A new recursive, continued fraction based, formula for digital lines $y = ax$ with irrational slopes $a = [0 ; a_1, a_2, a_3, ... ]$

The **index jump function** corresponding to $a$; **essential 1's**:

\[
[0; 1, a_2, \frac{1}{1}, a_5, \frac{1}{1}, a_8, a_9, \frac{1}{1}, a_{11}, a_{12}, \frac{1}{1}, \frac{1}{1}, a_{16}, a_{17}, \ldots ]
\]

(1, 2, 3, 5, 6, 8, 9, 10, 12, 13, 15, 17, ...)

The role of **essential 1's** in the construction of runs:

- $a_{i_a}(k+1) \geq 2 \implies S_k$ is the main run $S_k$,
- $a_{i_a}(k+1) = 1 \implies L_k$ is the main run $L_k$. 

The pixel-wise length of digital line segments

The denominators of the convergents:

For the convergents \( \frac{p_n}{q_n} = [0; a_1, a_2, \ldots, a_n] \) we have:
\[
q_0 = 1, \quad q_1 = a_1, \quad \text{and, for } n \geq 2, \quad q_n = a_nq_{n-1} + q_{n-2}.
\]

The pixel-wise length of runs on level \( n \) for the line \( y = ax \):
\[
|S_n| = q_{i_a(n+1)-1}, \quad |L_n| = q_{i_a(n+1)-1} + q_{i_a(n+1)-2}
\]
### Two equivalence relations on the set of slopes

<table>
<thead>
<tr>
<th><strong>Quantitative (length)</strong></th>
<th><strong>Qualitative (construction)</strong></th>
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</thead>
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<tr>
<td>★ Defined by run lengths (their cardinality)</td>
<td>★ Defined by the places of essential 1's</td>
</tr>
<tr>
<td>All lines from the same class have the same run lengths on all digitization levels.</td>
<td>All lines from the same class have the same construction in terms of long and short runs on all digitization levels.</td>
</tr>
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</table>

**An example (from paper 3):**

\[
||S_1|| = 1, ||S_2|| = 2, ||S_3|| = 2, ||S_4|| = 3.
\]

**Theorem (from paper 3):**

\[
\forall \ n \in \mathbb{N}^+ \ [ (\forall \ k \in [1, n-1] \mathbb{Z} \ s_k = 2k) \land (s_n > 2n \lor |J| = n - 1) ] \\
\Rightarrow \ \sup \{a; \ a \in [(s_j)_{j \in J}]_{\sim_{con}}\} = \frac{F_{2n-1}}{F_{2n}},
\]

where \((F_n)_{n \in \mathbb{N}^+}\) is the **Fibonacci** sequence and \((s_j)_{j \in J}\) is the sequence of the places of essential 1's.