1, a_8 - 1, a_9, a_{11} + 1, 1, a_{12} - 1, 2, 1, a_{16}, a_{17}, \ldots \right] \]
An informal introduction to the equivalence relations on CFs

\[ a = [0; 1, a_2, 1, 1, a_5, 1, 1, a_8, a_9, 1, a_{11}, a_{12}, 1, 1, 1, a_{16}, a_{17}, \ldots] \]

\[ (b_k)_{k=1}^\infty = (1, a_2, 1 + 1, a_5, 1 + 1, a_8, a_9, 1 + a_{11}, a_{12}, 1 + 1, 1 + a_{16}, a_{17}, \ldots) \]

\[ (b_k)_{k=1}^\infty = (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b_{11}, b_{12}, \ldots) \]

\[ a' = [0; 1, 1, a_2 - 1, 2, a_5, 1, 1, a_8 - 1, a_9, a_{11} + 1, 1, a_{12} - 1, 2, 1, a_{16}, a_{17}, \ldots] \]
An informal introduction to the equivalence relations on CFs

\[ a = \left[ 0; 1, a_2, 1, a_5, 1, a_8, a_9, 1, a_{11}, a_{12}, 1, 1, 1, a_{16}, a_{17}, \ldots \right] \]

\[ (b_k)_{k=1}^\infty = (1, a_2, 1 + 1, a_5, 1 + 1, a_8, a_9, 1 + a_{11}, a_{12}, 1 + 1, 1 + a_{16}, a_{17}, \ldots) \]

\[ (b_k)_{k=1}^\infty = (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b_{11}, b_{12}, \ldots) \]

\[ (s_k)_{k \in I} = (3, 6, 10, 13, 15, \ldots) \]
The index jump function

\[ a = [0; a_1, a_2, a_3, a_4, a_5, a_6, a_7, \ldots] \]

\[ i_a : \mathbb{N}^+ \rightarrow \mathbb{N}^+ \]

\[ i_a(1) = 1, \quad i_a(2) = 2, \quad \text{for } n \geq 2: \]

\[ i_a(n + 1) = i_a(n) + 1 + \delta_1(a_{i_a(n)}) \]
The index jump function: an example

\[ a = [0; \begin{array}{c} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \\ b_8 \\ a_{11} \\ a_{12} \\ \end{array}, \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \end{array}, a_{16}, a_{17}, \ldots ] \]

\[ (i_{a(k)})_{k\in \mathbb{N}^+} = (1, 2, 3, 5, 6, 8, 9, 10, 12, 13, 15, 17, \ldots) \]

An essential 1 is a CF-element equal to 1 and indexed by a value of the index jump function.
The index jump function - properties

Its values are positive integers

The function is increasing

For each slope $a$ and for each positive integer $n$

$$i_a(n + 1) - i_a(n) = 1 \quad \text{or} \quad i_a(n + 1) - i_a(n) = 2$$
$y = 2x - 2$

$y = x$
\[ y = 2x - 2 \]
\[ y = x \]
\[(i_a(n))_{n=1}^{10} = (1, 2, 4, 5, 6, 7, 9, 10, 12, 14)\]
\( a = [0; a_1, 1, a_3, a_4, a_5, a_6, 1, a_8, a_9, 1, a_{11}, 1, a_{13}, a_{14}, \ldots] \)

\[
\left( i_{a(n)} \right)_{n=1}^{10} = (1, 2, 4, 5, 6, 7, 9, 10, 12, 14)
\]

\( y = 2x - 2 \)

\( y = x \)
How $i_a(k + 1)$ and $a_{i_a(k+1)}$ describe the form of run$_{k+1}$

\[ a_{i_a(k+1)} \]

\[
L_k S_k^m \quad S_k^m L_k \quad L_k^m S_k \quad S_k L_k^m
\]

\[ i_a(k + 1) \]
How $i_a(k + 1)$ and $a_{i_a(k+1)}$ describe the form of $\text{run}_{k+1}$.
How $i_a(k + 1)$ and $a_{i_a(k+1)}$ describe the form of $\text{run}_{k+1}$
How $i_a(k + 1)$ and $a_{i_a(k+1)}$ describe the form of run$_{k+1}$
How $i_a(k + 1)$ and $a_{i_a(k+1)}$ describe the form of $r_{n+1}$

$m = a_{i_a(k+1)} - 1$

or

$m = a_{i_a(k+1)}$

$a_{i_a(k+1)}$

$m = a_{i_a(k+1)} + 1$

or

$m = a_{i_a(k+1)} + 1$

$L_k S^m_k$

$L_k^m S_k$

$L_k S^m_k$

$L_k^m S_k$

$i_a(k + 1)$
How $i_a(k+1)$ and $a_{i_a(k+1)}$ describe the form of $i_a(k+1)$.

$m = a_{i_a(k+1)} - 1$

Or $m = a_{i_a(k+1)}$

If $m > 1$:

$L_k S_k^m$

If $m = 1$:

$S_k^m L_k$

$m = a_{i_a(k+1)} + 1$

Or $m = a_{i_a(k+1)} + 1 + 1$

$i_a(k+1)$
The sequence of length specification for $a$

\[ b_1 = a_1 \quad \text{and, for} \quad n \geq 2: \]

\[ b_n = \begin{cases} 
    a_{i_a}(n), & \text{if } a_{i_a}(n) \neq 1 \\
    1 + a_{i_a}(n) + 1, & \text{if } a_{i_a}(n) = 1
\end{cases} \]
How $i_a(k + 1)$ and $a_{i_a(k+1)}$ describe the form of run $k+1$
How $i_a(k+1)$ and $a_{i_a(k+1)}$ describe the form of $\text{run}_{k+1}$
The index jump function: how it describes the runs

\[ a = [0; 1, 2, 1, 3, 1, 4, 1, 5, 1, 6, 1, 7, 1, 8, 1, 9, 1, 10, 1, 11, 1, 12, 1, 13, 1, 14, 1, 15, 1, 16, 1, 17, \ldots] \]

\[ (i_{a(k)})_{k \in \mathbb{N}^+} = (1, 2, 3, 5, 6, 8, 9, 10, 12, 13, 15, 17, \ldots) \]

**Essential 1's** are extremely important in description of runs.
## Digitization levels

<table>
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<tr>
<th>Level</th>
<th>1</th>
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\[
a = [0; 1, a_2, \underbrace{1, 1}_{b_3}, 1, b_5, 1, 1, b_8, a_9, 1, a_{11}, b_9, \underbrace{1, a_{12}}_{b_{10}}, 1, 1, a_{16}, b_{11}, a_{17}, \ldots] \\
(i_a(k))_{k \in \mathbb{N}^+} = (1, 2, 3, 5, 6, 8, 9, 10, 12, 13, 15, 17, \ldots)
\]
Short run length: the CF elements

\[ a = \left[ 0; b_1, b_2, 1, a_2, 1, a_5, 1, b_4, b_5, b_6, b_7, 1, a_8, a_9, a_{11}, a_{12}, 1, 1, a_{16}, a_{17}, \ldots \right] \]

\[ (i_a(k))_{k \in \mathbb{N}^+} = (1, 2, 3, 5, 6, 8, 9, 10, 12, 13, 15, 17, \ldots) \]

\[ \|S_k\| = b_k \]

\[ 1 \ a_2 \ 2 \ a_5 \ 2 \ a_8 \ a_9 \ 1 + a_{11} \ a_{12} \ 2 \ 1 + a_{16} \ a_{17} \]
The most frequent run: essential 1's

\[ a = [0; 1, a_2, \underbrace{1, 1}_b, a_5, \underbrace{1, 1}_b, b_6, a_8, b_7, \underbrace{1, a_{11}}_b, b_9, \underbrace{1, 1}_b, \underbrace{1, a_{16}}_b, b_{12}, \ldots] \]

\[(i_a(k))_{k \in \mathbb{N}^+} = (1, 2, 3, 5, 6, 8, 9, 10, 12, 13, 15, 17, \ldots)\]

\[ a_{i_a(k) + 1} > 1 \quad a_{i_a(k+1)} = 1 \]
The first run: parity of the function

$$a = [0; b_1, b_2, a_2, 1, b_3, a_5, 1, b_4, 1, b_5, a_8, b_6, b_7, 1, a_{11}, b_8, b_9, 1, a_{12}, \ldots]$$

$$(i_a(k))_{k \in \mathbb{N}^+} = (1, 2, 3, 5, 6, 8, 9, 10, 12, 13, 15, 17, \ldots)$$

$$_{i_a(k+1)}\text{even}$$

$$_{i_a(k+1)}\text{odd}$$
An illustration: for $a_2 = 2$ and $a_5 = 3$:

$P_n$ - the $n$th prefix according to the run hierarchy
$S_n$ - short run of level $n$
$L_n$ - long run of level $n$

$P_1 = S_1 = 1$, \quad $L_1 = 10$
$S_2 = S_1L_1$, \quad $P_2 = L_2 = S_1^2L_1$
$S_3 = L_2S_2$, \quad $P_3 = L_3 = L_2^2S_2$
$P_4 = S_4 = L_3S_3^2$, \quad $L_4 = L_3^2S_3^3$
$P_5 = S_5 = S_4L_4$, \quad $L_5 = S_4^2L_4^2$

\[\frac{41}{57} = [0; 1, 2, 1, 1, 3, 1, 1]\]

\[P_5 = S_5 = S_4L_4 = (L_3S_3^2)(L_3S_3^3) = (L_2^2S_2)(L_2S_2)^2(L_2^2S_2)(L_2S_2)^3\]
\[= (S_1^2L_1)^2S_1L_1(S_1^2L_1S_1L_1)^2(S_1^2L_1)^2S_1L_1(S_1^2L_1S_1L_1)^3\]
\[= (1110)^2110(1110110)^2(1110)^2110(1110110)^3\]
The sequence of length specification for $a$

\[ b_1 = a_1 \quad \text{and, for } \quad n \geq 2: \]

\[ b_n = \begin{cases} 
   a_{i_a}(n), & a_{i_a}(n) \neq 1 \\
   1 + a_{i_a}(n) + 1, & a_{i_a}(n) = 1
\end{cases} \]
Each class is generated by a sequence \((b_n)\) such that:

\[
b_1 \in \mathbb{N}^+ \quad \text{and, for } \quad n \geq 2, \quad b_n \in \mathbb{N}^+ \setminus \{1\}
\]

Each such \((b_n)\) is the sequence of length specification for some slope.
Two equivalence relations on the set of slopes

1. based on run length on all levels for \( s'(a) \):

\[
a \in \left[ (b_1, b_2, b_3, \ldots ) \right]_{\sim_{\text{len}}} \iff \\
\forall \ k \in \mathbb{N}^+ \ ||S_k|| = b_k
\]

2. based on run construction on all levels for \( s'(a) \):

\[
a \sim_{\text{con}} a' \iff \ i_a \equiv i_{a'}
\]
Done until now and to be done after a break:

1. Background information
2. Intuitions
3. Formal definitions and some motivation
4. Some results and open questions:
   - description of classes
   - fixed point theorem.
Quantitative equivalence relation (run length)

Defined by run lengths (their cardinality)

\[ ||S_1|| = 1, \quad ||S_2|| = 2, \quad ||S_3|| = 2, \quad ||S_4|| = 3. \]

\[ (b_n)_{n=1}^\infty = (1, 2, 2, 3, \ldots) \]

All lines from the same class have the same run lengths on all digitization levels.
Quantitative equivalence relation (run length)

Defined by run lengths (their cardinality)

\[
||S_1|| = 1, \quad ||S_2|| = 2, \quad ||S_3|| = 2, \quad ||S_4|| = 3.
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All lines from the same class have the same run lengths on all digitization levels.
Quantitative equivalence relation (run length)

Defined by run lengths (their cardinality)

\[
\begin{align*}
||S_1|| &= 1, & ||S_2|| &= 2, & ||S_3|| &= 2, & ||S_4|| &= 3.
\end{align*}
\]

\[
(b_n)_{n=1}^\infty = (1, 2, 2, 3, \ldots)
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Quantitative equivalence relation (run length)

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All lines from the same class have the same run lengths on all digitization levels.
Quantitative equivalence relation (run length)

Defined by run lengths (their cardinality)

$$\|S_1\| = 1, \ |\|S_2\| = 2, \ |\|S_3\| = 2, \ |\|S_4\| = 3.$$  

$$\left(b_n\right)_{n=1}^\infty = (1, 2, 2, 3, \ldots)$$

All lines from the same class have the same run lengths on all digitization levels.
How to compare continued fractions

\[ [a_0; a_1, a_2, \ldots] < [b_0; b_1, b_2, \ldots] \]

\[ \uparrow \]

\[ (a_0, -a_1, a_2, -a_3, a_4, -a_5, \ldots) \text{ lexic.} < (b_0, -b_1, b_2, -b_3, b_4, -b_5, \ldots) \]
Quantitative equivalence relation (run length)

The **least** element of the class:

\[
\min\{a \in ]0, 1[ \setminus \mathbb{Q}; \ a \in [(b_n)_{n \in \mathbb{N}^+}]_{\text{len}} \} = [0; b_1, 1, b_n - 1]_{n=2}^\infty.
\]

The **largest** element of the class:

\[
\max\{a \in ]0, 1[ \setminus \mathbb{Q}; \ a \in [(b_n)_{n \in \mathbb{N}^+}]_{\text{len}} \} = [0; b_1, b_2, 1, b_n - 1]_{n=3}^\infty.
\]
Qualitative equivalence relation (run construction)

Defined by the **index jump function**

Equivalently defined by the places of **essential 1's**

All lines from the same class have the same **construction** in terms of long and short runs on all digitization levels.

The **least** element in each class is 0.
A sequence \((t_j)_{j \in J}\) of positive integer numbers will be called an **essential sequence** iff:

- the set \(J\) is as follows: \(J = \emptyset, J = \mathbb{N}^+\) or \(J = [1, M]_\mathbb{Z}\) for some \(M \in \mathbb{N}^+\),

- the sequence \((t_j)_{j \in J}\) (if not empty) is a sequence of positive integers such that \(t_1 \geq 2\) and, for \(k \in J \setminus \{1\}\), \(t_k - t_{k-1} \geq 2\).
Each essential sequence defines an equivalence class under relation $\text{con}$.

An example:

If $t_n = 2n - 2$ for each $n \in \mathbb{N}^+$, then

$$[(t_n)_{n=1}^\infty]_{\text{con}} = [((\sqrt{5} - 1)/2)]_{\text{con}} = 
\{[0; c_1, 1, c_2, 1, c_3, 1, \ldots]; \ c_k \in \mathbb{N}^+\}. $$
Qualitative equivalence relation (run construction)

Supremum for each class:

\[ \forall \ n \in \mathbb{N}^+ \quad [(\forall \ k \in [1, n - 1]_\mathbb{Z}, \ t_k = 2k) \wedge (t_n > 2n \lor |J| = n - 1)] \]

\[ \Rightarrow \sup\{a; \ a \in [(t_j)_{j \in J}]_{\sim_{\text{con}}}\} = \frac{F_{2n-1}}{F_{2n}}, \]

where \((F_n)_{n \in \mathbb{N}^+}\) is the **Fibonacci** sequence

and \((t_j)_{j \in J}\) is any essential sequence.
How to compare continued fractions

\[ [a_0; a_1, a_2, \ldots] < [b_0; b_1, b_2, \ldots] \]

\( \uparrow \)

\((a_0, -a_1, a_2, -a_3, a_4, -a_5, \ldots)^{\text{lexic.}} < (b_0, -b_1, b_2, -b_3, b_4, -b_5, \ldots)\)
the candidates for max for any $J$: $[0; 1, a_2, \ldots]$

$t_1 > 2$ or $J = \emptyset$ (i.e., $a_2 \geq 2$)

$[0; 1, a_2^{(n)}, \ldots] \overset{a_2 \to \infty}{\longrightarrow} 1$

no max

$t_1 = 2$ (i.e., $a_2 = 1$)

the candidates for max: $[0; 1, 1, a_4, \ldots]$

$t_2 > 4$ or $|J| = 1$ (i.e., $a_4 \geq 2$)

$[0; 1, 1, 1, a_4^{(n)}, \ldots] \overset{a_4 \to \infty}{\longrightarrow} \frac{2}{3}$

no max

$t_2 = 4$ (i.e., $a_4 = 1$)

the candidates for max: $[0; 1, 1, 1, 1, a_6, \ldots]$

$t_3 > 6$ or $|J| = 2$ (i.e., $a_6 \geq 2$)

$[0; 1, 1, 1, 1, 1, a_6^{(n)}, \ldots] \overset{a_6 \to \infty}{\longrightarrow} \frac{5}{8}$

no max

$t_3 = 6$ (i.e., $a_6 = 1$)

the candidates for max: $[0; 1, 1, 1, 1, 1, 1, a_8, \ldots]$

$t_4 > 8$ or $|J| = 3$ (i.e., $a_8 \geq 2$)

$[0; 1, 1, 1, 1, 1, 1, 1, a_8^{(n)}, \ldots] \overset{a_8 \to \infty}{\longrightarrow} \frac{13}{21}$

no max

$t_4 = 8$ (i.e., $a_8 = 1$)

the candidates for max: $[0; 1, 1, 1, 1, 1, 1, 1, 1, a_{10}, \ldots]$

$t_5 > 10$ or $|J| = 4$ (i.e., $a_{10} \geq 2$)

$[0; 1, 1, 1, 1, 1, 1, 1, 1, 1, a_{10}^{(n)}, \ldots] \overset{a_{10} \to \infty}{\longrightarrow} \frac{34}{55}$

no max

$t_5 = 10$ (i.e., $a_{10} = 1$)

the candidates for max: $[0; 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, a_{12}, \ldots]$

and so on...
the candidates for max for any $J$:

$t_1 > 2$ or $J = \emptyset$ (i.e., $a_2 \geq 2$)

$[0; 1, a_2^{(n)}, \ldots] \xrightarrow{n \to \infty} 1$

no max

$t_1 = 2$ (i.e., $a_2 = 1$)

$[0; 1, 1, 1, a_4^{(n)}, \ldots] \xrightarrow{n \to \infty} \frac{2}{3}$

no max

the candidates for max: $[0; 1, 1, 1, a_4, \ldots]$

$t_2 > 4$ or $|J| = 1$ (i.e., $a_4 \geq 2$)

$t_2 = 4$ (i.e., $a_4 = 1$)

$[0; 1, 1, 1, 1, a_6^{(n)}, \ldots] \xrightarrow{n \to \infty} \frac{5}{8}$

no max

the candidates for max: $[0; 1, 1, 1, 1, a_6, \ldots]$

$t_3 > 6$ or $|J| = 2$ (i.e., $a_6 \geq 2$)

$t_3 = 6$ (i.e., $a_6 = 1$)

$[0; 1, 1, 1, 1, 1, a_8^{(n)}, \ldots] \xrightarrow{n \to \infty} \frac{13}{21}$

no max

the candidates for max: $[0; 1, 1, 1, 1, 1, a_8, \ldots]$

$t_4 > 8$ or $|J| = 3$ (i.e., $a_8 \geq 2$)

$t_4 = 8$ (i.e., $a_8 = 1$)

$[0; 1, 1, 1, 1, 1, 1, a_{10}^{(n)}, \ldots] \xrightarrow{n \to \infty} \frac{34}{55}$

no max

the candidates for max: $[0; 1, 1, 1, 1, 1, 1, a_{10}, \ldots]$

$t_5 > 10$ or $|J| = 4$ (i.e., $a_{10} \geq 2$)

$t_5 = 10$ (i.e., $a_{10} = 1$)

and so on...
the candidates for max for \( J \in \{0;1,a_2,\ldots\} \)

\[ t_1 > 2 \text{ or } J = \emptyset \text{ (i.e., } a_2 \geq 2) \]

\[ \left[0;1,a_2^{(n)},\ldots\right] \xrightarrow{a_2^{(n)} \to \infty} 1 \]

no max

\[ t_1 = 2 \]

(i.e., \(a_2 = 1\))

the candidates for max:

\[ 0;1,1,1,a_4,\ldots \]

\[ t_2 > 4 \text{ or } |J| = 1 \text{ (i.e., } a_4 \geq 2) \]

\[ \left[0;1,1,1,a_4^{(n)},\ldots\right] \xrightarrow{a_4^{(n)} \to \infty} \frac{2}{3} \]

no max

\[ t_2 = 4 \]

(i.e., \(a_4 = 1\))

the candidates for max:

\[ 0;1,1,1,1,a_6,\ldots \]

\[ t_3 > 6 \text{ or } |J| = 2 \text{ (i.e., } a_6 \geq 2) \]

\[ \left[0;1,1,1,1,a_6^{(n)},\ldots\right] \xrightarrow{a_6^{(n)} \to \infty} \frac{5}{8} \]

no max

\[ t_3 = 6 \]

(i.e., \(a_6 = 1\))

the candidates for max:

\[ 0;1,1,1,1,1,a_8,\ldots \]

\[ t_4 > 8 \text{ or } |J| = 3 \text{ (i.e., } a_8 \geq 2) \]

\[ \left[0;1,1,1,1,1,a_8^{(n)},\ldots\right] \xrightarrow{a_8^{(n)} \to \infty} \frac{13}{21} \]

no max

\[ t_4 = 8 \]

(i.e., \(a_8 = 1\))

the candidates for max:

\[ 0;1,1,1,1,1,1,a_{10},\ldots \]

\[ t_5 > 10 \text{ or } |J| = 4 \text{ (i.e., } a_{10} \geq 2) \]

\[ \left[0;1,1,1,1,1,1,a_{10}^{(n)},\ldots\right] \xrightarrow{a_{10}^{(n)} \to \infty} \frac{34}{55} \]

no max

\[ t_5 = 10 \]

(i.e., \(a_{10} = 1\))

the candidates for max:

\[ 0;1,1,1,1,1,1,1,a_{12},\ldots \]

and so on...
the candidates for max for any \( J : [0; 1, a_2, \ldots] \)

- \( t_1 > 2 \) or \( J = \emptyset \) (i.e., \( a_2 \geq 2 \))

\[ [0; 1, a_2^{(n)}, \ldots] a_2^{(n)} \rightarrow \infty 1 \]

no max

- \( t_1 = 2 \) (i.e., \( a_2 = 1 \))

\[ [0; 1, 1, 1, a_4, \ldots] a_4^{(n)} \rightarrow \infty \frac{5}{8} \]

the candidates for max: \( \{0; 1, 1, 1, a_4, \ldots\} \)

- \( t_2 > 4 \) or \( |J| = 1 \) (i.e., \( a_4 \geq 2 \))

\[ [0; 1, 1, 1, a_4^{(n)}, \ldots] a_4^{(n)} \rightarrow \frac{5}{8} \]

no max

- \( t_2 = 4 \) (i.e., \( a_4 = 1 \))

\[ [0; 1, 1, 1, 1, 1, a_6, \ldots] a_6^{(n)} \rightarrow \infty \frac{13}{21} \]

the candidates for max: \( \{0; 1, 1, 1, 1, 1, a_6, \ldots\} \)

- \( t_3 > 6 \) or \( |J| = 2 \) (i.e., \( a_6 \geq 2 \))

\[ [0; 1, 1, 1, 1, 1, a_6^{(n)}, \ldots] a_6^{(n)} \rightarrow \frac{13}{21} \]

no max

- \( t_3 = 6 \) (i.e., \( a_6 = 1 \))

\[ [0; 1, 1, 1, 1, 1, 1, 1, a_8, \ldots] a_8^{(n)} \rightarrow \infty \frac{34}{55} \]

the candidates for max: \( \{0; 1, 1, 1, 1, 1, 1, 1, a_8, \ldots\} \)

- \( t_4 > 8 \) or \( |J| = 3 \) (i.e., \( a_8 \geq 2 \))

\[ [0; 1, 1, 1, 1, 1, 1, a_8^{(n)}, \ldots] a_8^{(n)} \rightarrow \frac{34}{55} \]

no max

- \( t_4 = 8 \) (i.e., \( a_8 = 1 \))

\[ [0; 1, 1, 1, 1, 1, 1, 1, 1, 1, a_{10}, \ldots] a_{10}^{(n)} \rightarrow \infty \frac{34}{55} \]

the candidates for max: \( \{0; 1, 1, 1, 1, 1, 1, 1, 1, a_{10}, \ldots\} \)

- \( t_5 > 10 \) or \( |J| = 4 \) (i.e., \( a_{10} \geq 2 \))

\[ [0; 1, 1, 1, 1, 1, 1, 1, a_{10}^{(n)}, \ldots] a_{10}^{(n)} \rightarrow \frac{34}{55} \]

no max

- \( t_5 = 10 \) (i.e., \( a_{10} = 1 \))

\[ [0; 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, a_{12}, \ldots] \]

the candidates for max: \( \{0; 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, a_{12}, \ldots\} \)

and so on...
the candidates for max for any $J$: $[0; 1, a_2, \ldots]$

$t_1 > 2$ or $J = \emptyset$ (i.e., $a_2 \geq 2$)

$[0; 1, a_2^{(n)}, \ldots] a_2^{(n)} \to \infty 1$

no max

$t_1 = 2$ (i.e., $a_2 = 1$)

the candidates for max: $[0; 1, 1, 1, a_4, \ldots]$

$t_2 > 4$ or $|J| = 1$ (i.e., $a_4 \geq 2$)

$[0; 1, 1, 1, a_4^{(n)}, \ldots] a_4^{(n)} \to \infty \frac{5}{8}$

no max

$t_2 = 4$ (i.e., $a_4 = 1$)

the candidates for max: $[0; 1, 1, 1, 1, a_6, \ldots]$

$t_3 > 6$ or $|J| = 2$ (i.e., $a_6 \geq 2$)

$[0; 1, 1, 1, 1, a_6^{(n)}, \ldots] a_6^{(n)} \to \infty \frac{13}{21}$

no max

$t_3 = 6$ (i.e., $a_6 = 1$)

the candidates for max: $[0; 1, 1, 1, 1, 1, a_8, \ldots]$

$t_4 > 8$ or $|J| = 3$ (i.e., $a_8 \geq 2$)

$[0; 1, 1, 1, 1, 1, a_8^{(n)}, \ldots] a_8^{(n)} \to \infty \frac{34}{55}$

no max

$t_4 = 8$ (i.e., $a_8 = 1$)

the candidates for max: $[0; 1, 1, 1, 1, 1, 1, a_{10}, \ldots]$

$t_5 > 10$ or $|J| = 4$ (i.e., $a_{10} \geq 2$)

$[0; 1, 1, 1, 1, 1, 1, 1, 1, a_{10}^{(n)}, \ldots] a_{10}^{(n)} \to \infty \frac{34}{55}$

no max

$t_5 = 10$ (i.e., $a_{10} = 1$)

the candidates for max: $[0; 1, 1, 1, 1, 1, 1, 1, 1, 1, a_{12}, \ldots]$

and so on...
the candidates for max for any $J$: $[0;1,a_2,\ldots]$

$t_1 > 2$ or $J = \emptyset$ (i.e., $a_2 \geq 2$)

$[0;1,a_2^{(n)},\ldots] \overset{a_2^{(n)} \to \infty}{\rightarrow} 1$

no max

t_1 = 2
(i.e., $a_2 = 1$)

the candidates for max: $[0;1,1,1,\ldots]$

$t_2 > 4$ or $|J| = 1$ (i.e., $a_4 \geq 2$)

$[0;1,1,1,a_4^{(n)},\ldots] \overset{a_4^{(n)} \to \infty}{\rightarrow} \frac{2}{8}$

no max

t_2 = 4
(i.e., $a_4 = 1$)

the candidates for max: $[0;1,1,1,1,\ldots]$

$t_3 > 6$ or $|J| = 2$ (i.e., $a_6 \geq 2$)

$[0;1,1,1,1,a_6^{(n)},\ldots] \overset{a_6^{(n)} \to \infty}{\rightarrow} \frac{5}{8}$

no max

t_3 = 6
(i.e., $a_6 = 1$)

the candidates for max: $[0;1,1,1,1,1,\ldots]$

$t_4 > 8$ or $|J| = 3$ (i.e., $a_8 \geq 2$)

$[0;1,1,1,1,1,a_8^{(n)},\ldots] \overset{a_8^{(n)} \to \infty}{\rightarrow} \frac{13}{21}$

no max

t_4 = 8
(i.e., $a_8 = 1$)

the candidates for max: $[0;1,1,1,1,1,1,\ldots]$

$t_5 > 10$ or $|J| = 4$ (i.e., $a_{10} \geq 2$)

$[0;1,1,1,1,1,1,1,a_{10}^{(n)},\ldots] \overset{a_{10}^{(n)} \to \infty}{\rightarrow} \frac{34}{55}$

no max

t_5 = 10
(i.e., $a_{10} = 1$)

the candidates for max: $[0;1,1,1,1,1,1,1,1,\ldots]$

and so on...
the candidates for max for any \( J : [0;1,a_2,\ldots] \)

\[
t_1 > 2 \text{ or } J = \emptyset \quad (\text{i.e., } a_2 \geq 2)\]

\[
[0;1,a_2^{(n)},\ldots] a_2^{(n)} \rightarrow \infty 1
\]

no max

\[
t_1 = 2 \quad (\text{i.e., } a_2 = 1)\]

the candidates for max: \([0;1,1,1,a_4,\ldots]\)

\[
t_2 > 4 \text{ or } |J| = 1 \quad (\text{i.e., } a_4 \geq 2)\]

\[
[0;1,1,1,a_4^{(n)},\ldots] a_4^{(n)} \rightarrow \infty 2 \frac{1}{3}
\]

no max

\[
t_2 = 4 \quad (\text{i.e., } a_4 = 1)\]

the candidates for max: \([0;1,1,1,1,a_6,\ldots]\)

\[
t_3 > 6 \text{ or } |J| = 2 \quad (\text{i.e., } a_6 \geq 2)\]

\[
[0;1,1,1,1,a_6^{(n)},\ldots] a_6^{(n)} \rightarrow \infty \frac{5}{8}
\]

no max

\[
t_3 = 6 \quad (\text{i.e., } a_6 = 1)\]

the candidates for max: \([0;1,1,1,1,1,a_8,\ldots]\)

\[
t_4 > 8 \text{ or } |J| = 3 \quad (\text{i.e., } a_8 \geq 2)\]

\[
[0;1,1,1,1,1,a_8^{(n)},\ldots] a_8^{(n)} \rightarrow \infty \frac{13}{21}
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no max

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t_4 = 8 \quad (\text{i.e., } a_8 = 1)\]

the candidates for max: \([0;1,1,1,1,1,1,a_{10},\ldots]\)

\[
t_5 > 10 \text{ or } |J| = 4 \quad (\text{i.e., } a_{10} \geq 2)\]

\[
[0;1,1,1,1,1,1,a_{10}^{(n)},\ldots] a_{10}^{(n)} \rightarrow \infty \frac{34}{55}
\]

no max

\[
t_5 = 10 \quad (\text{i.e., } a_{10} = 1)\]

the candidates for max: \([0;1,1,1,1,1,1,1,a_{12},\ldots]\)

and so on...

A new fixed point theorem for words
The set of all right infinite words over \( \{1,2\} \):

\[
\{1, 2\}^\omega
\]

\( w : \mathbb{N}^+ \rightarrow \{1, 2\} \)

\[
w = w(1)w(2)w(3) \cdots \in \{1, 2\}^\omega
\]
Kolakoski word

The run-length encoding operator

$$\Delta_l: \{1, 2\}^\omega \rightarrow \mathbb{N}^\omega$$

$$w = \begin{cases} 1^{k_1}2^{k_2}1^{k_3}2^{k_4}\ldots, & \text{if } w \in 1 \cdot \{1, 2\}^\omega \\ 2^{k_1}1^{k_2}2^{k_3}1^{k_4}\ldots, & \text{if } w \in 2 \cdot \{1, 2\}^\omega \end{cases}$$

$$\Delta_l(w) = k_1k_2k_3\ldots$$
Kolakoski word

The run-length encoding operator - an example:

\[ w = 11112122222222112221212222222221 \ldots \]

\[ \Delta'(w) = 5, 1, 1, 7, 2, 3, 1, 1, 1, 9, \ldots \]
Kolakoski word

\[ w = 2 \ 2 \ 1 \ 1 \ 2 \ 1 \ 2 \ 2 \ 1 \ 2 \ 2 \ 1 \ 1 \ 2 \ldots \]

\[ \Delta_t(w) = 2 \ 2 \ 1 \ 1 \ 2 \ 1 \ 2 \ 2 \ 1 \ 2 \ 2 \ 1 \ 1 \ 2 \ldots \]
The constructional word $\gamma(a) \in \{0, 1\}^\omega$

Let $a = [0; a_1, a_2, \ldots]$. For $n \in \mathbb{N}^+$:

$\gamma_n(a) = i_a(n + 2) - i_a(n + 1) - 1$

$\gamma_n(a) = \delta_1(a_{i_a(n+1)})$

$\gamma_n(a) = \begin{cases} 
0, & S_n \text{ is the most frequent run on level } n \text{ for } s'(a) \\
1, & L_n \text{ is the most frequent run on level } n \text{ for } s'(a).
\end{cases}$
The constructional word \( \gamma(a) \in \{0, 1\}^\omega \)

Let \( a = [0; a_1, a_2, \ldots] \). For \( n \in \mathbb{N}^+ \):

\[
\gamma_n(a) = i_a(n + 2) - i_a(n + 1) - 1
\]

\[
\gamma_n(a) = \delta_1(\alpha_{i_a(n+1)})
\]

\[
\gamma_n(a) = \begin{cases} 
0, & S_n \text{ is the most frequent run on level } n \text{ for } s'(a) \\
1, & L_n \text{ is the most frequent run on level } n \text{ for } s'(a).
\end{cases}
\]
Fixed point theorem: the run-construction encoding operator

Definition: The run-construction encoding operator

\[ \Delta_c : \mathcal{UM}_0 \rightarrow \{0, 1\}^\omega \] is defined as \( \Delta_c = (1\gamma) \circ (s')^{-1} \).

\[ \begin{array}{c}
\mathcal{Q} \\
\left]0, 1\right[ \end{array} \xrightarrow{s'} \mathcal{UM}_0 \xrightarrow{1\gamma} \{0, 1\}^\omega \supset \mathcal{UM}_0 \]

where \( \mathcal{UM}_0 \) denotes the set of all upper mechanical words with irrational slope \( 0 < a < 1 \) and with intercept 0.
Let $a \in ]0, 1[ \setminus \mathbb{Q}$. The word $s'(a) = 1c(a)$ has balanced construction if

$$\exists \alpha \in \mathbb{R} \quad \gamma(a) = c(\alpha)$$

Sturmian-balanced construction if

$$\exists \alpha \in ]0, 1[ \setminus \mathbb{Q} \quad \gamma(a) = c(\alpha)$$

self-balanced construction

$$1\gamma(a) = \Delta_c(1c(a)) = 1c(a)$$
Balanced construction – some examples

Paper VI. Examples 2, 3, 4, 5.

- The words $s'(a)$ with $a = [0; a_1, a_2, a_3, \ldots]$, where $a_k \geq 2$ for all $k \geq 2$, have balanced construction.

- The words $s'(a)$ with $a = [0; a_1, 1, a_3, 1, a_5, 1, a_7, \ldots]$, where $a_{2k-1} \in \mathbb{N}^+$ for all $k \in \mathbb{N}^+$, have balanced construction.
A fixed-point theorem: exactly 1 fixed point in each equivalence class

Let \((b_n)_{n \in \mathbb{N}^+}\) be such that \(b_1 \in \mathbb{N}^+\) and \(b_n \in \mathbb{N}^+ \setminus \{1\}\) for all \(n \geq 2\). Then

\[
\exists 1 a \in ]0,1[ \setminus \mathbb{Q} \\
\quad a \in [(b_n)_{n \in \mathbb{N}^+}]_{\sim_{\text{len}}} \wedge s'(a) = \Delta_c(s'(a)).
\]
A fixed-point theorem:
exactly 1 fixed point in each equivalence class

Let \((b_n)_{n \in \mathbb{N}^+}\) be such that \(b_1 \in \mathbb{N}^+\) and \(b_n \in \mathbb{N}^+ \setminus \{1\}\) for all \(n \geq 2\). Then

\[
\exists a \in ]0,1[ \setminus \mathbb{Q} \quad a \in [(b_n)_{n \in \mathbb{N}^+}]_{\sim \text{len}} \land s'(a) = \Delta_c(s'(a)).
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A fixed-point theorem: exactly 1 fixed point in each equivalence class

Let \((b_n)_{n \in \mathbb{N}^+}\) be such that \(b_1 \in \mathbb{N}^+\) and \(b_n \in \mathbb{N}^+ \setminus \{1\}\) for all \(n \geq 2\). Then

\[
\exists 1 \ a \in ]0,1[ \setminus \mathbb{Q} \quad a \in [(b_n)_{n \in \mathbb{N}^+}]_{\sim_{\text{len}}} \land s'(a) = \Delta_c(s'(a)).
\]
Equivalence classes under the relation $\text{len}$
Equivalence classes under the relation \( \text{len} \)

- \( a_{\text{max}} = [0; b_1, b_2, 1, b_3 - 1, 1, b_4 - 1, 1, b_5 - 1, \ldots] \),
- \( a_{\text{min}} = a_{\text{long}} = [0; b_1, 1, b_2 - 1, 1, b_3 - 1, 1, b_4 - 1, \ldots] \),
- \( a_{\text{short}} = [0; b_1, b_2, b_3, b_4, \ldots] \),
- \( a_{\text{fix}} \) is the slope of the fixed point of the run-construction encoding operator \( \Delta_c \), i.e., \( \gamma(a_{\text{fix}}) = e(a_{\text{fix}}) \), where \( \gamma \) is the constructional word.
The set of all fixed points

No quadratic surd can be a fixed point!

Their constructional words have rational slopes, if any.

(Proposition 3 in Paper VI).
The set of all fixed points

**Theorem** Let $\text{Fix}(\Delta_c) \subseteq \mathcal{UM}_0$ denote the set of all fixed points of $\Delta_c$. Then:

1. $\text{Fix}(\Delta_c) \subseteq s'(\mathbb{Q}, \frac{2}{3})$; numbers $0$ and $\frac{2}{3}$ are accumulation points of $(s')^{-1}(\text{Fix}(\Delta_c))$.

2. $\text{card}(\text{Fix}(\Delta_c))$ is equal to that of the continuum.
Some combinatorial questions

Combinatorics on words - new classes of words

Iterations of the run-construction encoding operator

What can one say about the fixed points?
Formulate an iff condition for CFs of fixed points.

Two kinds of description: by the CF-elements and by the properties of real numbers (transcendental, algebraical)
Thank you for your attention