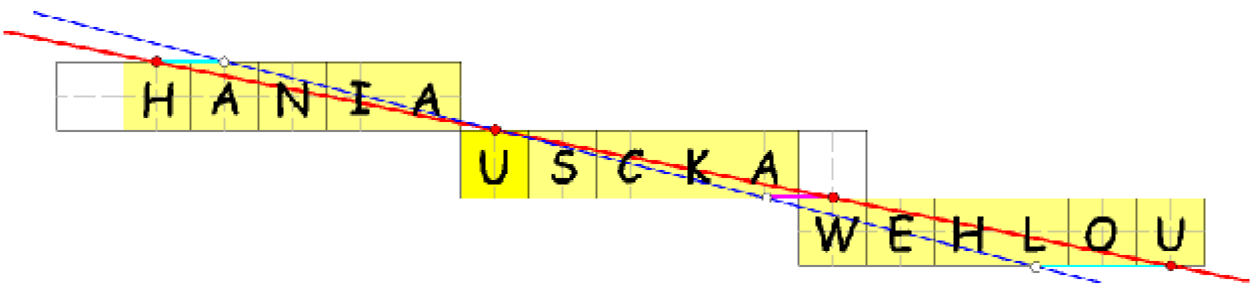
Digital Lines

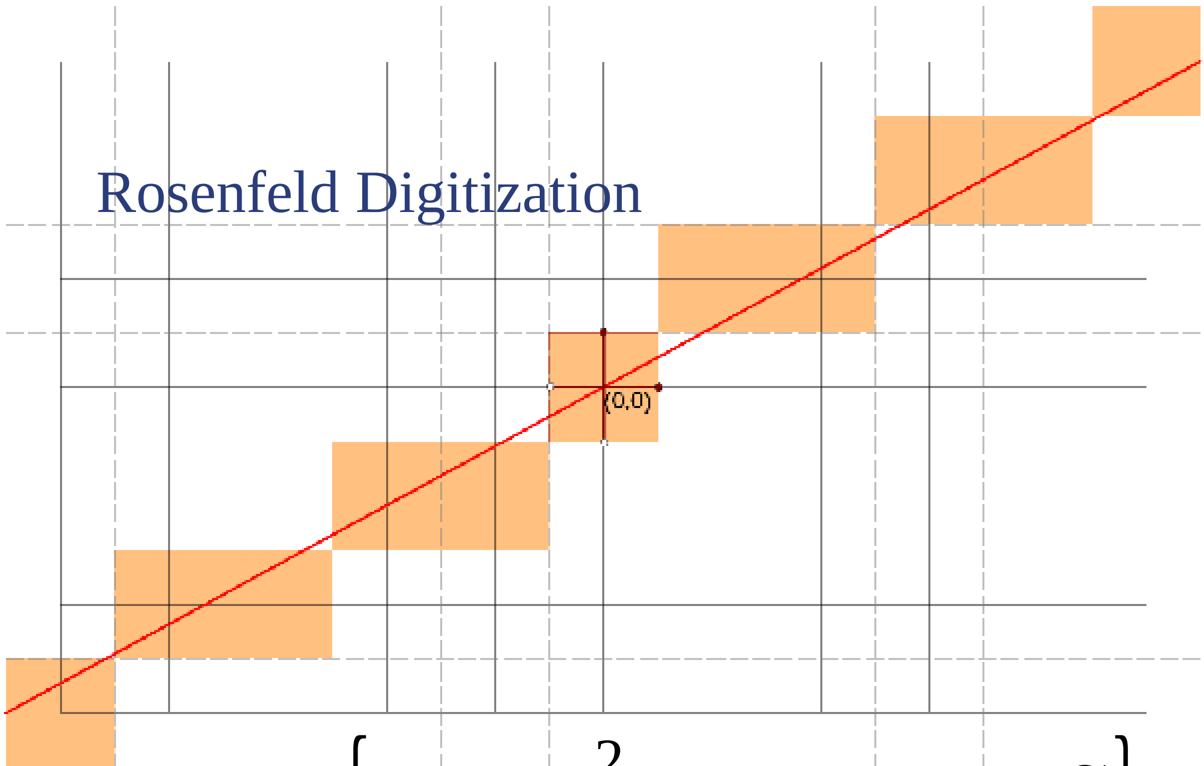
Theory for lines with irrational slope



(Christer Kiselman's research group: “Digital Geometry and Mathematical Morphology”)

One year ago, during my presentation for CBA I showed you a lot of pictures concerning digital lines according to Rosenfeld and Khalimsky. This presentation is a kind of continuation of the previous one, at least of a certain part of this.

Rosenfeld Digitization



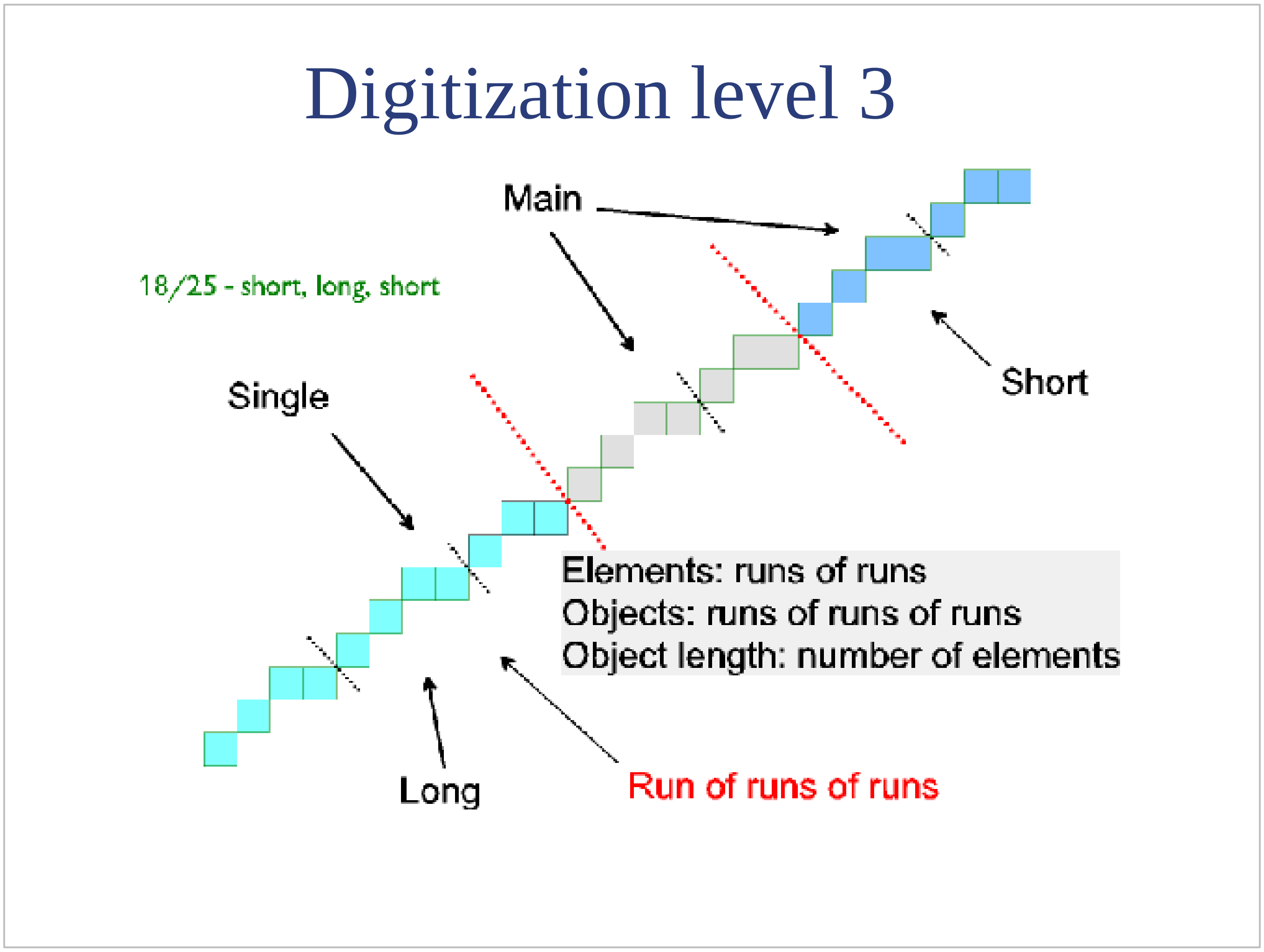
$$D_R(l)=\left\{(k,n)\in\mathbb{Z}^2;\quad l\cap C_R(k,n)\neq\emptyset\right\}$$

It is not necessary to explain for this public what it means to digitize a subset of R^2. I would only like to recall Rosenfeld's idea. To digitize a subset of R^2, we do the following:

first we put one-by-one crosses in all the grid points.
** Here is an example of such a cross in the origin.
** These two points belong to the cross, these two do not.
A grid point belongs to the digitization of an object if the object and the the grid point's cross have some points in common, so
** their intersection is not empty.
** We can think of such a square as a PIXEL, it gives a good picture of what digitization of an object means.

I will talk about a slight modification of this idea and I will call it "R'-digitization". The modification is moving the crosses half a unit downwards. It is only a small technical detail, but it makes the calculations and formulations easier.

It is only important to remember that the R'-digitization of a straight line $y=ax$ always includes the pixel(1,1). In the digitization of positive half lines it is the first pixel of the digitization.



On this picture you see the graphical explanation of the meaning of digitization level 3 and other terms connected with it. I use now the term ”run”, like everywhere in the literature, so no longer ”blocks” and ”sequences”.

Each digitization level, starting with level 1, has its objects and elements forming the objects. Object length is defined as the number of elements forming this object.

** For **level 1** elements are pixels (or grid points, which can be interpreted as addresses to pixels) and objects are runs. Here we have runs wih length 1 and runs with length 2. The long runs never appear more than once at a time, so we call them single. The objects which occur in multiples (in this case the short runs) we call main.
** For **level 2** elements are runs and objects are runs of runs.
** For **level 3** elements are runs of runs, objects are runs of runs of runs. On this level the short objects (length 2) are main (THEY OCCUR IN MULTIPLES) and long (length 3) are single (THEY NEVER APPEAR MORE THAN ONCE AT A TIME).

Generally, the elements of one level are the objects of the previous one. Those objects are composed of all the mains from the previous level and one single.

Two problems

- Generation
- Recognition

Basicly, there are 2 big problems we are dealing with: how to compute the digitization of a straight line and how to recognize a straight line in a set of pixels.

Kinds of description

- Algorithmic (you already know this)
- Theorems and proofs

There are 2 possible kinds of solutions of the problem: algorithmic one and one by means of theorems. You know a lot of examples of the first kind of solutions. I have tried to get a solution of the second kind.

Theorems and proofs

- Necessary condition (generation)
- Sufficient condition (recognition)

The solution of the problem of the generation of a digital straight line is included in the theorem ”necessary condition”. The solution of the second problem (recognition) in the theorem ”sufficient condition”.

Necessary conditions

First the theorem about necessary conditions for being a digital line. It is not my purpose to explain everything in detail. I only want to show what I have done and what I have used to get the formulation of the theorem. I have to present quite a big number of definitions to be able to write down the formal result. This means that it will probably be not so easy to follow 100% of the time. Do not worry.

Four restrictions

- $a > 0$
- $a < 1$
- a irrational
- $x > 0$

In order to get a relatively simple and clear formulation of the theorem, I had to make the four restrictions (SHOW THEM). Why?:

$a<0$ – you have to go backwards in the numeration of the objects (symmetry about y-axis solves the problem)
$a>1$ – you do not deal with a function **Z**->**Z** (symmetry about the diagonal $y=x$ solves the problem)
a rational – you get a lot of special cases; there is only a finite number of digitization levels, you get periodicity.
The problem for rational slopes has a nice complete solution with application of the euclidic algorithm (I showed this a year ago during my presentation here).
$x<0$ – you have to make the calculations with negative integers (a kind of point symmetry solves the problem).

Definition 1

For $y = ax$ where $0 < a < 1$ is irrational, the *digitization parameters* are:

$s_0 = a,$

$s_1 = \text{frac}\left(\frac{1}{s_0}\right),$

For $k > 1$:

$$s_k = \begin{cases} \text{frac}\left(\frac{1}{s_{k-1}}\right) & \text{if } s_{k-1} < \frac{1}{2} \\ \text{frac}\left(\frac{1}{1-s_{k-1}}\right) & \text{if } s_{k-1} > \frac{1}{2} \end{cases}$$

The most important parameters for the digitization of a straight line. In the case of irrational *a* there exist *sigma_k* for each natural *k*.

** On each digitization level it is very important if the digitization parameter for this level is greater or less than one half. This will decide about a lot.

Definition 2

$$s_0^* = s_0,$$

For $k \geq 1$:

$$s_k^* = \begin{cases} s_k & \text{if } s_k < \frac{1}{2} \\ 1 - s_k & \text{if } s_k > \frac{1}{2} \end{cases}$$

To get more homogeneous notation in some cases, it is good to introduce the modified digitization parameters as on this slide. We do not modify at all if the digitization parameter is less than one half and we take the complement to 1 in case the digitization parameter is greater than one half.

Definition 3

For a given straight line l with equation $y = ax$, where $0 < a < 1$ is irrational, we need a helper function:

$$\mathrm{Alter} : \mathbb{N} \to \{0,1\}$$

$$\mathrm{Alter}(i) = \begin{cases} 0 & \text{if } i = 0 \text{ or } i > 0 \text{ and } s_i > \frac{1}{2} \\ 1 & \text{if } i > 0 \text{ and } s_i < \frac{1}{2} \end{cases}$$

This alternation function keeps a register of the digitization parameters for all the digitization levels. A given level gets the value of this function equal to 1 if the sigma parameter of the level is less than one half and gets zero otherwise.

Why such a name of the function? Value 1 for the level i gives an alternation on the level i+1, value 0 gives no alternation. By alternation I mean the kind of the first object of the level (short or long).

Definition 4

The second helper function:

$$\mathrm{Sum} : \mathbb{N}^{+} \rightarrow \mathbb{N}$$

$$\mathrm{Sum}(k) = \sum_{i=0}^{k-1} \mathrm{Alter}(i) \quad \text{for } k \geq 1$$

The next helper function. This one shows for each given level *k*, how many times the alternation of the first happened on all the previous levels (because it is the sum of all the ones).

Lemma

If an irrational number s fulfills $0 < s < 1$ and:

case 1: $d = \mathrm{frac}\left(\frac{1}{s}\right) < \frac{1}{2}$, then

$$\left\lfloor \frac{i+1}{s} \right\rfloor - \left\lfloor \frac{i}{s} \right\rfloor = \begin{cases} \left\lfloor \frac{1}{s} \right\rfloor + 1 & \text{if} \quad \exists j \in \mathbb{N}^+, \ i = \left\lfloor \frac{j}{d} \right\rfloor \\ \left\lfloor \frac{1}{s} \right\rfloor & \text{otherwise} \end{cases}$$

case 2: $d = \mathrm{frac}\left(\frac{1}{s}\right) > \frac{1}{2}$, then

$$\left\lfloor \frac{i+1}{s} \right\rfloor - \left\lfloor \frac{i}{s} \right\rfloor = \begin{cases} \left\lfloor \frac{1}{s} \right\rfloor & \text{if} \quad \exists j \in \mathbb{N}^+, \ i = \left\lfloor \frac{j}{1-d} \right\rfloor \\ \left\lfloor \frac{1}{s} \right\rfloor + 1 & \text{otherwise} \end{cases}$$

This is the most important lemma for the whole theory. This lemma is the basis of the proof of the theorem ”necessary conditions”. It makes the proof really easy. You can recognize here two object lengths on each digitization level.

Definition 5

For a given straight line l with equation $y = ax$, where $0 < a < 1$ is irrational, we define the following functions:

- $\mathrm{run}_0 = \mathrm{Id} : \mathbb{N}^+ \to \mathbb{N}^+$, where Id denotes the identity function on $\mathbb{N}^+$.
- For $k \geq 1$: $\mathrm{run}_k : \mathbb{N}^+ \to \mathcal{P}\left(\mathrm{run}_{k-1}\left(\mathbb{N}^+\right)\right)$ defined for $j \geq 1$ as follows:

$$\mathrm{run}_k(1) = \left\{\mathrm{run}_{k-1}(i);\quad 1 \leq i \leq \left\lfloor \frac{1}{s^*_{k-1}} \right\rfloor + t(k)\right\}, \text{ and for } j \geq 2:$$

$$\mathrm{run}_k(j) = \left\{\mathrm{run}_{k-1}(i);\quad \left\lfloor \frac{j-1}{s^*_{k-1}} \right\rfloor + t(k) + 1 \leq i \leq \left\lfloor \frac{j}{s^*_{k-1}} \right\rfloor + t(k)\right\},$$

where

$$t(k) = \begin{cases} 0 & \text{if } \mathrm{Sum}(k) \text{ is even} \\ 1 & \text{if } \mathrm{Sum}(k) \text{ is odd,} \end{cases}$$

where s^*_k are the digitization parameters defined in def. 2, the function Sum is defined in def. 4 and $\mathcal{P}(A)$ denotes the power set of a set A

The formal definition of the digitization run on each digitization level. Run_*k* means ”run of runs of runs...” with the word ”runs” repeated *k* times (one time in singular and *k*-1 times in plural). Here you can see very clearly the first intuitive idea that the object of a given level is the element on the next levels. We define the runs as SETS with runs of the lower level as elements.

Pay also attention to the *t*-function: this one uses the Sum-function which decides about the first on the level.

Digitization level k

Let $k \geq 1$.

- Run of digitization level k (run_k):
  $\text{run}_k(j)$, where $j \geq 1$.
- Runs of digitization level k (runs_k):
  $\left\{\text{run}_k(i); i \in I\right\}$, where $I \in \mathcal{P}\left(\mathbb{N}^+\right)$.

Some terminology (just read it!)

Proposition

Let l be given by the equation $y = ax$ where $0 < a < 1$ is irrational. For each $k \geq 1$, the runs of the level k can have one of the two lengths:

$$\left\lfloor \frac{1}{s^*_{k-1}} \right\rfloor \text{ (short runs) or } \left\lfloor \frac{1}{s^*_{k-1}} \right\rfloor + 1 \text{ (long runs).}$$

This proposition states formally that there are two possible object lengths on each digitization level. The length of the runs on level *k* is expressed by the modified digitization parameter of the level *k*-1.

Definition 6

For a given straight line l with equation $y = ax$, where $0 < a < 1$ is irrational, we define the following functions for $k \in \mathbb{N}^+$:

$$\text{kind\_run}_k : \quad \mathbb{N}^+ \to \{S, L\},$$

where 'S' and 'L' are abbreviations for *short* and *long* respectively:

For $j \geq 1$:

$$\text{kind\_run}_k(j) = \begin{cases} S & \text{if } \operatorname{card}(\text{run}_k(j)) = \left\lfloor \dfrac{1}{s^*_{k-1}} \right\rfloor \\ L & \text{if } \operatorname{card}(\text{run}_k(j)) = \left\lfloor \dfrac{1}{s^*_{k-1}} \right\rfloor + 1 \end{cases}$$

where $\operatorname{card}(\text{run}_k(j))$ denotes the number of elements in $\text{run}_k(j)$ (the *length* of $\text{run}_k(j)$ being the number of runs_{k-1} forming it).

The proposition from the previous slide allows as to formulate the following definition. We have 2 kinds of runs on each level: short runs (S) and long runs (L).

Definition 7

For a given straight line l with equation $y = ax$, where $0 < a < 1$ is irrational, we define the *alternation - function :*

$$\mathrm{alt}: \{S, L\} \to \{S, L\}$$

as follows:

$$\mathrm{alt}(S) = L, \quad \mathrm{alt}(L) = S.$$

Another alternation function, which will be usefull in the formulation of the theorem. Alternation of a short run is a long run and vice versa.

Definition 8

For a given straight line l with equation $y = ax$, where $0 < a < 1$ is irrational, we define three level-fun ctions:

$$\text{single}_{(.)}, \text{main}_{(.)}, \text{first}_{(.)}: \mathbb{N}^+ \to \{S, L\}$$

as follows: for $k \geq 1$:

$$\text{single}_k = \begin{cases} S & \text{if } \{ j \in \mathbb{N}^+ ; \text{kind\_run}_k(j) = \text{kind\_run}_k(j+1) = S \} = \varnothing \\ L & \text{if } \{ j \in \mathbb{N}^+ ; \text{kind\_run}_k(j) = \text{kind\_run}_k(j+1) = L \} = \varnothing \end{cases}$$

$$\text{main}_k = \text{alt} \circ \text{single}_k$$

$$\text{first}_k = \text{kind\_run}_k(1)$$

Three level-functions:

-The first one determines the single run of each level (this one which never appears more than once at a time)

-The second one determins the main run of each level (this one which occurs in multiples). This is the opposite to the single.

-The third one determins the first run of each level (The one beginning with pixel(1,1)).

Theorem (necessary condition)

For a given straight line l with equation $y = ax$, where $0 < a < 1$ is irrational, the R'-digitization of the positive half line is the following subset of $\mathbb{Z}^2$:

$$D_{R'}(l) = \bigcup_{j \in \mathbb{N}^+} \{\mathrm{run}_1(j) \times \{j\}\}$$

where for $k \geq 1$ runs of the level k defined in def. 5 fulfill following con ditions:

$[\mathrm{M}]$: $s_k < \frac{1}{2} \Rightarrow \mathrm{main}_k = S, \qquad s_k > \frac{1}{2} \Rightarrow \mathrm{main}_k = L.$

S denotes a short run_k with length $\left\lfloor \frac{1}{s^*_{k-1}} \right\rfloor$ and L denotes a long run_k with length $\left\lfloor \frac{1}{s^*_{k-1}} \right\rfloor + 1.$

$[\mathrm{S}]$: For $j \geq 1$: if $\mathrm{main}_k = x$ where $x \in \{S, L\}$, then

$\mathrm{kind\_run}_k\left(\left\lfloor \frac{j}{s^*_k} \right\rfloor + 1\right) = \mathrm{alt}(x)$ and $\mathrm{kind\_run}_k(i) = x$ for all $i \neq \left\lfloor \frac{j}{s^*_k} \right\rfloor + 1.$

$[\mathrm{F}]$: $\mathrm{first}_k = S \Leftrightarrow \mathrm{Sum}(k)$ is even.

FINALLY the theorem: Necessary conditions for a set of pixels to be a digital straight line:

1. [M] – condition ”main” says that the main of the level with the digitization parameters less than one half is SHORT
and the main of the level with the digitization parameter greater than one half is LONG.

2. [S] – condition ”single” says that the numbers of the single runs are numbers like this (**)

3. [F] – condition ”first” shows that the first of the level is short iff the value of the Sum-function on the number of this level is even and long if the value is odd.

Necessary condition - construction

For the R'-digitization of a line l with equation $y = ax$, where $0 < a < 1$ is irrational, we have:

for each natural $j \geq 1$, $\text{run}_1(j)$ can have two possible lengths:

$$\left\lfloor \frac{1}{s_0} \right\rfloor \quad (S - \text{short}) \text{ and } \left\lfloor \frac{1}{s_0} \right\rfloor + 1 \quad (L - \text{long})$$

and the forms of runs_{k+1} (form_run_{k+1}) for $k \geq 1$ are as follows:

$$\text{form\_run}_{k+1} = \begin{cases} S \ldots SL & \text{iff Sum}(k+1) = \text{Sum}(k) + 1, \ \text{Sum}(k) \text{ is even} \\ SL \ldots L & \text{iff Sum}(k+1) = \text{Sum}(k), \quad \text{Sum}(k) \text{ is even} \\ LS \ldots S & \text{iff Sum}(k+1) = \text{Sum}(k) + 1, \ \text{Sum}(k) \text{ is odd} \\ L \ldots LS & \text{iff Sum}(k+1) = \text{Sum}(k), \quad \text{Sum}(k) \text{ is odd} \end{cases}$$

where S means run_k with length $\left\lfloor \frac{1}{s^*_{k-1}} \right\rfloor$, L means run_k with length $\left\lfloor \frac{1}{s^*_{k-1}} \right\rfloor + 1$

and the function Sum is defined in def. 4.

The corollary. Follows directly from the theorem from the previous slide. I show this one to you because it is the constructive one. It says HOW to compute the whole structure of the digitization of any line *y=ax* with our 4 restrictions. As you can see, everything is determined by the digitization parameters (function Sum also!) and we start the construction in pixel(1,1).

Sufficient conditions

Now some words about sufficient conditions to be a digital line.

Theorem (sufficient condition)

Each subset of $(\mathbb{N}^+)^2$ containing $(1,1)$ and fulfilling the conditions $[M]$, $[S]$ and $[F]$ on all the levels is the R'-digitization of the positive half line of some line $y = ax$, where $0 < a < 1$ is irrational.

Note: $[M]$, $[S]$ and $[F]$ are formulated in the theorem "Necessary conditions".

This theorem states that the necessary conditions ”main”, ”single” and ”first” are also sufficient. So we get en iff condition for digital lines, which is a very nice thing for the recognition.

Proposition for recognition

Let $n \geq 1$ be a natural number. Each sequence of non-zero natural numbers $(k_1, k_2, \ldots, k_n)$ such that $k_i > 1$ for $1 < i \leq n$ generates m lines with rational coefficients, where

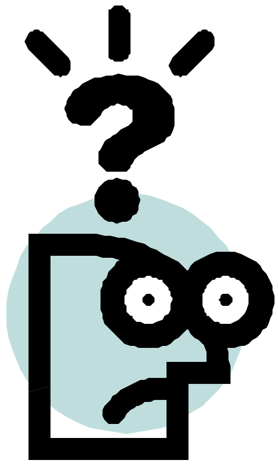
$$m = \begin{cases} 2^{n\text{-}1} & \text{if } k_n \neq 2 \\ 2^{n\text{-}2} & \text{if } k_n = 2 \end{cases}$$

The digitizations fulfill the following n conditions:

for $i = 1, \ldots, n$ the short object's length on digitization level i is k_i

This proposition leads to a sufficient condition to be a digital line. It helps us recognize a properly digitized straight line from a sequence of pixels.
k_i is the length of the short object at digitization level i. The proposition says that all the sequences of natural numbers greater than 1 give a digitization of a straight line according to the necessary condition formulated in the theorem “necessary conditions”.
The proof for rational slopes is based on the euclidic algorithm and the proof for irrational slopes – on the theory for continued fractions.

Any questions?



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http://www.wehlou.com/hania/diggeom.htm

Thank you for your attention. Questions, remarks, comments? E-mail, the presentation will come on our web-site soon.

I will appreciate very much all the remarks, because the content of this presentation is a part of a paper I am working on now.