Digital lines, Sturmian words, and continued fractions

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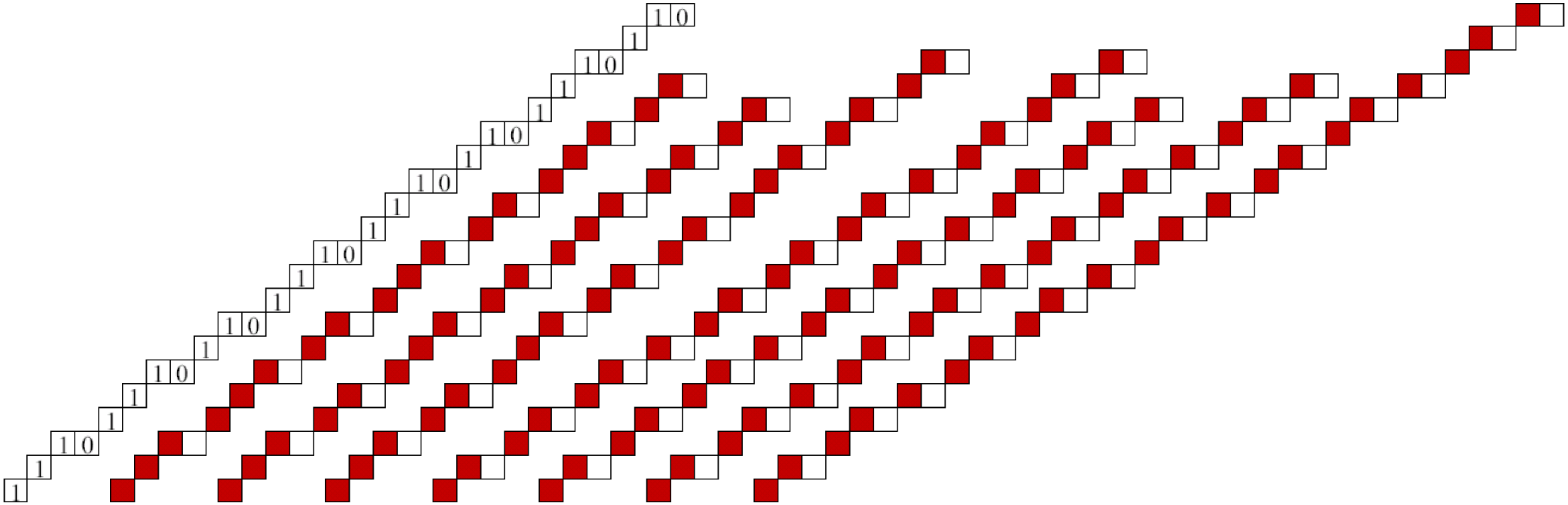
Hanna Uscka-Wehlou



25 September 2009

Four short stories:

about continued fractions

about words

about digital lines

about my contribution

1

continued fractions

Continued fractions - notation

$$a = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} = [0; a_1, a_2, a_3, \ldots]$$

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Continued fractions – a definition

$$a = [a_0; a_1, a_2, a_3, \ldots]$$

$$\alpha_0 = a; \qquad \text{for} \quad n \geq 0 :$$

$$a_n = \lfloor \alpha_n \rfloor, \quad \alpha_{n+1} = \frac{1}{\alpha_n - a_n}$$

$$= \frac{1}{\text{frac}(\alpha_n)}$$

Why do we need continued fractions?

This works for irrational numbers as well

An excellent tool for approximations

They give a better understanding of a number

Numerous geometrical interpretations and applications

Solution's techniques in number theory (Pell's equation)

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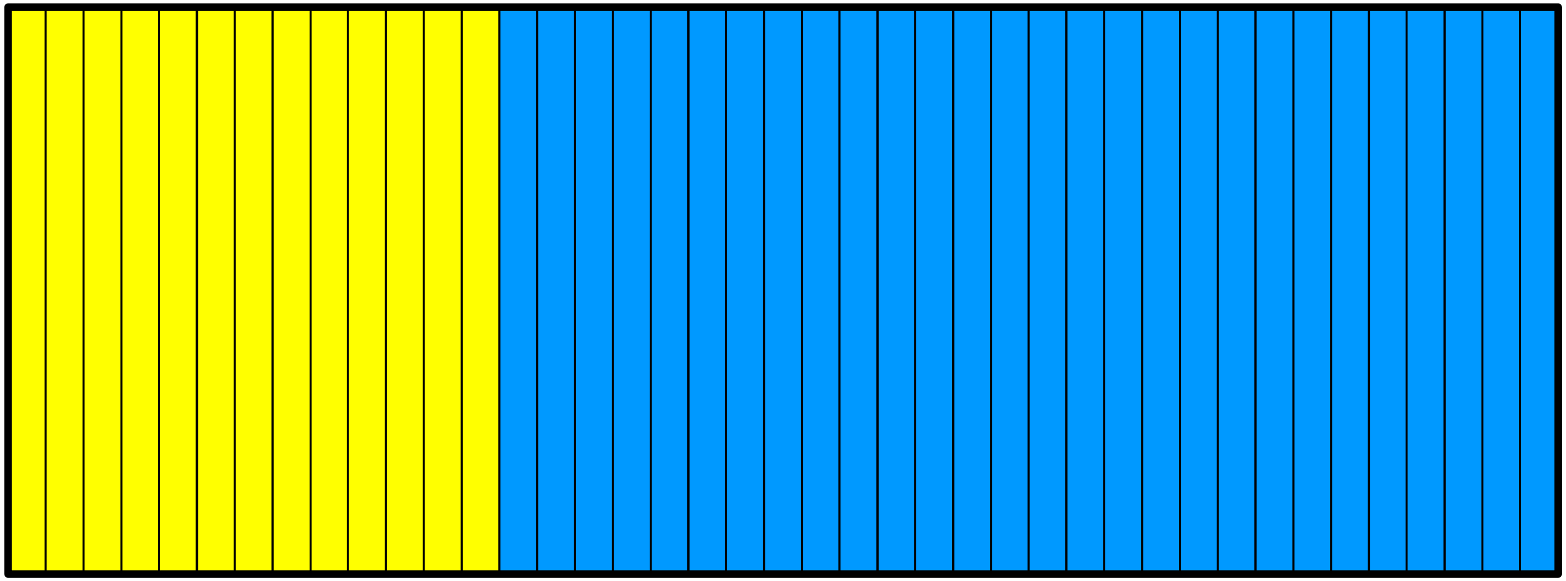
A fraction – an interpretation

$$\frac{13}{41}$$



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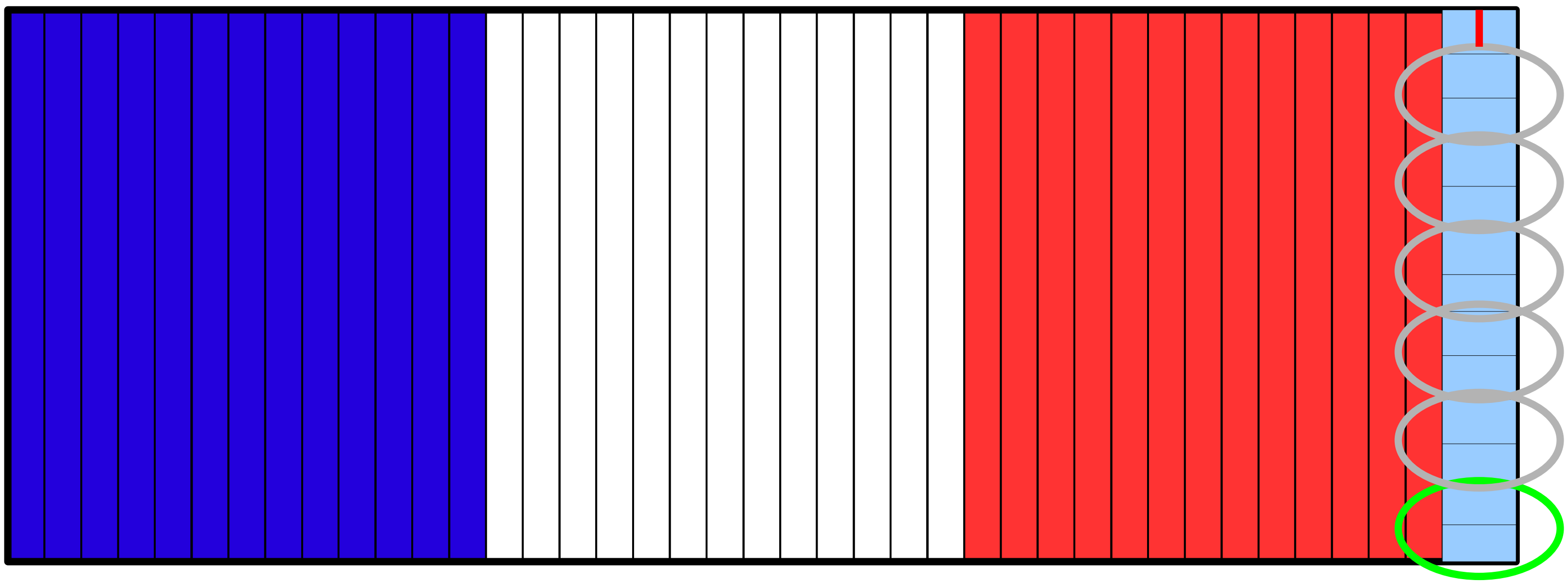
A continued fraction – an interpretation

$$\frac{13}{41} = \frac{1}{\frac{41}{13}} = \frac{1}{3+\frac{2}{13}} = \frac{1}{3+\frac{1}{\frac{13}{2}}} = \frac{1}{3+\frac{1}{6+\frac{1}{2}}} = [0;3,6,2].$$



Continued fractions and their relatives

Euclid's algorithm

The Gauss map

The Stern – Brocot tree

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words

Words – the alphabet

$\mathcal{A}$ = {a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w,x,y,z}

dziewczynka, vriendschap, dataspel, ciotka, siostrzyczka

$\mathcal{A}$ = {1,2}

122111121, 1, 12121212, 2222

$\mathcal{A}$ = {#,¤,&,@}

¤¤¤&@#¤@@@@&&&#¤@& ...

Finite words

A – alphabet (a set of symbols)

A^* - the set of finite words over A

$(A^*, +)$ - is a monoid :

- concatenation (+) is associative $(u+v)+w=u+(v+w)$

  $101010+1111=1010101111$
- the empty word is the neutral element

$(A^*, +)$ is called the free monoid on the set A.

- no inverse operation, no commutativity

Kolakoski word

The set of all right infinite words over {1,2}:

$$\{1,2\}^\omega$$

$$w\colon \mathbf{N}^+ \to \{1,2\}$$

$$w = w(1)w(2)w(3)\cdots \in \{1,2\}^\omega$$

The run-length encoding operator

$$\Delta_l\colon \{1,2\}^\omega \to \mathbf{N}^\omega$$

$$w = \begin{cases} 1^{k_1}2^{k_2}1^{k_3}2^{k_4}\cdots, & \text{if } w \in 1\cdot\{1,2\}^\omega \\ 2^{k_1}1^{k_2}2^{k_3}1^{k_4}\cdots, & \text{if } w \in 2\cdot\{1,2\}^\omega \end{cases}$$

$$\Delta_l(w) = k_1k_2k_3\cdots$$

Kolakoski word

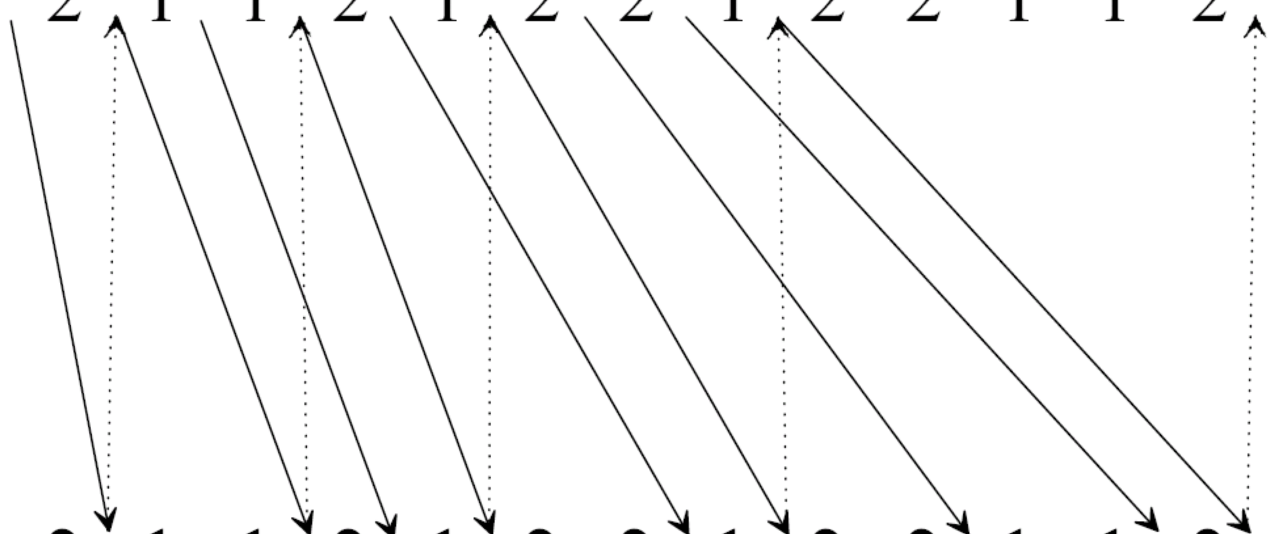
The run-length encoding operator – an example:

$$w = 11111212222222112221212222222221\ \ldots$$

$$\Delta_l(w) = 5,1,1,7,2,3,1,1,1,9,\ \ldots$$

Kolakoski word

$$w = 2\ 2\ 1\ 1\ 2\ 1\ 2\ 2\ 1\ 2\ 2\ 1\ 1\ 2 \ldots$$



$$\Delta_l(w) = 2\ 2\ 1\ 1\ 2\ 1\ 2\ 2\ 1\ 2\ 2\ 1\ 1\ 2 \ldots$$

Sturmian words

The word w is called a factor of a word u if there exist words x, y such that $u=x+w+y$.

1222 is a factor of 0001222111132131101010001

10101 is a factor of 10101010101010101

ABCDA is a factor of CBBBDACADBCAABCDA

10101 is a factor of 10101

Sturmian words

Sturmian words are infinite words which have exactly $m+1$ different factors of length m for every natural m.

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10101001010010101001010010101001010010100...

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$m=4$

1010, 0101, 0010, 1001, 0100.

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1010100101001010100101001010100101001010100 ...

$m=4$

1010, 0101, 0010, 1001, 0100.

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Balanced words (binary)

n – the length of the word

m – any positive natural number less than n

Each m-letter long factor of this word can contain either k or k+1 1's

An example:

n = 41

m = 16

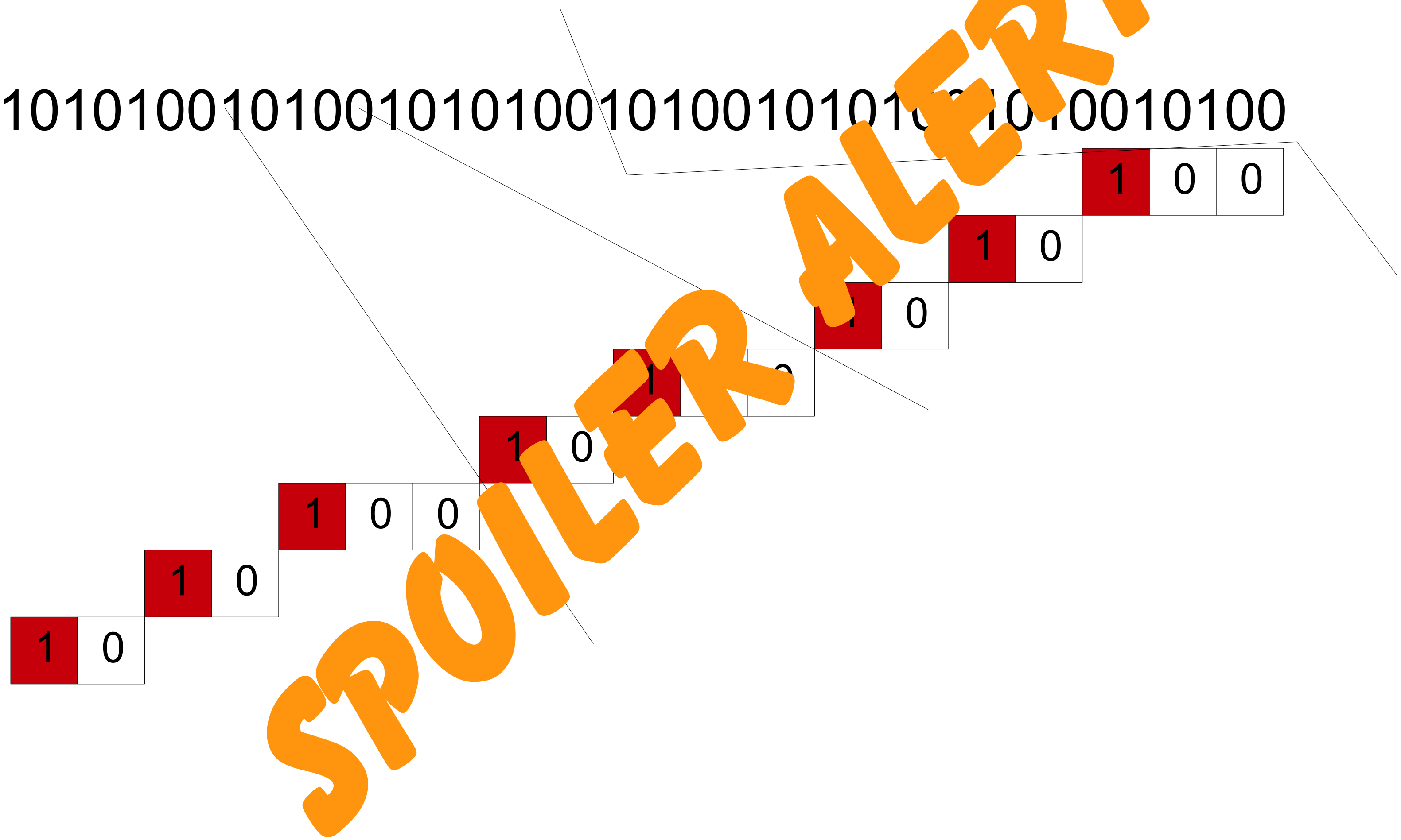
10101001010010101001010010101001010010100

7 6

k = 6

Balanced words give straight lines

10101001010010101001010010101[illegible]10100010100



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Upper and lower mechanical, characteristic words

$$s'(a), s(a) \colon \mathbf{N} \longrightarrow \{0,1\}$$

$$\forall\, n \in \mathbf{N} \qquad s'_n(a) = \lceil a(n+1) \rceil - \lceil an \rceil,$$

$$s_n(a) = \lfloor a(n+1) \rfloor - \lfloor an \rfloor$$

$$c(a) \colon \mathbf{N}^+ \longrightarrow \{0,1\}$$

$$\forall\, n \in \mathbf{N}^+ \qquad c_n(a) = \lfloor a(n+1) \rfloor - \lfloor an \rfloor$$

Sturmian words : different characterizations

Theorem Let s be an infinite word. The following are equivalent:

- s is Sturmian;
- s is balanced and aperiodic;
- s is irrational (lower or upper) mechanical.

Palindromes

kajak, Hannah, wow, SOS, 123454321, ?*#@#*?

Square-free words

adbcbabcbdabdcbabcdbabdcb …

Overlap-free words

abbabaabbaababbabaab … (the Thue-Morse word)

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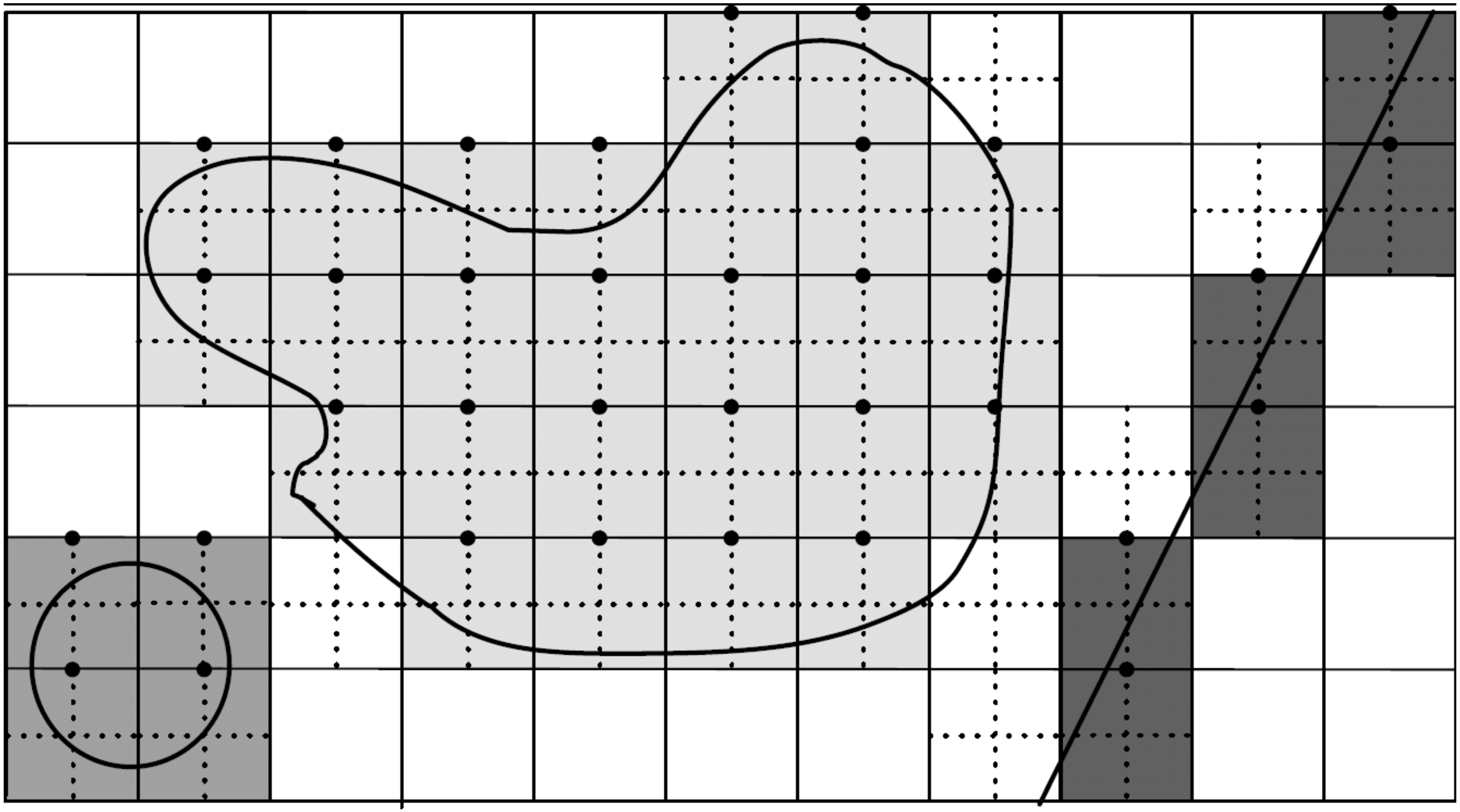
digital lines

Digital geometry – R'-digitization



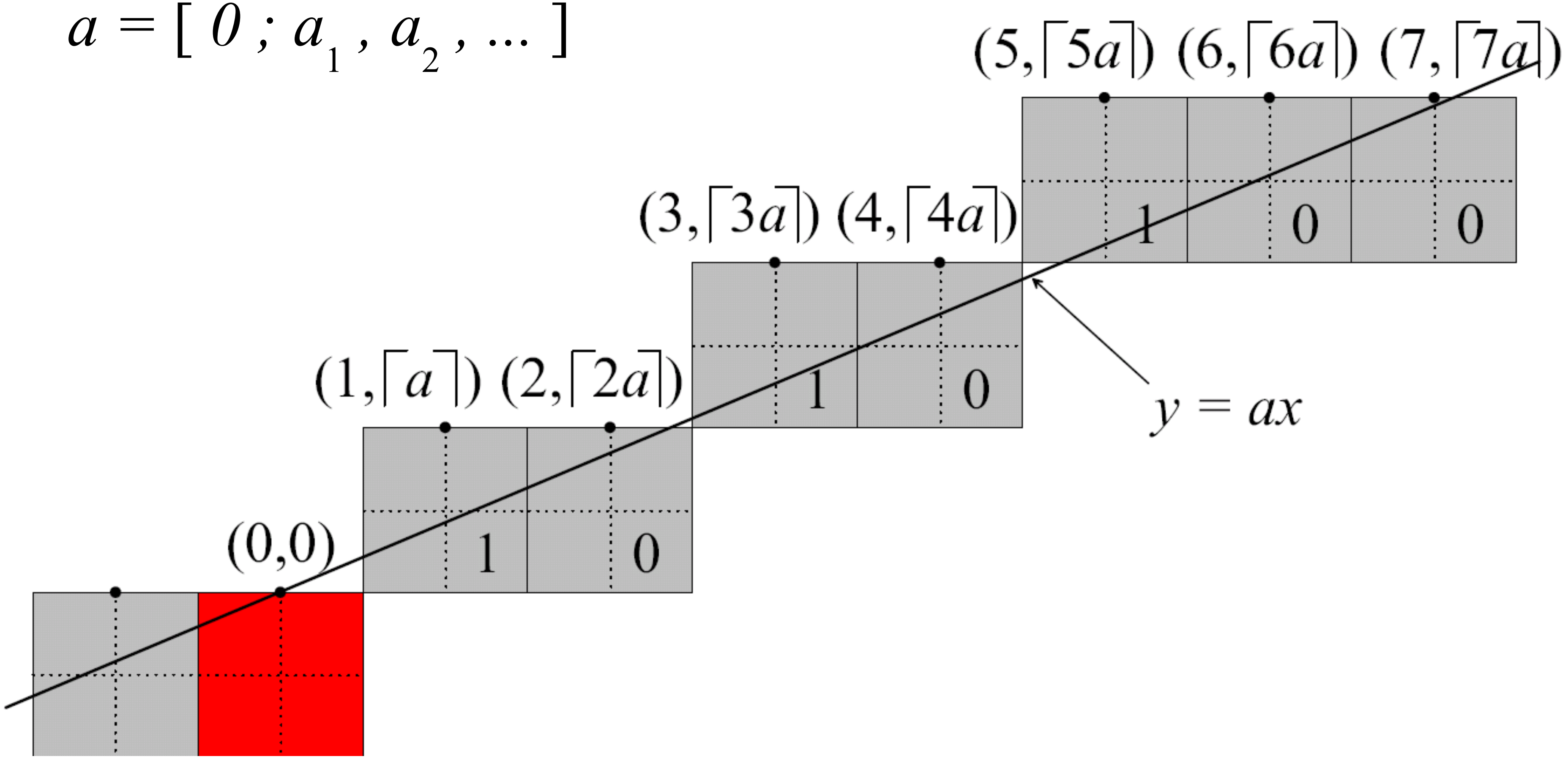
Digital geometry – straight lines

The arithmetical expression of the R'-digitization of the line $y = ax$ for irrational positive a less than 1 :

$$D_{R'}(y = ax) = \{(k, \lceil ak \rceil);\ k \in \mathbf{Z}\}$$

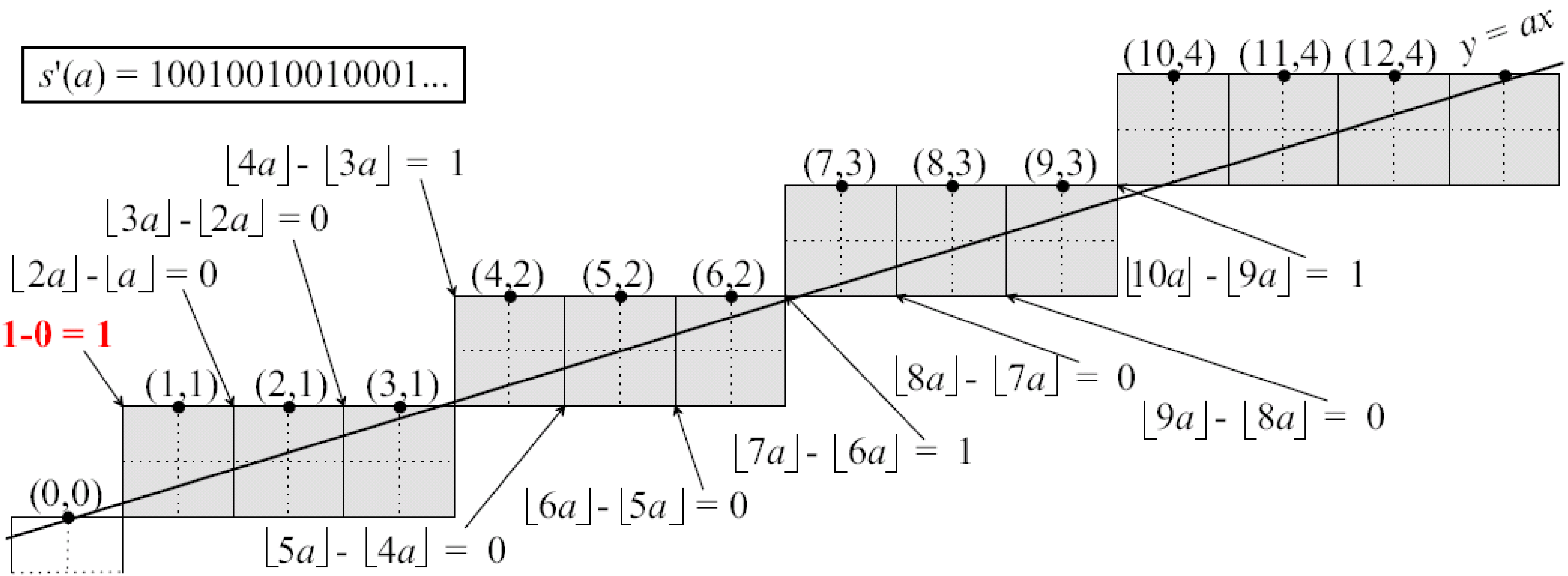
The R'-digital line $y = ax$ with irrational slope

$a = [\,0\,;\,a_1\,,\,a_2\,,\,\ldots\,]$

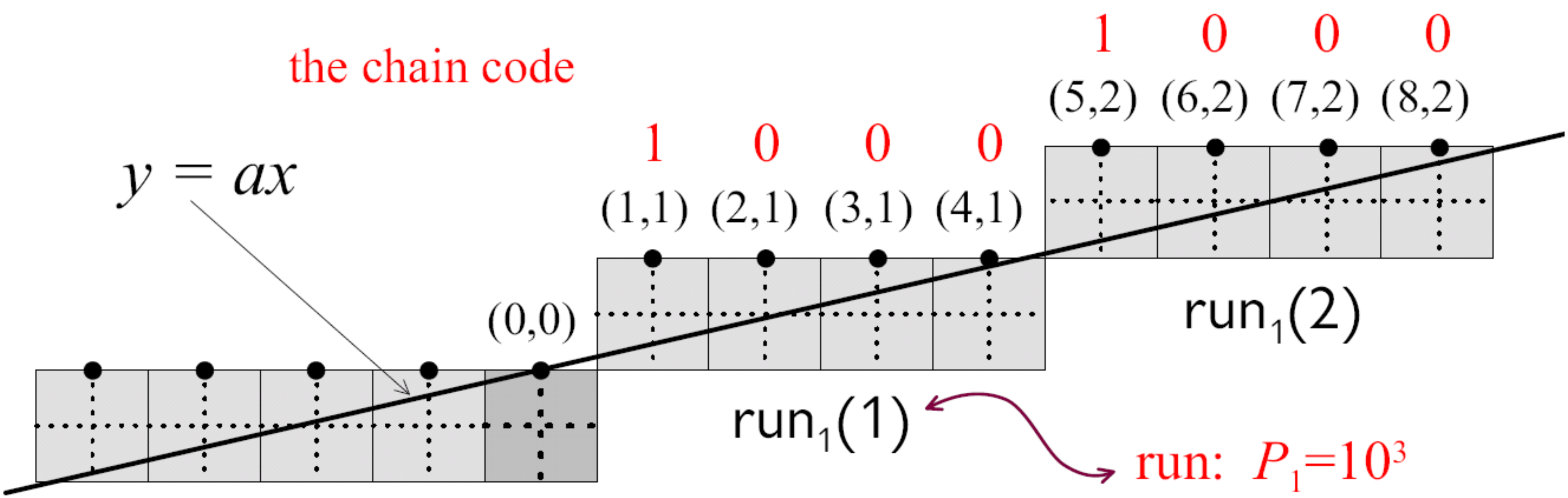


Digital geometry – straight lines and mechanical words

The R'-digital line $y = ax$ with slope $a = [\,0\,;\,a_1\,,\,a_2\,,\,\ldots\,]$ and the corresponding upper mechanical word $s'(a)$:



Digital geometry – the concept of run



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Digital geometry – the concept of run

P_n - the n^{th} prefix according to the run hierarchy

S_n - short run of level n

L_n - long run of level n

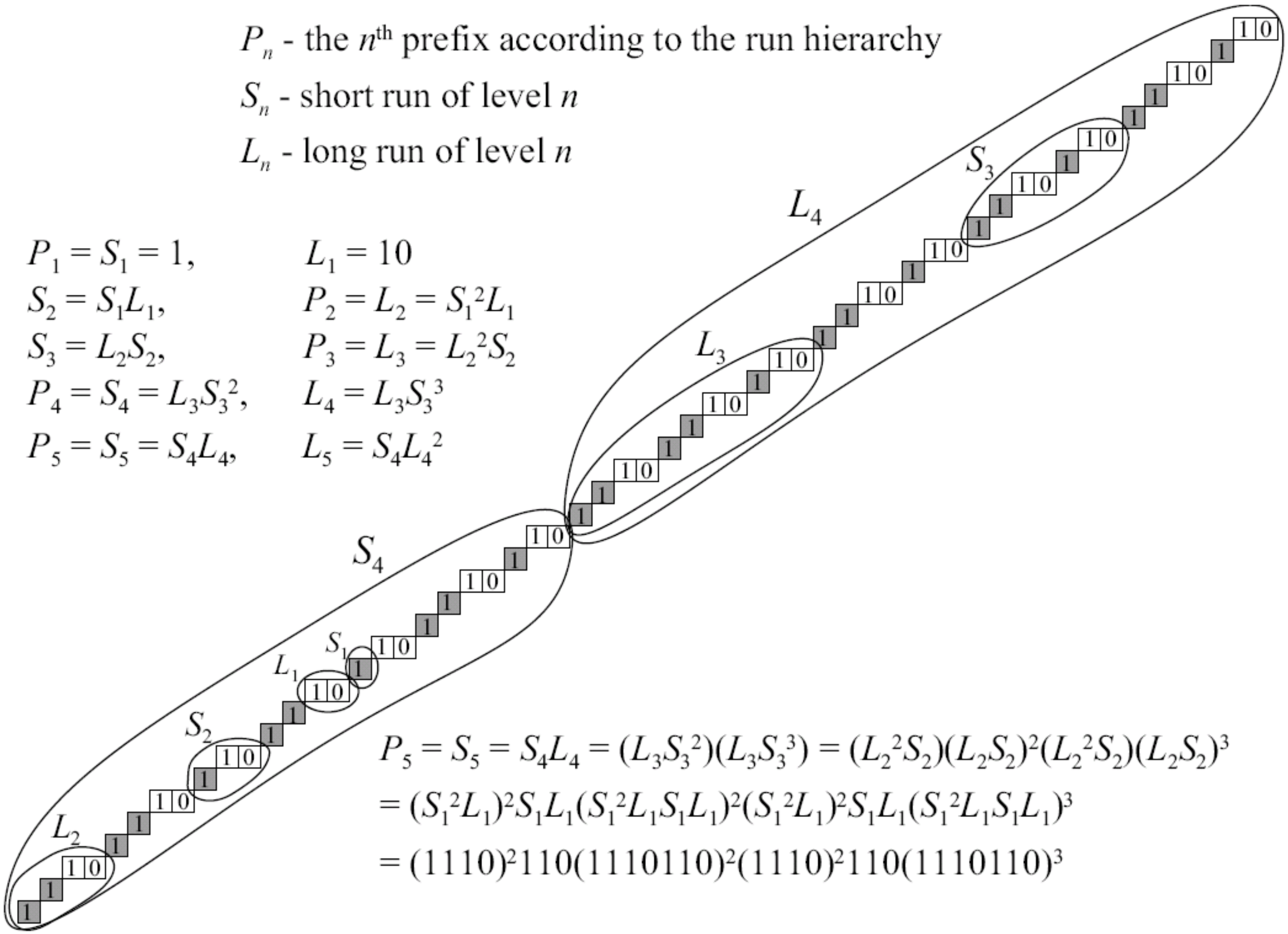
$P_1 = S_1 = 1,$ $\qquad L_1 = 10$

$S_2 = S_1 L_1,$ $\qquad P_2 = L_2 = S_1^2 L_1$

$S_3 = L_2 S_2,$ $\qquad P_3 = L_3 = L_2^2 S_2$

$P_4 = S_4 = L_3 S_3^2,$ $\qquad L_4 = L_3 S_3^3$

$P_5 = S_5 = S_4 L_4,$ $\qquad L_5 = S_4 L_4^2$



$$P_5 = S_5 = S_4 L_4 = (L_3 S_3^2)(L_3 S_3^3) = (L_2^2 S_2)(L_2 S_2)^2 (L_2^2 S_2)(L_2 S_2)^3$$

$$= (S_1^2 L_1)^2 S_1 L_1 (S_1^2 L_1 S_1 L_1)^2 (S_1^2 L_1)^2 S_1 L_1 (S_1^2 L_1 S_1 L_1)^3$$

$$= (1110)^2 110 (1110110)^2 (1110)^2 110 (1110110)^3$$

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Hierarchy of runs – runs on level $k+1$

$$L_k S_k^m \qquad S_k^m L_k \qquad L_k^m S_k \qquad S_k L_k^m$$

Three questions. About:

the run length on level $k+1$

the main run on level k

the first run on level k

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my contribution

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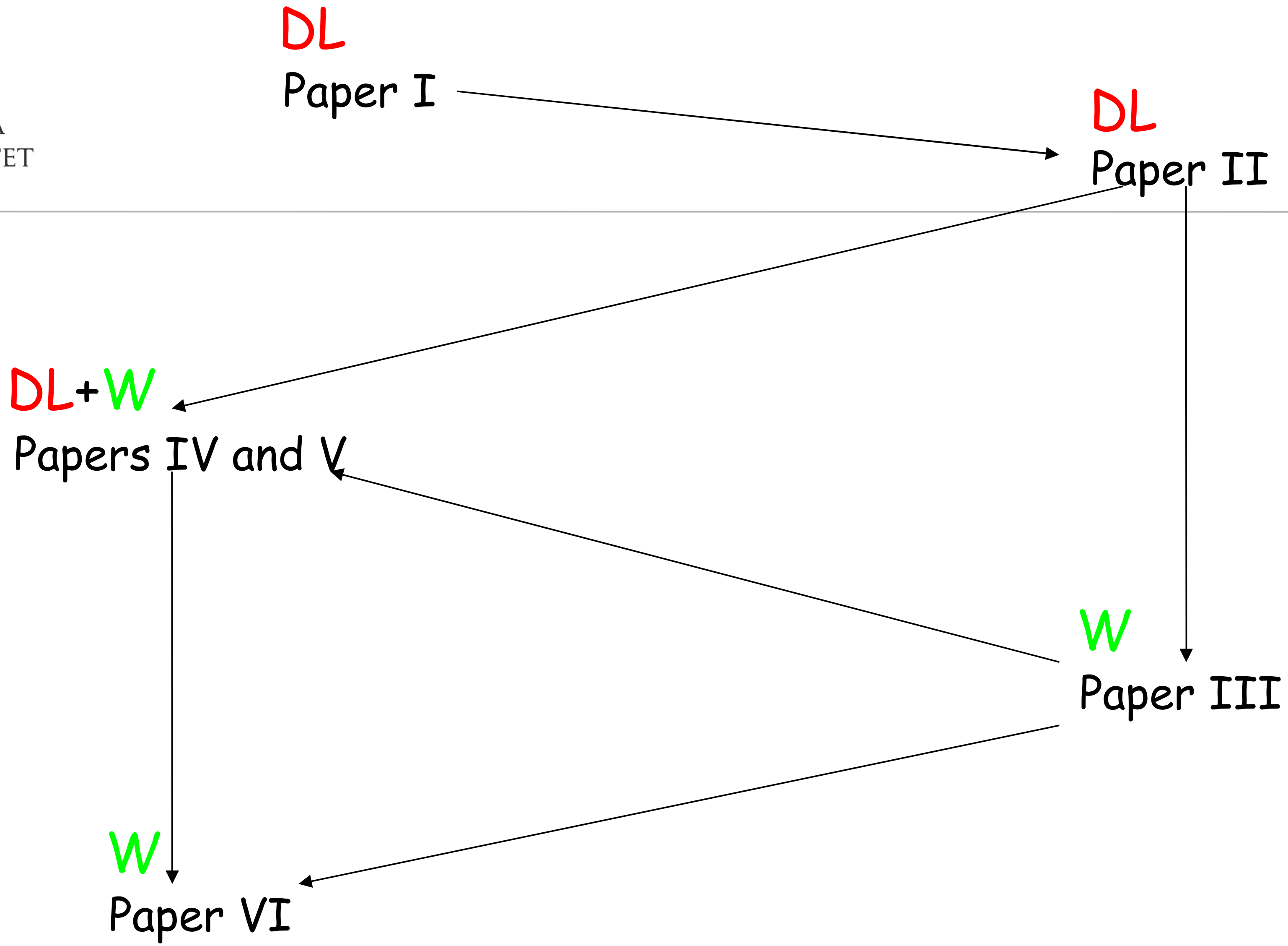

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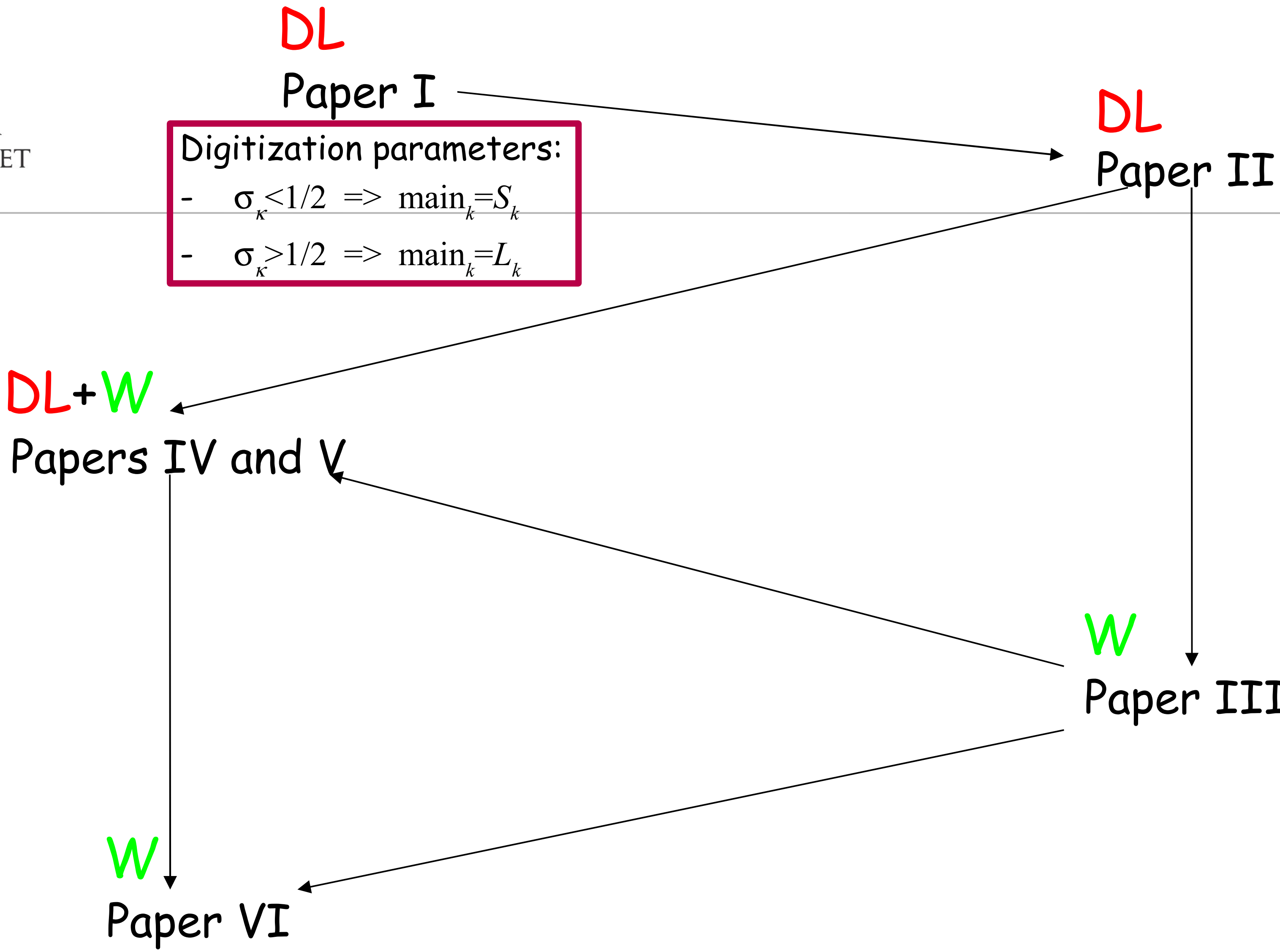

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DL

Paper I

Digitization parameters:

- $\sigma_\kappa < 1/2 \Rightarrow main_k = S_k$
- $\sigma_\kappa > 1/2 \Rightarrow main_k = L_k$

DL

Paper II

CFs and digital lines:

- index jump function:
  - $a_{i(k+1)} > 1 \Rightarrow main_k = S_k$
  - $a_{i(k+1)} = 1 \Rightarrow main_k = L_k$
- σ and the Gauss map

DL+W

Papers IV and V

W

Paper III

W

Paper VI

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CF-based description of upper mechanical words

W

Paper VI

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Papers IV and V

Two equivalence relations:

- by run length
- by the index jump function

W

Paper III

CF-based description of upper mechanical words

W

Paper VI

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DL+W

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- by run length
- by the index jump function

W

Paper III

CF-based description of upper mechanical words

W

Paper VI

- the constructional word
- balanced construction
- fixed point theorem

What I have done

A new CF-description (essential 1's, run hierarchy)

Two equivalence relations on the set of slopes

A new fixed point theorem for words

The index jump function

$$a = [0; a_1, a_2, a_3, a_4, a_5, a_6, a_7, \ldots]$$
$$i_a : \mathbf{N}^+ \to \mathbf{N}^+$$

$$i_a(1) = 1,\ i_a(2) = 2, \text{ for } n \geq 2:$$
$$i_a(n+1) = i_a(n) + 1 + \delta_1(a_{i_a(n)})$$

How $i_a(k+1)$ and $a_{i_a(k+1)}$ describe the form of run_{k+1}

$$a_{i_a(k+1)}$$

$$L_k S_k^m \qquad S_k^m L_k \qquad L_k^m S_k \qquad S_k L_k^m$$

$$i_a(k+1)$$

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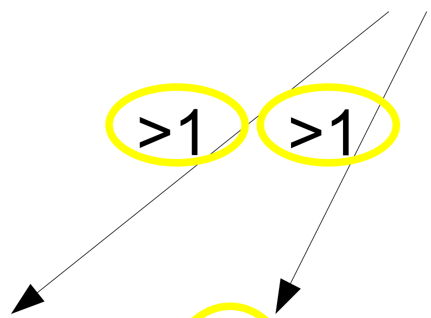

$$L_k S_k^m \qquad S_k^m L_k \qquad L_k^m S_k \qquad S_k L_k^m$$

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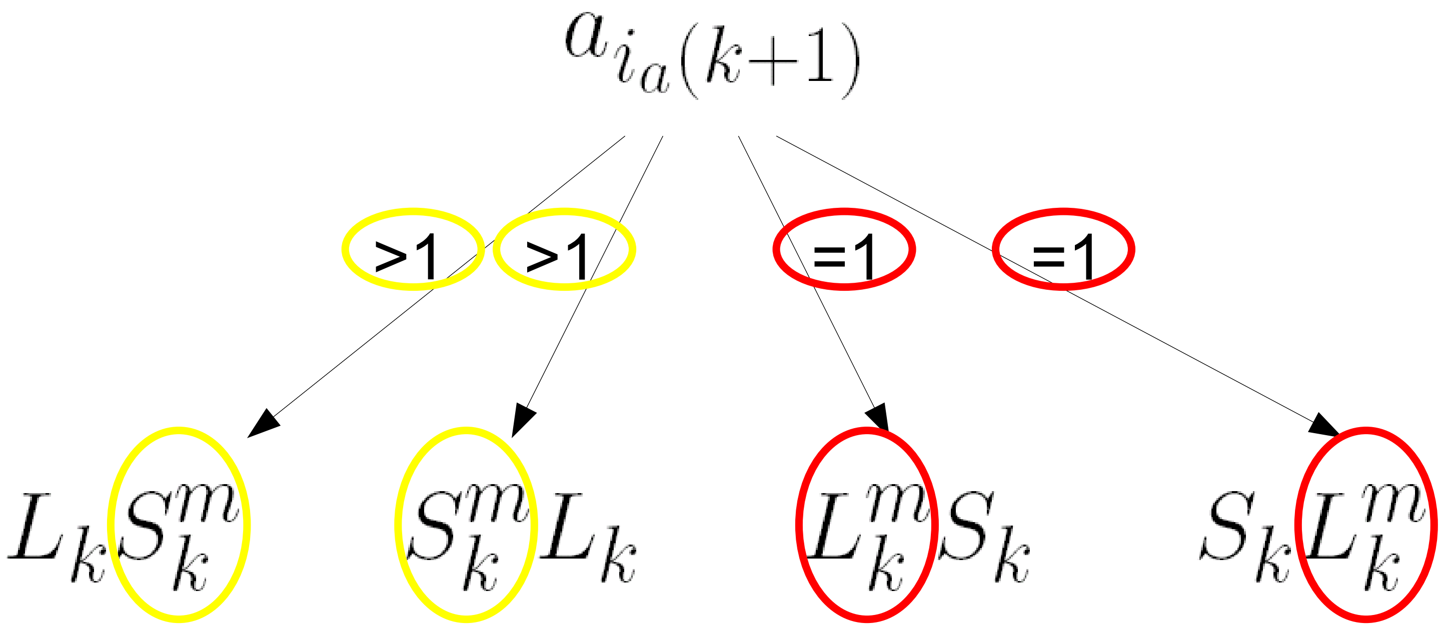

$$i_a(k+1)$$

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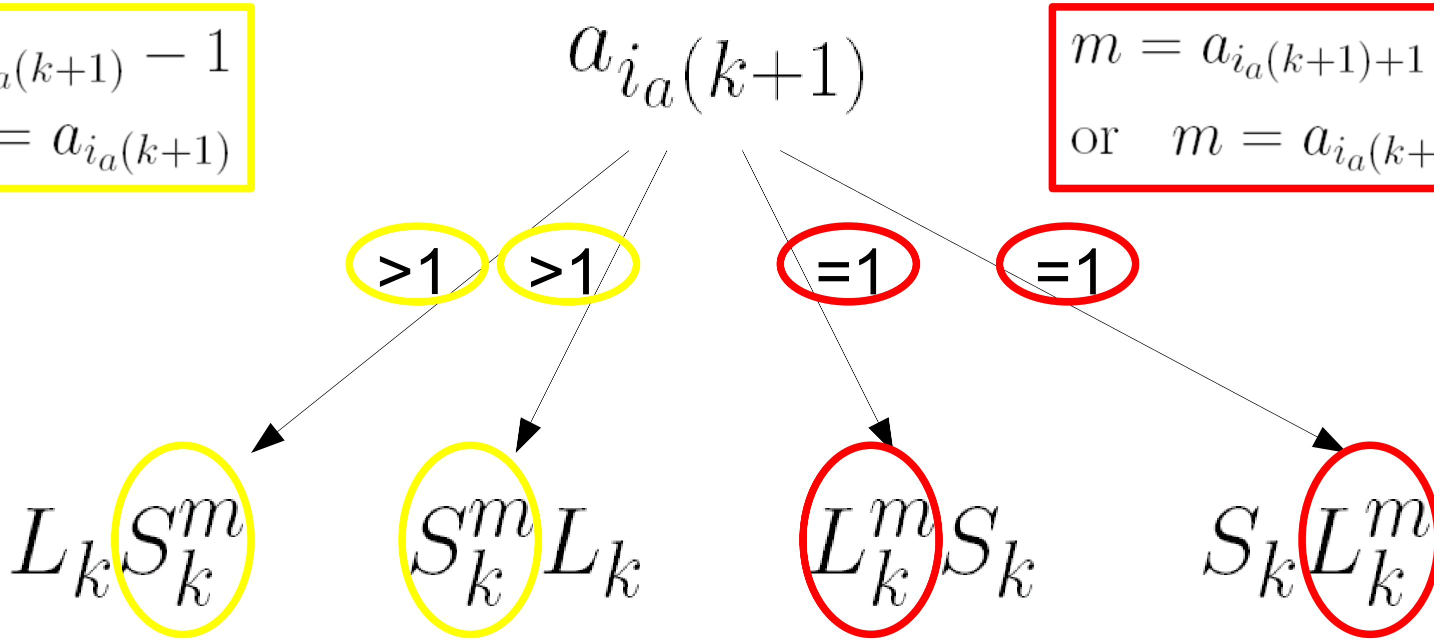
How $i_a(k+1)$ and $a_{i_a(k+1)}$ describe the form of run_{k+1}

$m = a_{i_a(k+1)} - 1$

or $m = a_{i_a(k+1)}$

$m = a_{i_a(k+1)+1}$

or $m = a_{i_a(k+1)+1} + 1$



$i_a(k+1)$

How $i_a(k+1)$ and $a_{i_a(k+1)}$ describe the form of run_{k+1}

$$a_{i_a(k+1)}$$

$$L_k S_k^m \qquad S_k^m L_k \qquad L_k^m S_k \qquad S_k L_k^m$$

odd

odd

$$i_a(k+1)$$

How $i_a(k+1)$ and $a_{i_a(k+1)}$ describe the form of run_{k+1}

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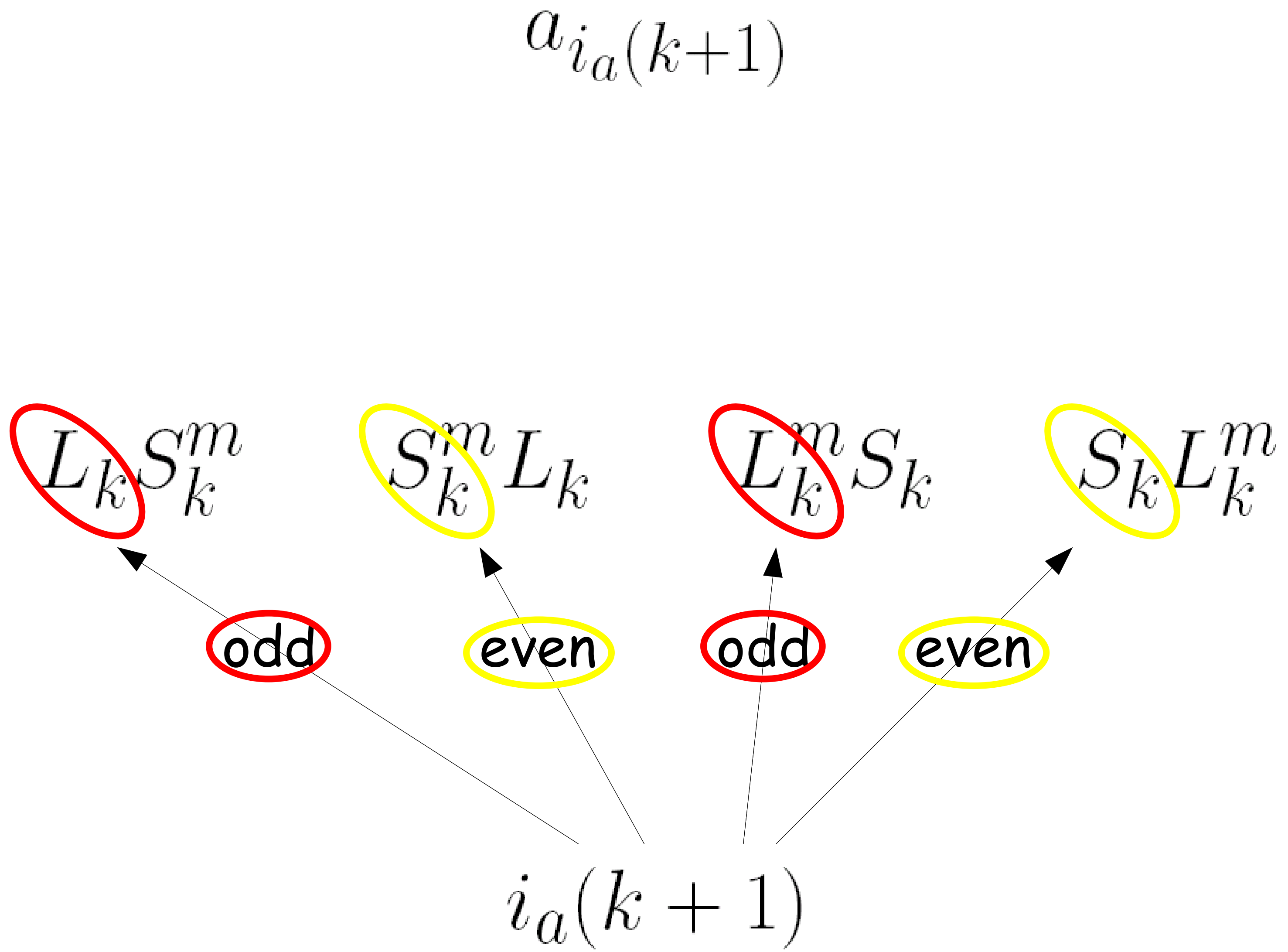

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The index jump function: how it describes the runs

$$a = [0;\ \overset{b_1}{1},\ \overset{b_2}{a_2},\ \overbrace{\underline{1},\ 1}^{b_3},\ \overset{b_4}{a_5},\ \overbrace{\underline{1},\ 1}^{b_5},\ \overset{b_6}{a_8},\ \overset{b_7}{a_9},\ \overbrace{\underline{1},\ a_{11}}^{b_8},\ \overset{b_9}{a_{12}},\ \overbrace{\underline{1},\ 1}^{b_{10}},\ \overbrace{\underline{1},\ a_{16}}^{b_{11}},\ \overset{b_{12}}{a_{17}},\ \ldots]$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \ldots$$

$$(i_a(k))_{k \in \mathbf{N}^+} = (\ 1,\ 2,\ 3,\ 5,\ 6,\ 8,\ 9,\ 10,\ 12,\ 13,\ 15,\ 17,\ \ldots)$$

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Digitization levels

Level :	1	2	3	4	5	6	7	8	9	10	11	12	
$a = [0;$	$\overset{b_1}{1},$	$\overset{b_2}{a_2},$	$\overbrace{\underline{1},\ 1}^{b_3},$	$\overset{b_4}{a_5},$	$\overbrace{\underline{1},\ 1}^{b_5},$	$\overset{b_6}{a_8},$	$\overset{b_7}{a_9},$	$\overbrace{\underline{1},\ a_{11}}^{b_8},$	$\overset{b_9}{a_{12}},$	$\overbrace{\underline{1},\ 1}^{b_{10}},$	$\overbrace{\underline{1},\ a_{16}}^{b_{11}},$	$\overset{b_{12}}{a_{17}},$	$\ldots]$
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\ldots$
$(i_a(k))_{k\in\mathbf{N}^+} = ($	$1,$	$2,$	$3,$	$5,$	$6,$	$8,$	$9,$	$10,$	$12,$	$13,$	$15,$	$17,$	$\ldots)$

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Short run length: the CF elements

Level :	1	2	3	4	5	6	7	8	9	10	11	12	
$a =$	$[0;\ \overset{b_1}{1},$	$\overset{b_2}{a_2},$	$\overbrace{\underline{1},\ 1}^{b_3},$	$\overset{b_4}{a_5},$	$\overbrace{\underline{1},\ 1}^{b_5},$	$\overset{b_6}{a_8},$	$\overset{b_7}{a_9},$	$\overbrace{\underline{1},\ a_{11}}^{b_8},$	$\overset{b_9}{a_{12}},$	$\overbrace{\underline{1},\ 1}^{b_{10}},$	$\overbrace{\underline{1},\ a_{16}}^{b_{11}},$	$\overset{b_{12}}{a_{17}},$	$\ldots]$
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\ldots$
$(i_a(k))_{k\in\mathbf{N}^+} = ($	$1,$	$2,$	$3,$	$5,$	$6,$	$8,$	$9,$	$10,$	$12,$	$13,$	$15,$	$17,$	$\ldots)$

$$\|S_k\| = b_k$$

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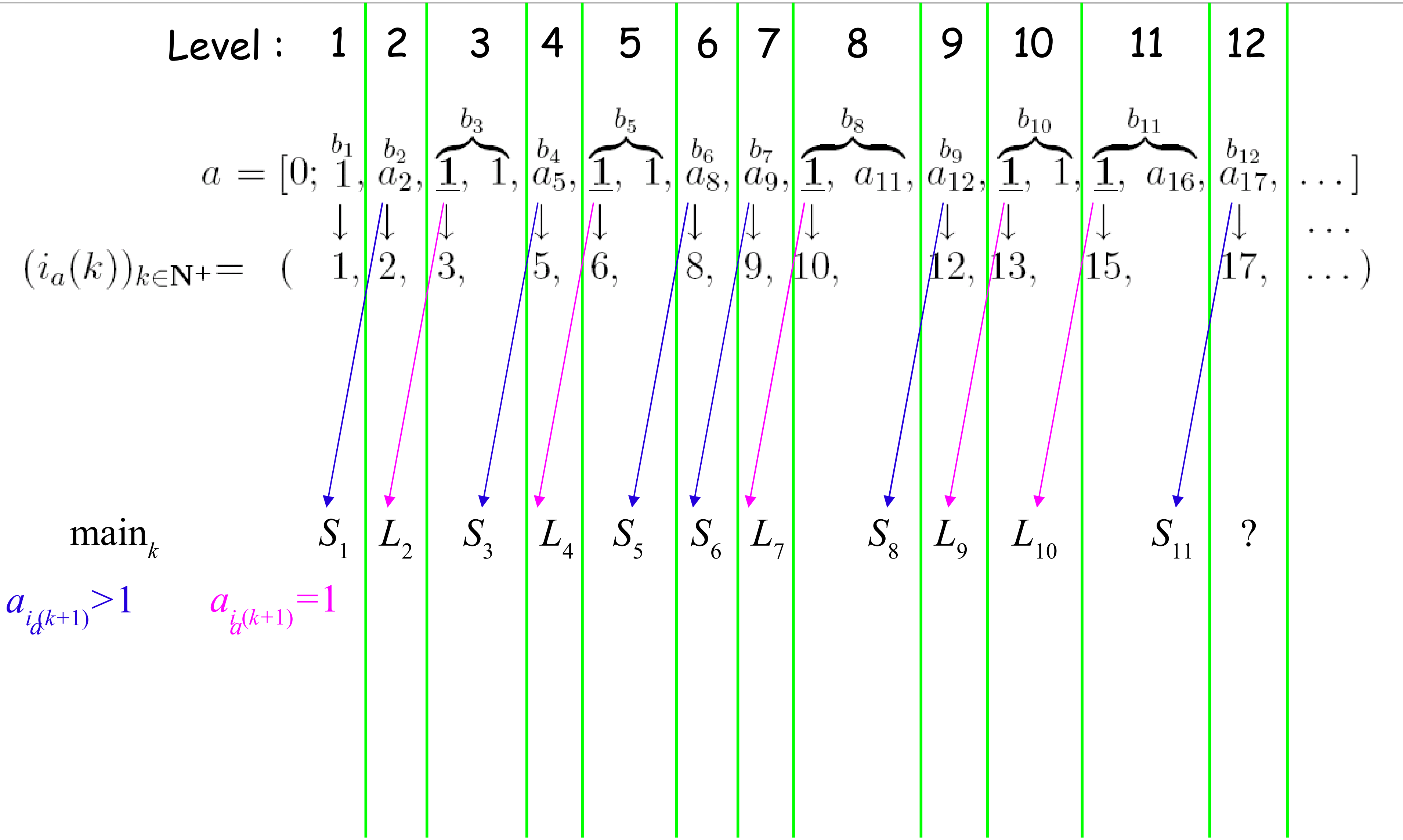
The most frequent run: essential 1's



The first run: parity of the function

$$a = [0;\ \overbrace{1}^{b_1},\ \overbrace{a_2}^{b_2},\ \overbrace{\underline{1},\ 1}^{b_3},\ \overbrace{a_5}^{b_4},\ \overbrace{\underline{1},\ 1}^{b_5},\ \overbrace{a_8}^{b_6},\ \overbrace{a_9}^{b_7},\ \overbrace{\underline{1},\ a_{11}}^{b_8},\ \overbrace{a_{12}}^{b_9},\ \overbrace{\underline{1},\ 1}^{b_{10}},\ \overbrace{\underline{1},\ a_{16}}^{b_{11}},\ \overbrace{a_{17}}^{b_{12}},\ \ldots]$$

$$(i_a(k))_{k\in\mathbf{N}^+} = (\ 1,\ 2,\ 3,\ 5,\ 6,\ 8,\ 9,\ 10,\ 12,\ 13,\ 15,\ 17,\ \ldots)$$

Level :	1	2	3	4	5	6	7	8	9	10	11	12
b_k	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}
a	1	a_2	$\underline{1},\ 1$	a_5	$\underline{1},\ 1$	a_8	a_9	$\underline{1},\ a_{11}$	a_{12}	$\underline{1},\ 1$	$\underline{1},\ a_{16}$	a_{17}
$i_a(k)$	1	2	3	5	6	8	9	10	12	13	15	17
first$_k$	S_1	L_2	L_3	S_4	S_5	L_6	S_7	S_8	L_9	L_{10}	L_{11}	?

$i_a(k+1)$ even

$i_a(k+1)$ odd

Two equivalence relations on the set of slopes

1. based on run length on all levels for $s'(a)$:

$$a \in [(b_1, b_2, b_3, \ldots)]_{\sim_{\text{len}}} \quad \Leftrightarrow$$

$$\forall \; k \in \mathbf{N}^+ \;\; \|S_k\| = b_k$$

2. based on run construction on all levels for $s'(a)$:

$$a \sim_{\text{con}} a' \quad \Leftrightarrow \quad i_a \equiv i_{a'}$$

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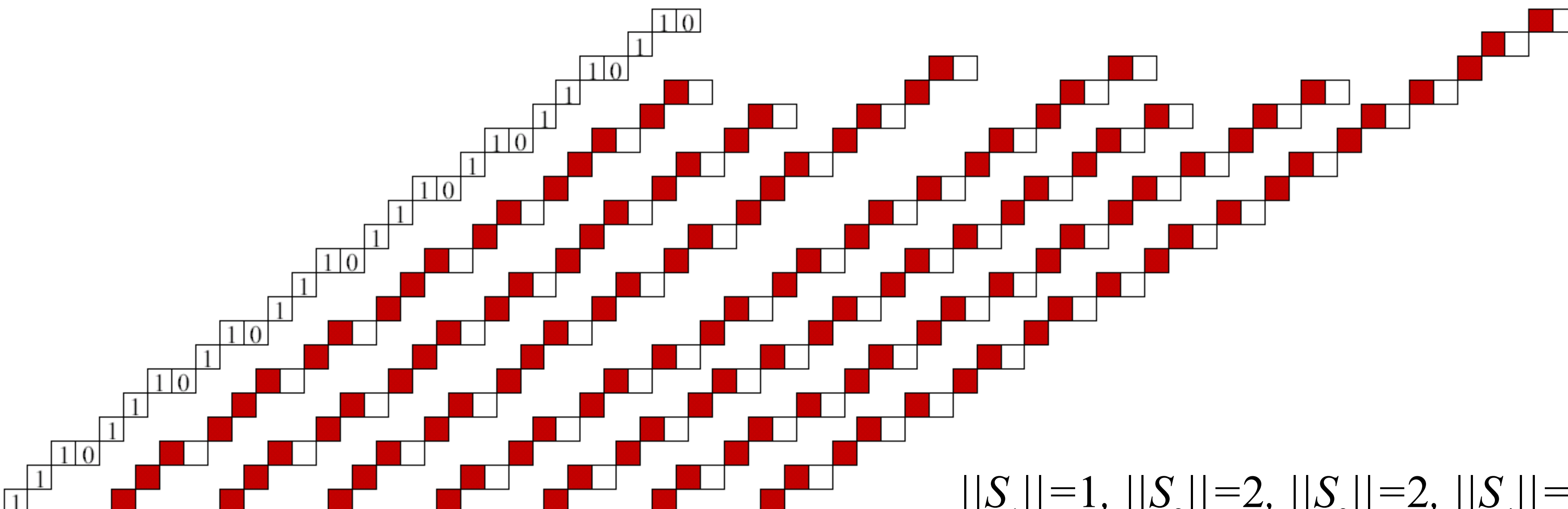
Quantitative equivalence relation (run length)

Defined by run lengths (their cardinality)



$$||S_1||=1,\ ||S_2||=2,\ ||S_3||=2,\ ||S_4||=3.$$

All lines from the same class have the same run lengths on all digitization levels.

Quantitative equivalence relation (run length)

The least element of the class :

$$\min\{a \in\,]0,1[\, \setminus \mathbf{Q};\;\; a \in [(b_n)_{n\in\mathbf{N}^+}]_{\sim_{\text{len}}}\} = [0; b_1, \overline{1, b_n - 1}]_{n=2}^{\infty}.$$

The largest element of the class :

$$\max\{a \in\,]0,1[\, \setminus \mathbf{Q};\;\; a \in [(b_n)_{n\in\mathbf{N}^+}]_{\sim_{\text{len}}}\} = [0; b_1, b_2, \overline{1, b_n - 1}]_{n=3}^{\infty}.$$

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Qualitative equivalence relation (run construction)

Defined by the places of essential 1's

Equivalently defined by the index jump function

All lines from the same class have the same construction in terms of long and short runs on all digitization levels.

The least element in each class is 0.

Qualitative equivalence relation (run construction)

Largest elements of the classes:

$$\forall \; n \in \mathbf{N}^+ \;\; [(\forall \;\; k \in [1, n-1]_{\mathbf{Z}} \;\; s_k = 2k)$$

$$\wedge \;\; (s_n > 2n \;\; \vee \;\; |J| = n-1)]$$

$$\Rightarrow \;\; \sup\{a; \;\; a \in [(s_j)_{j \in J}]_{\sim_{\text{con}}}\} = \frac{F_{2n-1}}{F_{2n}},$$

where $(F_n)_{n \in \mathbf{N}^+}$ is the **Fibonacci** sequence and

$(s_j)_{j \in J}$ is the sequence of the places of essential 1's.

Fixed point theorem: the constructional word

The constructional word $\gamma(a) \in \{0,1\}^\omega$

Let $a = [0; a_1, a_2, \ldots]$. For $n \in \mathbf{N}^+$:

$$\gamma_n(a) = i_a(n+2) - i_a(n+1) - 1$$

$$\gamma_n(a) = \delta_1(a_{i_a(n+1)})$$

$$\gamma_n(a) = \begin{cases} 0, & S_n \text{ is the most frequent run on level } n \text{ for } s'(a) \\ 1, & L_n \text{ is the most frequent run on level } n \text{ for } s'(a). \end{cases}$$

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Fixed point theorem: the run-construction encoding operator

Definition The *run-construction encoding operator* $\Delta_c : \mathcal{UM}_0 \longrightarrow \{0,1\}^\omega$ is defined as $\Delta_c = (1\gamma) \circ (s')^{-1}$.

$$
\begin{array}{ccc}
]0,1[\setminus \mathbf{Q} & \xrightarrow{\quad s' \quad} & \mathcal{UM}_0 \\
& \searrow^{1\gamma} & \downarrow \Delta_c \\
& & \{0,1\}^\omega \supset \mathcal{UM}_0
\end{array}
$$

where $\mathcal{UM}_0$ denotes the set of all upper mechanical words with irrational slope $0 < a < 1$ and with intercept 0.

Balanced construction

Let $a \in]0,1[\setminus \mathbf{Q}$. The word $s'(a) = 1c(a)$ has

balanced construction if

$$\exists\, \alpha \in \mathbf{R} \quad \gamma(a) = c(\alpha)$$

Sturmian-balanced construction if

$$\exists\, \alpha \in]0,1[\setminus \mathbf{Q} \quad \gamma(a) = c(\alpha)$$

self-balanced construction

$$1\gamma(a) = \Delta_c(1c(a)) = 1c(a)$$

A fixed-point theorem: exactly 1 fixed point in each equivalence class

Let $(b_n)_{n\in\mathbf{N}^+}$ be such that $b_1 \in \mathbf{N}^+$ and $b_n \in \mathbf{N}^+ \setminus \{1\}$ for all $n \geq 2$. Then

$$\exists^1_{a\in]0,1[\setminus\mathbf{Q}}$$

$$a \in [(b_n)_{n\in\mathbf{N}^+}]_{\sim_{\text{len}}} \ \wedge \ s'(a) = \Delta_c(s'(a)).$$

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What to do with it?

Digital geometry – operations defined on classes?

Combinatorics on words – new classes of words

Iterations of the run-construction encoding operator

What can one say about the fixed points?

Data compression?

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Dziękuję za uwagę

Tack för er uppmärksamhet

Je vous remercie de votre attention

Thank you for your attention

Dank u voor uw aandacht