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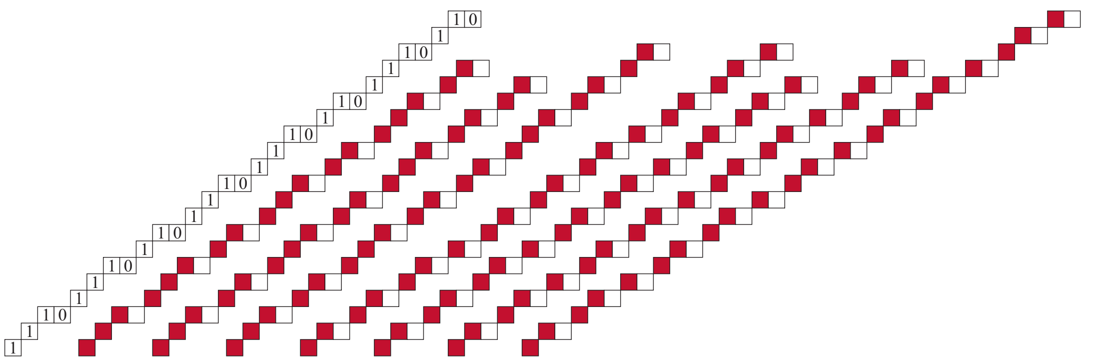
Digital lines, Sturmian words, and continued fractions

Hanna Uscka-Wehlou



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Dissertation presented at Uppsala University to be publicly examined in Polhemsalen, Lägerhyddsvägen 1, Uppsala, Ångström Laboratory, Friday, September 25, 2009 at 10:15 for the degree of Doctor of Philosophy. The examination will be conducted in English.

Abstract
Uscka-Wehlou, H. 2009. Digital lines, Sturmian words, and continued fractions. Matematiska institutionen. *Uppsala Dissertations in Mathematics* 65. 59 pp. Uppsala.
ISBN 978-91-506-2090-0.

In this thesis we present and solve selected problems arising from digital geometry and combinatorics on words. We consider digital straight lines and, equivalently, upper mechanical words with positive irrational slopes $a < 1$ and intercept 0. We formulate a continued fraction (CF) based description of their run-hierarchical structure.

Paper I gives a theoretical basis for the CF-description of digital lines. We define for each irrational positive slope less than 1 a sequence of digitization parameters which fully specifies the run-hierarchical construction.

In Paper II we use the digitization parameters in order to get a description of runs using only integers. We show that the CF-elements of the slopes contain the complete information about the run-hierarchical structure of the line. The index jump function introduced by the author indicates for each positive integer k the index of the CF-element which determines the shape of the digitization runs on level k.

In Paper III we present the results for upper mechanical words and compare our CF-based formula with two well-known methods, one of which was formulated by Johann III Bernoulli and proven by Markov, while the second one is known as the standard sequences method. Due to the special treatment of some CF-elements equal to 1 (*essential* 1's in Paper IV), our method is currently the only one which reflects the run-hierarchical structure of upper mechanical words by analogy to digital lines.

In Paper IV we define two equivalence relations on the set of all digital lines with positive irrational slopes $a < 1$. One of them groups into classes all the lines with the same run length on all digitization levels, the second one groups the lines according to the run construction in terms of long and short runs on all levels. We analyse the equivalence classes with respect to minimal and maximal elements. In Paper V we take another look at the equivalence relation defined by run construction, this time independently of the context, which makes the results more general.

In Paper VI we define a *run-construction encoding operator*, by analogy with the well-known *run-length encoding operator*. We formulate and present a proof of a fixed-point theorem for Sturmian words. We show that in each equivalence class under the relation based on run length on all digitization levels (as defined in Paper IV), there exists exactly one fixed point of the run-construction encoding operator.

Keywords: digital geometry, digital line, hierarchy of runs, combinatorics on words, Sturmian word, upper mechanical word, characteristic word, irrational slope, continued fraction, Gauss map, fixed point

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ISSN 1401-2049
ISBN 978-91-506-2090-0
urn:nbn:se:uu:diva-107274 (http://urn.kb.se/resolve?urn=urn:nbn:se:uu:diva-107274)

To "the kids": Milena, Julian, Andrzej, Danielle and Charline,
and to my sisters: Joanna, the first of us to do her Ph.D., and
Maria, who is still working on hers. Good luck!

Cover image: Eight digital straight line segments illustrating the equivalence relation based on the run length on all digitization levels (as defined in Paper IV), restricted to the first four levels. The eight possible forms of S_4 with the length specification $(1, 2, 2, 3)$ are presented. The 0's and 1's on the leftmost digital straight line segment give an understanding of the relationship between digital lines and the corresponding upper mechanical words (chain codes).

List of Papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I Uscka-Wehlou, Hanna, 2007. Digital lines with irrational slopes. *Theoretical Computer Science* **377**, pp. 157–169.

II Uscka-Wehlou, Hanna, 2009. Run-hierarchical structure of digital lines with irrational slopes in terms of continued fractions and the Gauss map. *Pattern Recognition* **42**, pp. 2247–2254.

III Uscka-Wehlou, Hanna, 2008. A Run-hierarchical Description of Upper Mechanical Words with Irrational Slopes Using Continued Fractions; 15 pp. In *Proceedings of 12th Mons Theoretical Computer Science Days (Mons, Belgium), 27–30 August 2008.* http://www.jmit.ulg.ac.be/jm2008/index-en.html.

IV Uscka-Wehlou, Hanna, 2009. Two equivalence relations on digital lines with irrational slopes. A continued fraction approach to upper mechanical words. *Theoretical Computer Science* **410** (38–40), pp. 3655–3669.

V Uscka-Wehlou, Hanna, 2009. Continued fractions, Fibonacci numbers, and some classes of irrational numbers; 14 pp. Manuscript submitted to a journal.

VI Uscka-Wehlou, Hanna, 2009. Sturmian words with balanced construction; 12 pp. In *Proceedings of Words 2009, the 7th International Conference on Words (Salerno, Italy), 14–18 September 2009.* http://words2009.dia.unisa.it/accepted.html.

Contents

1. Introduction

This thesis is based on a number of different domains of mathematics. The two main domains are digital geometry and combinatorics on words. The problems we treat in this thesis appear also in symbolic dynamics, crystallography, and astronomy. This interdisciplinary character can be both very frustrating and very rewarding. The usual reason for frustration is that you can be surprised that the problem you are working on has already been solved in another domain than your own. You must check a large number of disciplines of natural science to ascertain that your work is original. The rewarding aspect is the fact that once you have managed to formulate and solve a really interesting problem, it can be applied in many different ways, which gives an enormous sense of satisfaction. In this introduction we would like to sketch the problem, present the different terms associated with it, and describe the circumstances in which it appears.

Before we present the tools we have used in this thesis and the domains we were working in, we will introduce some terms related to the problem of interest, i.e., to descriptions of the sequence $(\lfloor na \rfloor)_{n\in\mathbf{N}}$ for a positive irrational a less than 1. These are:

1. the *β-sequence* defined by a (Bernoulli, Markov, Venkov)
$$\beta(n) = \lfloor (n+1)a \rfloor - \lfloor na \rfloor;$$
see Nillsen et al. (1999) [66],
2. the *Beatty sequence* associated with a: $\mathcal{B}_a = (\lfloor an \rfloor)_{n\in\mathbf{N}^+}$; see Beatty (1926) [6], de Bruijn (1989) [25], Komatsu (1995) [57],
3. the *characteristic word*, the *upper (lower) mechanical word* with slope a; see Definition 6 and formula (1.8) in this thesis,
4. *rotation* on a circle (*Sturmian trajectory* defined by a); see Arnoux et al. (1999) [3] and Definition 8 in this thesis,
5. Freeman *chain code* of the line $y = ax$; see Freeman (1970) [38], and Figure 1.5 and formula (1.6) in this thesis,
6. the *cutting sequence* of the line $y = ax$; see Figure 1.5 in this thesis,
7. the *billiard word* with slope a; see Borel and Reutenauer (2005) [15] and the brief description in Section 1.3.1 of this thesis.

Other terms connected with our problem are: *Rauzy rules*, *standard sequences*, *balanced words*, words with *minimal complexity*, and *Christoffel words*.

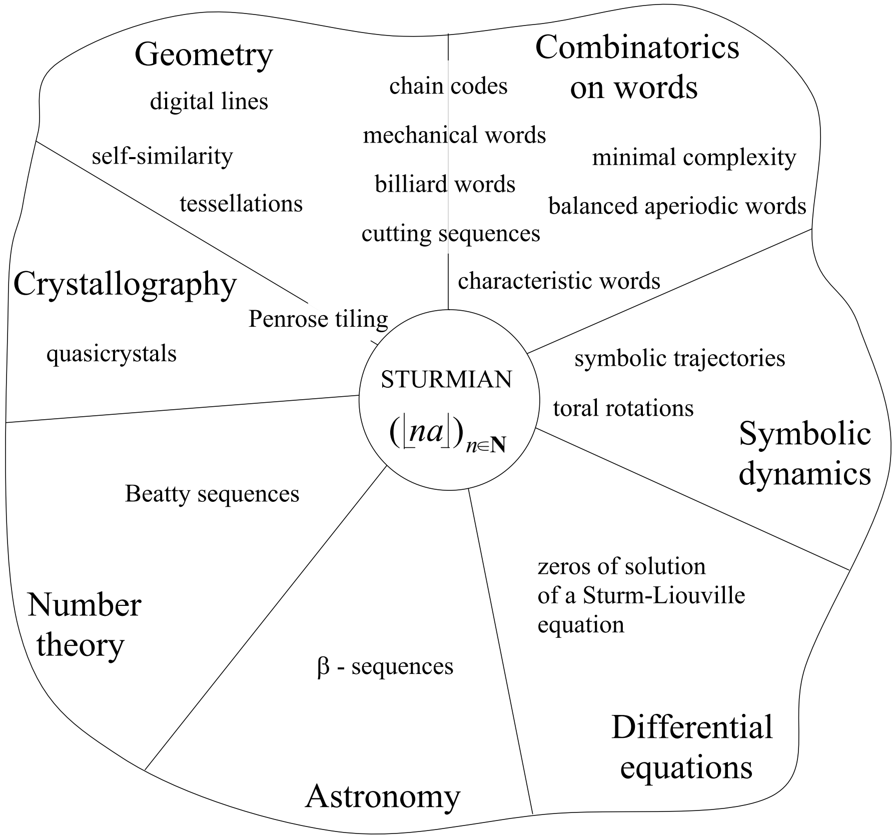

Figure 1.1: The main domains and terms related to our problem.

In Section 1.1 we will give a brief introduction to the number-theoretical tools which we will use in this thesis. In Figure 1.1 we present the main domains and terms related to our problem of describing the sequence $(\lfloor na \rfloor)_{n\in\mathbf{N}}$ for $a \in]0,1[\setminus \mathbf{Q}$. In Sections 1.2 and 1.3 we will try to link the above mentioned terms to each other.

Examples of other places in the literature where the interdisciplinary character of our problem is illuminated are Stolarsky (1976) [80], Bruckstein (1991) [24], Lothaire (2002) [60, pp. 45–60], Pytheas Fogg (2002) [69, pp. 143–198], Berthé et al. (2005) [12], Berthé (2009) [11], and Harris and Reingold (2004) [44].

1.1 Our number-theoretical tools

1.1.1 A brief introduction to continued fractions (CF)

The history of the use of continued fractions (CF) is as long as the history of the use of Euclid's algorithm (*ca* 300 BC), because the process of finding the greatest common divisor for two natural positive numbers n and k is the same process as calculating the CF-expansion of $\frac{n}{k}$. A list of important dates and names in the history of CFs can be found in Wikipedia (http://en.wikipedia.org/wiki/Continued_fraction). Here we will only mention Aryabhatta (499), Rafael Bombelli (1579), Pietro Cataldi (1613)—first notation for CFs, John Wallis (1695)—introduction of the term *continued fraction*, Leonard Euler, Johann Lambert, Joseph Louis Lagrange, Karl Friedrich Gauss, and Bill Gosper (1972) [42]—first exact algorithms for CF-arithmetic. For more historical information see Brezinski (1991) [18], Flajolet et al. (2000) [36], and Vardi (1998) [86].

To illustrate the first statement of this section with an example, we will run Euclid's algorithm for the numbers 17 and 31:

$$\begin{aligned} 31 &= \mathbf{1}\cdot 17 + 14 \\ 17 &= \mathbf{1}\cdot 14 + 3 \\ 14 &= \mathbf{4}\cdot 3 + 2 \\ 3 &= \mathbf{1}\cdot 2 + 1 \\ 2 &= \mathbf{2}\cdot 1 \end{aligned}$$

and compare it with the following:

$$\frac{17}{31} = \frac{1}{\frac{31}{17}} = \frac{1}{1+\frac{14}{17}} = \frac{1}{1+\cfrac{1}{\frac{17}{14}}} = \frac{1}{1+\cfrac{1}{1+\frac{3}{14}}} = \frac{1}{1+\cfrac{1}{1+\cfrac{1}{\frac{14}{3}}}}$$

$$= \frac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\frac{2}{3}}}} = \frac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{\frac{3}{2}}}}} = \frac{1}{\mathbf{1}+\cfrac{1}{\mathbf{1}+\cfrac{1}{\mathbf{4}+\cfrac{1}{\mathbf{1}+\cfrac{1}{\mathbf{2}}}}}}.$$

We notice the numbers 1, 1, 4, 1, 2 (in boldface in both calculations), which are the integer parts of the quotients in the performed divisions, appear in both operations.

After this first example we will formally define a CF-expansion of a number. We will do this for irrational numbers, because they are at the absolute center of our attention in this thesis. The algorithm for rational numbers is the same as for irrational numbers, but with the difference that it ends after a certain number of steps.

Let a be an irrational number. The following algorithm gives the regular (or simple) continued fraction (CF) for a:

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\dots}}}} = [a_0; a_1, a_2, a_3, \dots].$$

We define a sequence of integers (a_n) and a sequence of real numbers (α_n) by

$$\alpha_0 = a; \quad a_n = \lfloor \alpha_n \rfloor \quad \text{and} \quad \alpha_{n+1} = \frac{1}{\alpha_n - a_n} \quad \text{for} \quad n \geq 0.$$

Then $a_n \geq 1$ and $\alpha_n > 1$ for $n \geq 1$. The natural numbers $a_0, a_1, a_2, a_3, \dots$ are called the *elements* of the CF. They are also called the *terms* of the CF, see Beskin (1986) [13, p. 20]; or *partial quotients*, see Venkov (1970) [87, p. 40]. We will use the word *elements*, following Khinchin (1997) [49, p. 1].

If a is irrational, so is each α_n, and the sequences (a_n) and (α_n) are infinite. In case of a rational a, we get $\alpha_m = a_m$ for some m and then we cannot proceed and the algorithm stops, a_m is the last CF-element of a.

We notice that

$$\alpha_{n+1} = [a_{n+1}; a_{n+2}, \dots] \quad \text{and} \quad \frac{1}{\alpha_{n+1}} = [0; a_{n+1}, a_{n+2}, \dots] = \alpha_n - a_n,$$

which is an example of an easy calculation of the inverse of a CF. Another is subtracting a CF from 1, which is presented as Lemma 5 and Lemma 6 in Paper II.

A CF-expansion exists for all $a \in \mathbf{R}$ (we have just presented an algorithm) and is unique if we impose the additional condition (for rational slopes) that the last element cannot be 1, because

$$\frac{1}{n} = \frac{1}{n - 1 + \frac{1}{1}},$$

so allowing 1 as the last element would destroy the uniqeness of the expansion. In our work, however, we even need not bother about the last elements, because we are dealing with irrational numbers, and these always have a CF-expansion with an infinite numbers of elements. For more details see Khinchin (1997) [49, p. 16].

We call $[a_0; a_1, a_2, \dots, a_n]$, for each $n \in \mathbf{N}$, the n^{th} *convergent* of the CF $[a_0; a_1, a_2, \dots]$. If we define

$$p_0 = a_0, \;\; p_1 = a_1 a_0 + 1, \;\; \text{and} \;\; p_n = a_n p_{n-1} + p_{n-2} \;\; \text{for} \;\; n \geq 2$$

and

$$q_0 = 1, \quad q_1 = a_1, \quad \text{and} \quad q_n = a_n q_{n-1} + q_{n-2} \quad \text{for} \quad n \geq 2,$$

then

$$[a_0; a_1, a_2, \ldots, a_n] = \frac{p_n}{q_n} \qquad \text{for} \quad n \in \mathbf{N};$$

see for example Vajda (2008) [84, pp. 158–159]. All convergents are irreducible. Even-order convergents form an increasing sequence and odd-order convergents a decreasing. Every odd-order convergent is greater than any even-order convergent; see Khinchin (1997) [49, p. 6, Th. 4].

CFs form a very important tool in the approximation of irrational numbers. The following justification of this statement comes from Khinchin (1997) [49, pp. 21–22]. A rational fraction m/n (for $n > 0$) is called a *best approximation* of a real number a if every other rational fraction s/t with the same or smaller denominator differs from a by the same or a greater amount, thus:

$$0 < t \leq n \quad \Rightarrow \quad \left| a - \frac{s}{t} \right| \geq \left| a - \frac{m}{n} \right|.$$

For each $a = [a_0; a_1, a_2, \ldots]$, for each $k \geq 2$ and $1 \leq j \leq a_k - 1$, the fractions

$$\frac{p_{k-2} + j p_{k-1}}{q_{k-2} + j q_{k-1}}$$

we call *intermediate fractions*. The following theorem [49, p. 22, Theorem 15] shows the importance of CF-expansion in the approximation of irrational numbers.

Theorem 1 *Every best approximation of a number $a \in \mathbf{R} \setminus \mathbf{Q}$ is a convergent or an intermediate fraction of the CF expressing that number.*

If we know the CF-expansion of a real number, we can determine the value of that number with an *a priori* chosen degree of accuracy. More about approximations can be found in Khinchin (1997) [49, pp. 16–50].

In Paper II of this thesis we give some examples of irrational numbers with very simple CF-expansions. We recall the theorem (Lagrange 1770, Euler 1737) which states that

quadratic surds, and only they, are represented by periodic CFs;

see Beskin (1986) [13, pp. 66–71]. We also show some transcendental numbers with a periodical pattern in the CF-expansion.

For more information about CFs, see Khinchin (1997) [49], Flajolet et al. (2000) [36], and, for CF-arithmetic, Gosper (1972) [42].

CFs have a beautiful geometrical interpretation, which was formulated by Felix Klein in 1895 and explored by Vladimir Igorevich Arnold. Before we formulate the theorem by Klein, we will cite Korkina (1996) [58].

> In geometric representation, continued fractions are associated with the boundaries of the convex hull of all integer points in some angles in the plane. The coefficients of a continued fraction are equal to the integer lengths of segments belonging to the boundaries of convex hulls.

In Debled (1995) [30, p. 63] we find the following formulation of Klein's theorem.

Theorem 2 (Klein 1895; the formulation from [30, p. 63]) *Let a be an irrational number with the CF-expansion $a = [a_0; a_1, a_2, \ldots]$ and let us denote its convergents by $p_n/q_n = [a_0; a_1, a_2, \ldots, a_n]$ for $n \in \mathbf{N}$. We have the following:*

- *the integer points on the boundary of the convex hull of all integer points lying under the line $y = ax$ are*

$$(q_0, p_0) = (1, a_0), (q_2, p_2), \ldots, (q_{2k}, p_{2k}), \ldots,$$

- *the integer points on the boundary of the convex hull of all integer points lying over the line $y = ax$ are*

$$(q_3, p_3), (q_5, p_5), \ldots, (q_{2k+1}, p_{2k+1}), \ldots.$$

In our work we use CFs to describe digital straight lines in the plane (see Section 1.2). In order to describe lines (and planes) in higher dimensions, one can use multidimensional CFs, as in the Ph.D. thesis of Thomas Fernique from 2007 [35].

1.1.2 The Gauss map

In both Euclid's algorithm and the process of finding the CF-expansion of a number, the integer and fractional part of numbers are very much involved. Each positive irrational a less than 1 can be represented in the following way:

$$\frac{1}{\frac{1}{a}} = \frac{1}{\left\lfloor \frac{1}{a} \right\rfloor + \mathrm{frac}\left(\frac{1}{a}\right)} = \frac{1}{\left\lfloor \dfrac{1}{a} \right\rfloor + \dfrac{1}{\left\lfloor \dfrac{1}{\mathrm{frac}\left(\frac{1}{a}\right)} \right\rfloor + \mathrm{frac}\left(\dfrac{1}{\mathrm{frac}\left(\frac{1}{a}\right)}\right)}} \tag{1.1}$$

which could be continued in the same way as the process of finding the CF-expansion of $\frac{17}{31}$ presented in Section 1.1.1 of this introduction. It would, however, become quite monstrous and would lose its readability very quickly (if it has not lost it already). To make the notation more elegant, one can use the Gauss map, which leads to a more compact notation of (1.1). Let us recall that the Gauss map $G\colon [0,1] \to [0,1]$ is defined as follows:

$$G(x) = \begin{cases} 0, & \text{if } x = 0 \\ \operatorname{frac}\left(\frac{1}{x}\right) & \text{if } 0 < x \leq 1. \end{cases}$$

For $a = [0; a_1, a_2, \ldots]$ we have obviously

$$G^n(a) = [0; a_{n+1}, a_{n+2}, \ldots] \quad \text{for} \quad n \in \mathbf{N}^+, \tag{1.2}$$

where $G^1(a) = G(a)$ and $G^{k+1}(a) = G(G^k(a))$ for $k \in \mathbf{N}^+$. This means that

$$a = \left[0; \left\lfloor \frac{1}{a} \right\rfloor, \left\lfloor \frac{1}{G(a)} \right\rfloor, \left\lfloor \frac{1}{G^2(a)} \right\rfloor, \ldots \right], \tag{1.3}$$

which expresses the same idea as (1.1), but in a much nicer way. One can read more about the relationships between CFs and the Gauss map in Bates et al. (2005) [5]. In Paper II of this thesis we discuss the relationship between the Gauss map and the process of digitization of straight lines with positive irrational slopes less than 1.

1.1.3 The Stern–Brocot tree

Another important number-theoretical tool, which is strongly related to CFs, is the *Stern–Brocot tree*; see Figure 1.2. This section is heavily based on Graham, Knuth and Patashnik (2006) [43, pp. 116–123].

The Stern–Brocot tree was discovered independently by the German mathematician Moritz Stern in 1858 and by the French clockmaker Achille Brocot three years later, in 1861. It is constructed as follows; cf. Graham et al. (2006) [43, pp. 116–117]:
We start with two fractions, $\frac{0}{1}$ and $\frac{1}{0}$, where the latter is meant formally as a pair $(1, 0)$ symbolizing the infinity, and gives us a possibility of a homogeneous definition of all nodes.

The dashed lines between $\frac{0}{1}$ and $\frac{1}{1}$ and between $\frac{1}{1}$ and the empty place [1] for $\frac{1}{0}$ (which represents infinity) symbolize that $\frac{0}{1}$ lies to the left of all nodes of the tree and $\frac{1}{0}$ lies to the right of all nodes of the tree as they

[1] When I was drawing Figure 1.2 I felt very bad about putting $\frac{1}{0}$ in it (it is probably caused by some childhood trauma related to division by zero). Then I changed my mind and used this *formally meant* fraction anyway and then... my computer crashed when I tried to export this new picture to .eps format. It felt like a supernatural confirmation of my initial doubts, so I decided to leave the place for $\frac{1}{0}$ in Figure 1.2 empty.

are respectively the smallest and the largest (the latter is, again, meant symbolically) numbers used as labels for nodes. The node labeled by $\frac{m}{n}$ is a *descendant* of the node $\frac{k}{l}$ (and $\frac{k}{l}$ is then called an *ancestor* of $\frac{m}{n}$) if there is a path leading upwards from $\frac{m}{n}$ to $\frac{k}{l}$.

In order to construct the Stern–Brocot tree, we repeat the following operation as many times as needed:

$$\text{Insert } \frac{m+m'}{n+n'} \text{ between two adjacent fractions } \frac{m}{n} \text{ and } \frac{m'}{n'}.$$

This way we get, after starting with $\frac{0}{1}$ and $\frac{1}{0}$, one more fraction in the first step and continue with three fractions, $\frac{0}{1}$, $\frac{1}{1}$, and $\frac{1}{0}$, which includes two pairs of consecutive fractions. This means that we can produce two new fractions in the next step and get $\frac{0}{1}$, $\frac{1}{2}$, $\frac{1}{1}$, $\frac{2}{1}$, and $\frac{1}{0}$, which contains four pairs of consecutive fractions, giving four more fractions in the next step, etc. Each fraction in the Stern–Brocot tree, except the first two, is of the form $\frac{m+m'}{n+n'}$, where $\frac{m}{n}$ is the nearest ancestor above and to the left, and $\frac{m'}{n'}$ is the nearest ancestor above and to the right.

The result is shown in Figure 1.2. By putting the dashed lines going down towards $\frac{1}{1}$ from the left and from the right we try to justify the usage of the word *between* in the definition of the nodes of the Stern–Brocot tree. All the nodes in Figure 1.2 are placed in such a way that each node lies between the two ancestors it arises from. However, it is clear that it is physically impossible to continue drawing in this way and preserve some readability, so we restrict our picture in Figure 1.2 to the top of the Stern–Brocot tree.

The left-hand side of the Stern–Brocot tree is also called the *Farey tree*; see Lagarias (1992) [59, p. 42]. The Stern–Brocot tree has a number of interesting properties, which can be found in Graham et al. (2006) [43, pp. 116–120]. For example, all the fractions appearing in it are in lowest terms. Each positive fraction $\frac{m}{n}$ where m and n are relatively prime, appears in the tree exactly once and none of them is omitted. The construction of the tree preserves the order $<$ in $\mathbf{Q}$.

Let us denote traversing down the left or right branch by L and R respectively. Beginning from $\frac{1}{1}$ and following the path to a particular fraction, we get a unique coding by string of L's and R's for this fraction. The $\frac{1}{1}$-fraction itself is represented by the empty string called I (identity). For example, $RLRL$ represents $\frac{8}{5}$. The first element of the string is the letter which codes our first move downwards from $\frac{1}{1}$ in the direction of the fraction we want to reach.

For each string S of L's and R's, let us denote by $f(S)$ the fraction corresponding to the route coded by S in the Stern–Brocot tree. We have

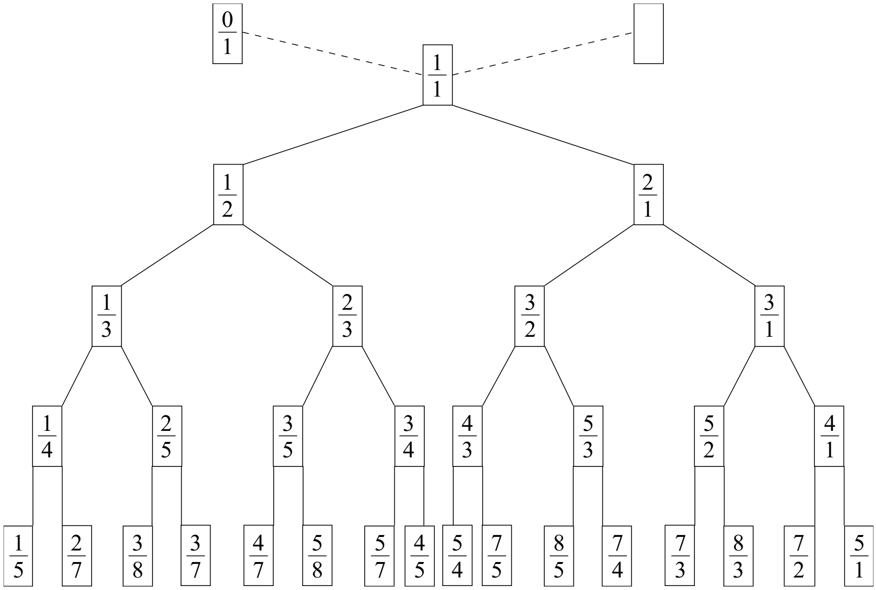

Figure 1.2: The top of the Stern–Brocot tree.

for each $a_0 \in \mathbf{N}$, $a_1, \ldots, a_{n-2} \in \mathbf{N}^+$, $a_{n-1} \geq 0$, and an even n

$$f(R^{a_0} L^{a_1} R^{a_2} \cdots L^{a_{n-1}}) = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ldots + \cfrac{1}{a_{n-1} + \cfrac{1}{1}}}}}; \tag{1.4}$$

see Graham et al. (2006) [43, pp. 301–307]. This can also be used for infinite CFs, thus for irrational numbers. We have for example

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1, \ldots]$$

$$e = R^2 L R^2 L R L^4 R L R^6 L R L^8 R L R^{10} L R L^{12} R L \cdots.$$

The formula for the CF-expansion of e comes from Euler (1737), but, according to Brezinski (1991) [18, p. 97], it was already formulated by R. Cotes in the *Philosophical Transactions* in 1714. The L-R-string for e can be found in Graham et al. (2006) [43, p. 122].

Both CFs and the Stern–Brocot tree have been intensively used in connection with descriptions of digital straight lines. Isabelle Debled, in her Ph.D. thesis from 1995 [30], used (1.4) to find the convergents of the rational slopes of the lines to be digitized. She also used Klein's theorem

(Theorem 2 in this thesis). This resulted in an algorithm for digital lines with rational slopes; see Debled (1995) [30, p. 65].

1.2 Digital geometry

Digital geometry is a very young branch of mathematics; it is only about 50 years old and arose out of the need for making computer graphics. Following Kiselman (2008) [53], we can call it the *geometry of the computer screen*. In the abstract of McIlroy (1992) [62] we find the following statement: *Computer graphics is geometry on a grid*, with further explanation as follows:

> Computers make drawings by coloring pixels in a bitmap, which may be thought of as the points of a plane integer lattice. People made digital pictures for thousands years before computers and well before number theory, too [McIlroy refers to a discretized image from Chicama Valley, Peru, ca 2000 BC, which is now in the American Museum of Natural History]. But with computers, where algorithm supplants artistry, the mathematics becomes more important. Drawing a figure becomes a problem in two-dimensional Diophantine approximation: picking points of the lattice to get a good fit.

Digital geometry is of course much more than finding a good integer-number approximation of $\mathbf{R}^2$-sets. It is also about how to express movements with integer numbers only, how to define digital convexity, connectivity, and other well-known properties and objects from euclidic geometry. To learn more about it, one can read Klette's and Rosenfeld's textbook on digital geometry from 2004 [54] or the proceedings of the DGCI conferences (Discrete Geometry for Computer Imagery).

Another excellent source of information about digital geometry and its topics, this time in Swedish and written for everybody who is interested in mathematics, not only for specialists in digital geometry, is the chapter about the geometry of computer screen by Christer Kiselman [53] in a popular-science book about selected topics in mathematics. A much more advanced source of material is the lecture notes (in English) in digital geometry and mathematical morphology by the same author; see Kiselman (2004) [52]. Debled-Rennesson (2007) [31] provides deep insight into very recent developments in numerous subjects of digital geometry, both from a theoretical standpoint and with a strong focus on applications (many algorithms are presented).

In the Ph.D. thesis of Erik Melin from 2008 [64], the focus is on the *Khalimsky topology* and digital continuous functions. Convex functions on discrete sets are studied by Kiselman (2004) [51]. A lot of work on the topic of discrete rotations has been done by Nouvel and Rémila; see [67] and references there. In [50], Christer Kiselman formulates and proves a discrete version of the Jordan curve theorem. A very important part of digital geometry is the development of theoretical mathematical tools for image processing, like for example distance functions; see Borgefors (2003) [16], Strand (2008) [81]. There exists also an enormous number of publications concerning 3D discrete geometry. Digital planes and 3D-lines have been examined in [29, 30, 89, 19, 46, 1, 47, 35, 2] and elsewhere.

In this introduction to digital geometry we will focus on the digitization process in 2D and descriptions of digital straight lines, because this part of digital geometry is essential for our results.

1.2.1 Digital lines

To *digitize* a subset of $\mathbf{R}^2$ means to find its representation in $\mathbf{Z}^2$, i.e., to approximate it with a set of integer-coordinate points. There are different ways to do so, but in all of them the main aim is the same: to select pixels that are close to the $\mathbf{R}^2$-set they approximate.

The first attempts to do that were algorithmic. The most well-known algorithm for plotting lines on a grid is due to Bresenham (1965) [17]. This has become an excellent tool in computer graphics, but it does not give a description of digital lines as mathematical objects.

Freeman (1974) [39] gave a more mathematical description of how one can make a choice of pixels for approximation of a continuous curve with integer-coordinate points. The *Freeman approximation* of a curve is the set of points $(k, n) \in \mathbf{Z}^2$ for which the curve intersects either of the closed unit segments, horizontal $H(k, n)$ and vertical $V(k, n)$ centered on (k, n), where

$$H(k, n) = \{(x, n);\ k - 0.5 \leq x \leq k + 0.5\},$$

$$V(k, n) = \{(k, y);\ n - 0.5 \leq y \leq n + 0.5\}.$$

The Freeman approximation chooses, for each intersection of the curve with a grid line of the lattice, the point nearest to the intersection. Figure 1.3 shows Freeman points (the integer-coordinate points approximating the curve) as dots, the segments $H(k, n)$ and $V(k, n)$ as bars, and the pixels centered in Freeman points as shadowed squares; cf. McIlroy (1992) [62, pp. 106–108].

The Freeman approximation is invariant under symmetry operations of the integer lattice: integer translations, half turns, quarter turns, and reflections, but it also has some drawbacks. In some cases it can give thick digitizations, for example when a curve passes exactly halfway between two adjacent lattice points; see the pixels number 1 and 2 and the pixels number 3 and 4 in Figure 1.3.

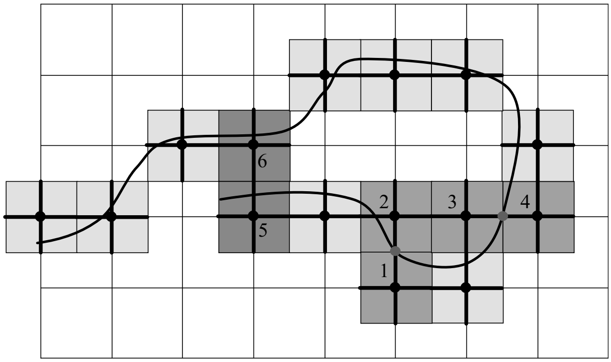

Figure 1.3: Freeman approximation of a curve. The explanation for the shaded squares follows in the text.

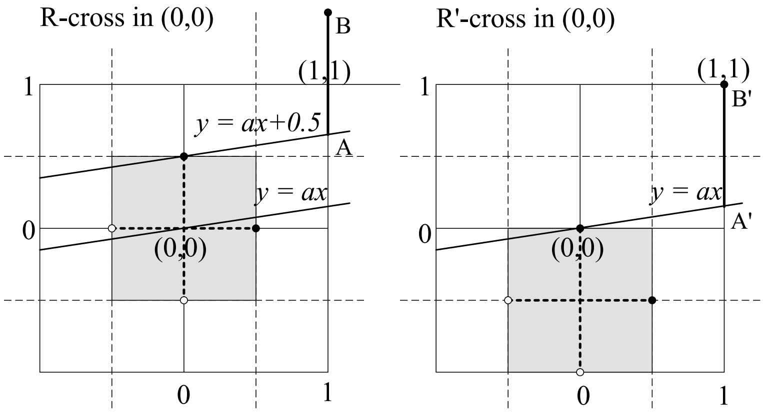

Figure 1.4: Rosenfeld R-cross and the modified Rosenfeld R′-cross.

Another way to do integer-point approximations is the classical digitization defined by Azriel Rosenfeld in 1974 [72]. In this thesis we call it the R-digitization. We remove the point $(k-0.5, n)$ from $H(k,n)$ and the point $(k, n-0.5)$ from $V(k,n)$ and in this way we get the following *R-crosses* centered in (k,n) for each $(k,n) \in \mathbf{Z}^2$ (see also Figure 1.4):

$$C_R(k,n) = (\{k\} \times]n-0.5, n+0.5]) \cup (]k-0.5, k+0.5] \times \{n\})\,.$$

The R′-crosses illustrated in Figure 1.4 and the R′-digitization will be discussed later in this section (on page 23).

Rosenfeld's digitization of a subset A of $\mathbf{R}^2$, which we will denote by $D_R(A)$, is the set of all (k,n) in $\mathbf{Z}^2$ for which the intersection of A and $C_R(k,n)$ is not empty:

$$D_R(A) = \{(k,n) \in \mathbf{Z}^2;\;\; A \cap C_R(k,n) \neq \emptyset\}.$$

We can see that this way of representing $\mathbf{R}^2$-sets by grid points allows us to reduce thick digitizations in some cases. For example, in Figure 1.3 we get rid of the pixel number 4. However, the solution is not perfect. We can eliminate none of the pixels numbered 1, 2 and 3, and we do not handle the pixels numbered 5 and 6 correctly either; the discrete curve contains a loop while the continuous curve is simple (injective).

If the subset A to digitize is a straight line $y = ax$ with $a \in \,]0, 1[\, \setminus \mathbf{Q}$, then the R-digitization can be described arithmetically as follows:

$$D_R(A) = \{(k, \lfloor ak + 0.5 \rfloor);\;\; k \in \mathbf{Z}\}. \tag{1.5}$$

The resulting lines can be very thick for rational slopes, like for example $y = x + 0.5$, but there is a solution for this—we choose the upper or the lower half of the digitization. We can form the sequence of differences of the consecutive n-values (the discrete counterpart of y-values) of $D_R(A)$ in the following way:

$$c_a(k) = \lfloor a(k+1) + 0.5 \rfloor - \lfloor ak + 0.5 \rfloor, \tag{1.6}$$

which forms the *chain code* of the line $y = ax$. In this way we code the move upwards (in the grid) with 1 and a horizontal move with 0; see Figure 1.5. Actually, chain codes of a digital straight line were discussed even before the definition of R-digitization was formulated. They were described in Freeman (1970) [38], where we can find necessary conditions (F1)–(F3) for self-similarity of digital straight lines:

> To summarize, we thus have the following three specific properties which all chains of straight lines must possess:
> (F1) at most two types of elements can be present, and these can differ only by unity, modulo eight;
> (F2) one of the two element values always occurs singly;
> (F3) successive occurrences of the element occurring singly are as uniformly spaced as possible.

The chain codes of lines with rational slopes are periodic, in case of irrational slopes we get aperiodicity; see Klette and Rosenfeld (2004a) [54, p. 312, Th. 9.3]. One can also define a *digital ray* with slope a, which is $c_a(0)c_a(1)c_a(2)\cdots$; see Klette and Rosenfeld (2004a) [54, p. 310].

In Figure 1.5 we also illustrate the *cutting sequence* of a straight line $y = ax$ for $a \in \,]0, 1[\, \setminus \mathbf{Q}$ (call it l). Cutting sequences were described by Caroline Series in 1985 [74]. We consider the intersections of the line l with the grid lines passing through integer-coordinate points. We create the following sequence of labels: use V if the grid line crossed by the line l is vertical and H if it is horizontal. The sequence of labels, read from origin out, is called the *cutting sequence* of l. The relationship between

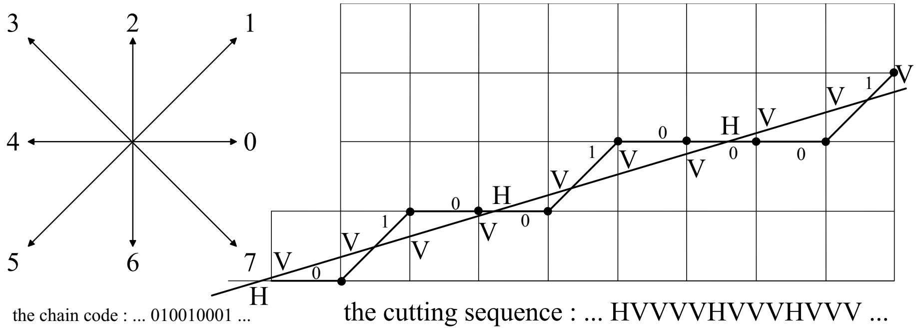

Figure 1.5: The chain code (small 0's and 1's) and the cutting sequence (large V's and H's) of a straight line.

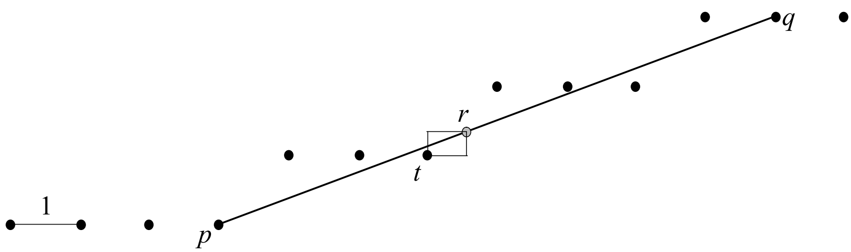

Figure 1.6: The chord property.

cutting sequences and characteristic words is examined in Crisp et al. (1993) [28] and is illustrated in Figure 1.5.

In 1974, Azriel Rosenfeld introduced the following definition.

Definition 1 *A set M of grid points satisfies the* chord property *iff, for any two distinct p and q in M and any point r on the (real) line segment pq, there exists a grid point $t \in M$ such that the l^∞-distance between r and t is less than* 1*, i.e.,* $\max(|r_1 - t_1|, |r_2 - t_2|) < 1$*.*

The chord property is illustrated in Figure 1.6.

This definition and the theorem which says that digital straightness and chord property are equivalent [54, pp. 316–317, Th. 9.7], allowed Rosenfeld to formulate the following necessary conditions on the chain codes of digital straight line segments; see Rosenfeld (1974) [72]. These conditions had already been (partly) presented by Freeman in 1970, but they were not precisely formulated—(F3) is quite informal—and not proven. There is an intimate connection between them and the *self-similarity* of digital lines discussed in Bruckstein (1991) [24]. The conditions presented by Rosenfeld are stated in terms of the *runs* in the chain code. By run we mean a maximum-length factor a^n or b^n of chain codes (thus

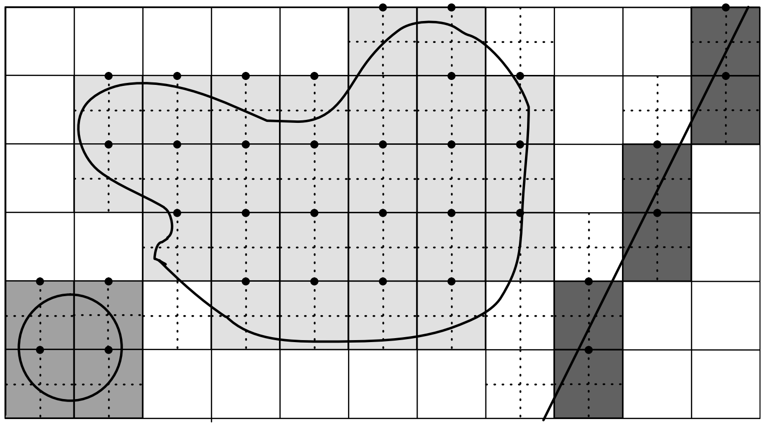

Figure 1.7: Subsets of $\mathbf{R}^2$ and their R′-digitization; three examples.

$a, b \in \{0, 1, 2, \ldots, 7\}$). The concept of run is absolutely essential for this thesis. Rosenfeld's conditions (R1)–(R4) are the following.

(R1) The runs have at most two directions, differing by 45° [see Figure 1.5; for slopes between 0 and 1 we get symbols 0 and 1, we can thus take $a = 0$ and $b = 1$], and for one of these directions the run length must be 1.
(R2) The runs can have only two lengths, which are consecutive integers.
(R3) One of the runs can occur only one at a time.
(R4) ..., for the run length that occurs in runs, these runs can themselves have only two lengths, which are consecutive integers, and so on.

In our work, we modified the R-digitization by introducing R′-crosses, which are R-crosses translated by $(0, -0.5)$ as in Figure 1.4. In Figure 1.7 we can see an example of three subsets of $\mathbf{R}^2$, a disc, a surface inside a closed curve, and a straight line segment, together with their R′-digitizations which are represented by the shadowed squares. The bullets on the grid symbolize integer-coordinate points, which form a digital representation of the $\mathbf{R}^2$-sets.

The R′-digitization of the line $y = ax$ is the following:

$$D_{R'}(\{(x, y) \in \mathbf{R}^2;\ y = ax\}) = \{(k, \lceil ak \rceil);\ \ k \in \mathbf{Z}\}. \tag{1.7}$$

We notice that the modification of Rosenfeld's definition lets us eliminate the +0.5 term from the formula (1.5).

Jean-Pierre Reveillès gave in his Ph.D. thesis [71] in 1991 a definition of the arithmetic discrete line, which is very widely used. He defined digital

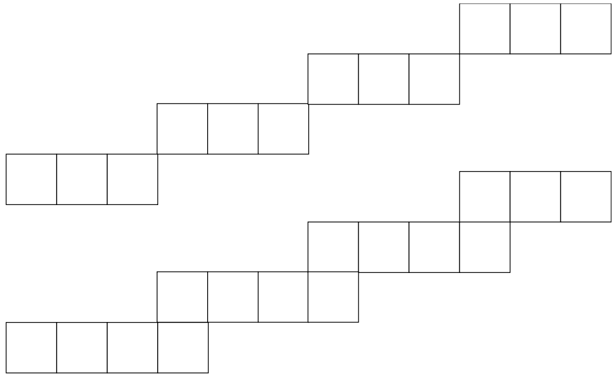

Figure 1.8: Top: a naive line (8-connected), bottom: a standard line (4-connected).

lines by pairs of linear Diophantine (involving only integers) inequalities.

Definition 2 *Let a, b, μ and ω be integers such that a and b are relatively prime and $\omega > 0$. Then*

$$D_{a,b,\mu,\omega} = \{(i,j) \in \mathbf{Z}^2;\ \ \mu \leq ai + bj < \mu + \omega\}$$

is called an arithmetic line *with the* normal vector (a, b)*, the* translation parameter *(or the* inferior bound*)* μ*, and the* arithmetic thickness ω*. If $\omega = \|(a,b)\|_\infty = \max(|a|, |b|)$, then $D_{a,b,\mu,\omega}$ is called* naive*, and if $\omega = \|(a,b)\|_1 = |a| + |b|$, then $D_{a,b,\mu,\omega}$ is called* standard.

In Figure 1.8 we show examples of a naive line and a standard line. Naive lines are 8-connected which means that each grid point belonging to a naive line has a horizontal, vertical or diagonal neighbor which also belongs to the same line. Standard lines are 4-connected (the same condition, but with the restriction that the neighbors must be horizontal or vertical). In Debled (1995) [30, pp. 70–71] one can find more pictures and a deeper discussion of different kinds of connectivity.

1.2.2 CF-based descriptions of digital lines

In this thesis we will present a new CF-description of digital lines. The use of CFs in modelling digital lines was discussed by Brons [22] as early as 1974. The algorithm provided by Brons is only valid for rational slopes.

In 1982, Wu formulated a theorem describing digital straightness by a set of conditions (called the *DSS property*) which the corresponding chain code must fulfill; see Klette and Rosenfeld (2004b) [55, pp. 208–209, Th. 14]. Proofs of this theorem based on CFs were published in 1991 independently by Bruckstein and Voss. Bruckstein (1991) [24] described digital straightness by a number of transformations preserving it. Some of these transformations were defined by means of CFs.

Dorst and Duin (1984) [32] presented an algorithm for drawing digital straight lines, which is also valid for irrational slopes. Pitteway (1985) [68] described the relationship between Euclid's algorithm (which is, as we already mentioned, equivalent with the process of finding the CF-expansion when dealing with rational numbers) and run-length encoding.

Voss (1993) [90, pp. 153–157] described a method of constructing the digitization of straight lines with rational slopes using the so called *splitting formula* to split slopes into elementary slopes $\frac{1}{k}$, where $k \in \mathbf{N}^+ \setminus \{1\}$. The splitting formula is recursive. The idea of the formula is based on the characteristic triangles determined by CFs.

Troesch (1993) [82] discussed the relationship between Euclid's algorithm and digitization runs.

Debled presented in her Ph.D. thesis from 1995 a CF-based description of digital lines. As we already mentioned, she used the Stern–Brocot tree (defined in Section 1.1.1) and Klein's theorem (Theorem 2 on p. 14 in this thesis); see [30, pp. 59–66]. Reveillès (1991) [71, p. 157] formulated a CF-based condition to describe intersections of digital naive lines. Sivignon et al. (2004) [76] used the results by Debled and by Reveillès, named above, for investigating the character of digital intersections (e.g., their connectivity and periodicity).

The solution to the problem of describing runs given by Stephenson in his Ph.D. thesis from 1998 [77] was formulated as an algorithm. The run-hierarchical structure of digital lines with rational slopes is also described in Stephenson and Litow (2000, 2001) [78, 79].

De Vieilleville and Lachaud published their combinatoric approach to digital lines in 2006; see [88].

In the text above we listed only the researchers which described digital straight lines through CFs. There are, of course, lots of authors examining the problem with tools other than CFs, which are less relevant to the subject of this thesis. Melin (2005) [63] defined a continuous digitization of straight lines in the Khalimsky plane, Samieinia (2007) [73] further elaborated this subject.

For more information about digital straightness see the review by Klette and Rosenfeld from 2004 [55].

1.3 Combinatorics on words

This part of the introduction is heavily based on Lothaire (2002) [60], Pytheas Fogg (2002) [69], Karhumäki (2004) [48], Berstel and Perrin (2007) [10], and the Ph.D. thesis of Amy Glen from 2006 [41].

Combinatorics on words is a relatively new domain of discrete mathematics. It has grown enormously in last few decades, because of its numerous applications in

- mathematics (symbolic dynamics, digital geometry),
- computer science (pattern recognition, digital straightness, algorithmic number theory),
- physics (quasicrystal modelling); see Duneau and Katz (1985) [33], and de Bruijn (1990) [26] and the references there,
- biology (molecular biology); see for example Van Vliet, Hoogeboom and Rozenberg (2006) [85].

In de Bruijn (1990) [26] one can find the discussion of relationships between quasicrystals, Penrose tilings, and Beatty sequences; see also Lagarias (1992) [59, pp. 59–66].

According to Karhumäki (2004) [48], the 1906 paper by Axel Thue (1863–1922) on repetition-free words is considered a starting point of mathematical research on words. In Berstel and Perrin (2007) [10] we read that the concept of repetitions is nowadays familiar to biologists under the name of *tandem repeats*. They occur in DNA when a pattern of two or more nucleotides is repeated and the repetitions are directly adjacent to each other. An example can be ATTCGATTCGATTCG in which the sequence ATTCG appears in a multiple.

The modern systematic research on words, in particular words as elements of free monoids, was initiated by M. P. Schützenberger in the sixties.

The main object in combinatorics on words is a word, i.e., a (finite or infinite) sequence of symbols from a finite set $\mathcal{A}$ called the *alphabet*. We denote by $\mathcal{A}^\star$ the set of all *finite words* over $\mathcal{A}$ (i.e., finite sequences of elements from $\mathcal{A}$). The set $\mathcal{A}^\star$ is a *free monoid* (non-empty set equipped with an associative binary operation and an identity element) under the operation of *concatenation*. The concatenation of two finite words u and v, written uv, is obtained by juxtaposing their letters. This operation is clearly not commutative, for example, if $\mathcal{A} = \{0, 1\}$, $u = 10$ and $v = 01$ then $uv = 1001$ and $vu = 0110$, so $uv \neq vu$. The identity ε of $\mathcal{A}^\star$ is the empty word. The *free semigroup* over $\mathcal{A}$ is defined by $\mathcal{A}^+ := \mathcal{A}^\star \setminus \{\varepsilon\}$. The *length* $|u|$ of a word $u \in \mathcal{A}^\star$ is the total number of letters forming it. As an example of finite words we can give actual DNA molecules. These are words over the four-letter alphabet $\mathcal{N} = \{\mathrm{A}, \mathrm{C}, \mathrm{G}, \mathrm{T}\}$, where the name of the alphabet $\mathcal{N}$ symbolizes the word *nucleotides*, while A, C, G, and T stand for *Adenine*, *Cytosine*, *Guanine*, and *Thymine* respectively; see Van Vliet, Hoogeboom and Rozenberg (2006) [85].

We denote by $\mathcal{A}^\omega$ the set of (right) infinite words (i.e., sequences of symbols in $\mathcal{A}$ indexed by non-negative integers) and we also use the notation $\mathcal{A}^\infty = \mathcal{A}^\star \cup \mathcal{A}^\omega$. The three most common ways of presenting an infinite word are as follows:

- an infinite word $\mathbf{x}$ can be given by a map $\mathbf{x}\colon \mathbf{N} \to \mathcal{A}$; we define $x_i = \mathbf{x}(i)$ for any $i \geq 0$ (thus each x_i is a letter from $\mathcal{A}$) and write $\mathbf{x} = x_0x_1x_2\cdots$,
- an infinite word w can be formed by concatenating infinitely many finite words C_i and we denote $w = C_1C_2\cdots$, or

$$w = \prod_{j=1}^{\infty} C_j$$

(as in Theorem 4 in Paper III),
- an infinite word w can be defined as the limit of an infinite sequence of finite words $P_1, P_2, P_3, \ldots$ such that P_i is a proper prefix of P_{i+1} for each $i \geq 1$, i.e., $w = \lim_{n\to\infty} P_n$ is the unique infinite word having the words $P_1, P_2, \ldots$ as prefixes; we have such a situation in Theorem 3 in Paper III.

A finite word w is a *factor* of a (finite or infinite) word x if a word y and a finite word u exists such that $x = uwy$. The set of factors of x is denoted by $F(x)$ and the set of factors of length n is denoted by $F_n(x)$.

Definition 3 [60, p. 7] *The* complexity function *of a finite or infinite word x over some alphabet $\mathcal{A}$ is the function that counts, for each integer $n \geq 0$, the number $P(x,n)$ of different factors of length n in x:*

$$P(x,n) = \mathrm{card}(F_n(x)),$$

i.e., for each $x \in \mathcal{A}^\infty$, its complexity function is $P(x,\cdot)\colon \mathbf{N} \to \mathbf{N}$, where for each $n \in \mathbf{N}$ the value $P(x,n)$ is the number of different factors of length n in x.

Clearly, for each word x we have $P(x,0) = 1$, because $F_0(x) = \{\varepsilon\}$, and $P(x,1)$ is the number of letters appearing in x. If x is right infinite, every factor can be extended to the right, so we have for each $n \in \mathbf{N}$

$$P(x,n) \leq P(x,n+1).$$

If $x \in \mathcal{A}^\star$, then $P(x,n) = 0$ for $n > |x|$.

The research in combinatorics on words has, as mentioned above, numerous applications in computer science, biology (genes), crystallography etc. A very good source of information about current research—examining palindromes, overlap-free words, repetitions in words, avoidable patterns, episturmian words, and other topics—is the Ph.D. thesis of Amy Glen from 2006 [41] and the recent survey by Jean Berstel from 2007 [9].

1.3.1 Sturmian words

Infinite binary sequences called *Sturmian words* are a very intensely researched kind of words. They have been studied by many researchers from combinatorial, algebraic, and geometric points of view; see Lothaire (2002) [60], Pytheas Fogg (2002) [69], Berthé et al. (2005) [12].

In Pytheas Fogg (2002) [69, p. 142], in chapter *Sturmian Sequences* by P. Arnoux, we find the following:

> In this chapter, we will study symbolic sequences generated by an irrational rotation. Such sequences appear each time a dynamic system has two rationally independent periods; this is a very typical situation, arising for example in astronomy (with the rotation of the moon around the earth, and of the earth around the sun), or in music (with the building of musical scales, related to the properties of $\log 3/\log 2$), and such sequences have been studied for a long time.

Some explanation for the examples above can be found in Barrow (2000) [4].

The first detailed investigation of Sturmian words can be found in Morse and Hedlund (1940) [65]. They introduced the term *Sturmian*, named for the mathematician Charles François Sturm (1803–1855). The reason was the following: Sturm is best remembered for the Sturm–Liouville problem, an eigenvalue problem in second order differential equations. When we examine the zeros of solutions of a homogeneous second order differential equation

$$y'' + \phi(x)y = 0$$

where ϕ is a continuous function of period 1 and denote by k_n the number of zeros of a solution in the interval $[n, n+1[$, then the infinite word $ab^{k_0}ab^{k_1}\cdots$ is either Sturmian (which we will define in a moment) or eventually periodic (Definition 5 on p. 29 in this thesis); see Morse and Hedlund (1940) [65, p. 1], and Berstel and Perrin (2007) [10, p. 1004].

The pioneering 1940 paper by Morse and Hedlund [65] contains many properties of Sturmian words. Even though the majority of facts we will present in this introduction comes from [65], we will cite more recent sources to make the notation and the names of terms conform to those from Lothaire (2002) [60].

There are many equivalent ways of defining Sturmian words. One of them is (combinatorial) characterization by the number of factors (i.e., by the complexity function recalled in Definition 3 in this thesis); see Coven and Hedlund (1973) [27].

Definition 4 *A* Sturmian *word is an infinite word s such that*

$$P(s, n) = n + 1$$

for any integer $n \geq 0$.

Taking in particular $n = 1$, we get from Definition 4, that $P(x, 1) = 2$ for any Sturmian word x, so any Sturmian word is a binary word, i.e., a word over a two letter alphabet, because $P(x, 1)$ is the number of letters appearing in x. We can, without loss of generality, call our letters 0 and 1, thus $\mathcal{A} = \{0, 1\}$.

Let us compare Definition 4 with the following definition and lemma formulated for sequences of two symbols a and b in Morse and Hedlund (1940) [65, p. 10]:

> CONDITION A. *A set of m-chains will be said to satisfy Condition A if, n being any positive integer not exceeding m, the b-lengths of the sub n-chains of the given set of m-chains assume at most two values.*
>
> LEMMA 3.2. *If a set of m-chains satisfies Condition A, the number of chains in the set cannot exceed $m + 1$.*

By an *m-chain* one should understand a sequence of letters a and b of the form $ab^{k_1}ab^{k_2}a \cdots ab^{k_m}a$, where k_i for $i = 1, 2, \ldots, m$ is a natural number (not necessarily positive), while the *b-length* is the number of symbols b in the m-chain, which is equal to $k_1 + k_2 + \cdots + k_m$.

To be able to list some properties of Sturmian words, we first introduce the following definition; the formulation is from Lothaire (2002) [60, p. 9]:

Definition 5 *A word $x \in \mathcal{A}^\omega$ is*
- periodic *if it is of a form $x = z^\omega$ for some $z \in \mathcal{A}^+$,*
- eventually periodic *if it is of a form $x = yz^\omega$ for some $y, z \in \mathcal{A}^+$,*
- aperiodic *if it is not eventually periodic.*

The following theorem, Lothaire (2002) [60, Th. 1.3.13], shows the relation between the periodicity and the complexity function of a word. Its proof can be found in Lothaire (2002) [60, p. 19].

Theorem 3 *Let x be an infinite word. The following are equivalent:*
(1) x is eventually periodic,
(2) $P(x, n) = P(x, n + 1)$ for some $n \in \mathbf{N}$,
(3) $P(x, n) < P(x, 1) + n - 1$ for some $n \in \mathbf{N}^+$,
(4) $\{P(x, n);\ n \in \mathbf{N}\}$ is bounded.

This gives us the following proposition; cf. Lothaire (2002) [60, p. 46].

Proposition 1 *Sturmian words are aperiodic infinite words of minimal complexity, i.e., x is Sturmian iff it is such an aperiodic infinite word that its complexity function $P(x, \cdot)\colon \mathbf{N} \to \mathbf{N}$ is minimal.*

In order to make the description of Sturmian words more explicit, we will define characteristic, and upper and lower mechanical words. Mechanical

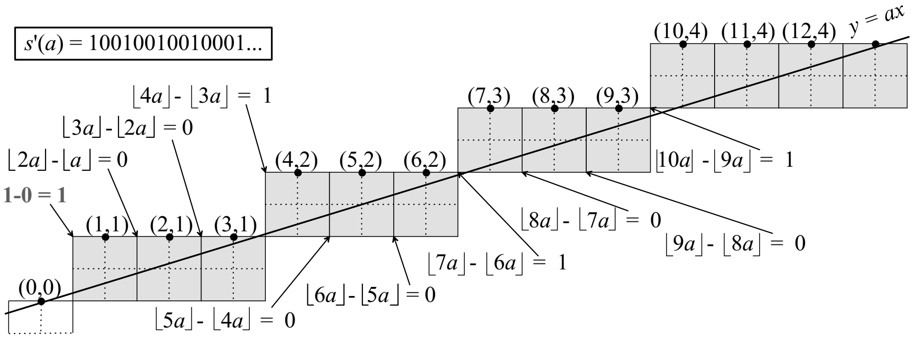

Figure 1.9: An R$'$-digital line $y = ax$ with irrational slope $a = [0; a_1, a_2, \ldots]$ and corresponding upper mechanical word $s'(a)$.

sequences were introduced in Morse and Hedlund (1940) [65, p. 13], but the following formulation comes from Lothaire (2002) [60, p. 53].

Definition 6 *For each $a \in \,]0,1[\, \setminus \mathbf{Q}$ we define two binary words in the following way: $s(a)\colon \mathbf{N} \to \{0,1\}$ and $s'(a)\colon \mathbf{N} \to \{0,1\}$ are such that for each $n \in \mathbf{N}$*

$$s_n(a) = \lfloor a(n+1) \rfloor - \lfloor an \rfloor \;\; \textit{and} \;\; s'_n(a) = \lceil a(n+1) \rceil - \lceil an \rceil.$$

The word $s(a)$ is the lower mechanical word *and $s'(a)$ is the* upper mechanical word *with* slope a *and* intercept 0.

We have $s_0(a) = \lfloor a \rfloor = 0$ and $s'_0(a) = \lceil a \rceil = 1$ and, because $\lceil x \rceil - \lfloor x \rfloor = 1$ for irrational x (we can thus use both $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ in our descriptions), we have

$$s(a) = 0c(a) \;\; \text{and} \;\; s'(a) = 1c(a),$$

meaning 0, respectively 1, concatenated to $c(a)$. The word $c(a)$ is called the *characteristic word* of a. For each $a \in \,]0,1[\setminus \mathbf{Q}$, the characteristic word associated with a is thus the following infinite word $c(a)\colon \mathbf{N}^+ \to \{0,1\}$:

$$c_n(a) = \lfloor a(n+1) \rfloor - \lfloor an \rfloor = \lceil a(n+1) \rceil - \lceil an \rceil, \quad n \in \mathbf{N}^+. \qquad (1.8)$$

It is clear that the problem of finding upper or lower mechanical or characteristic words for any $a \in \,]0,1[\, \setminus \mathbf{Q}$ is equivalent to the problem of finding the Beatty (or β) sequence for this a.

When comparing Figure 1.5 with Figure 1.9, and (1.6) and (1.7) with (1.8), one can easily see that the chain code of $y = ax$ (according to the R$'$-digitization) and the characteristic word $c(a)$ is the same sequence.

Another geometrical interpretation of Sturmian words is the following. We shoot a ball in a square billiards, with initial irrational slope a. The ball bounces on the sides of the billiards according to the laws of elastic shock, and we write V when the ball touches a vertical side, and H for a horizontal side. In this way we form a binary word over the alphabet $\{H, V\}$. For given a we call such a word the *billiard word* with slope a. We notice the equivalence between this word and the cutting sequence for the line $y = ax$; see Borel and Reutenauer (2005) [15] and Figure 1.5 in Section 1.2.1 of this thesis.

Another description of Sturmian words is connected with the balance property. The formulation of the following definition comes from Lothaire (2002) [60, p. 48].

Definition 7 *Let x, y be* 0-1-*words.*

- *The* height *of a finite word x is the number $h(x)$ of letters* 1 *in x.*
- *Given two finite words x and y of the same length, their* balance *$\delta(x, y)$ is the number*

$$\delta(x, y) = |h(x) - h(y)|.$$

- *A set of finite words X is* balanced *if*

$$x, y \in X \quad \wedge \quad |x| = |y| \quad \Rightarrow \quad \delta(x, y) \leq 1.$$

- *A finite or infinite word is itself balanced if the set of its factors (thus, finite words) is balanced.*

In terms of digital geometry we can formulate the balance property in the following way: for each two digital naive straight line segments with the same slope $a \in]0, 1[\setminus \mathbf{Q}$ and the same length (i.e., which contain the same number of pixels), their height can differ at most by 1. Actually, we will see in a moment (Theorem 4 on p. 32) that this formal property of balance has the same meaning as the (R1)–(R4)-conditions formulated by Rosenfeld in 1974 [72] and in Section 1.2.1 of this thesis.

The fact that Sturmian sequences are balanced was established in 1940 by Morse and Hedlund [65, p. 22, Th. 7.1]. The balance property described in [65] bears the name of *Condition S*.

> CONDITION S. *Under Condition S the a-lengths (b-lengths) of two m-blocks with the same m shall differ by at most one.*

By an *m-block* one should understand a sequence $c_1 c_2 \cdots c_m$ of letters a and b (i.e., $c_i \in \{a, b\}$ for $i = 1, 2, \ldots, m$), while a-length (b-length) is the number of a's (b's) in this sequence. The a-length (or the b-length) corresponds to the height $h(x)$ from Definition 7.

In order to characterize Sturmian words by their complexity function, Coven and Hedlund made use of the equivalence between balanced aperiodic and Sturmian words. In their 1973 paper [27] the balance property is called the *Sturmian Block Condition*. More about the balance property can be found in Vuillon (2003) [91] and, about generalized balances, in Fagnot and Vuillon (2000) [34].

The following theorem, Lothaire (2002) [60, p. 51], shows the connection between Definitions 4, 5, 6, and 7.

Theorem 4 *Let s be an infinite word. The following are equivalent:*
(1) s is Sturmian;
(2) s is balanced and aperiodic;
(3) s is irrational (lower or upper) mechanical.

Balanced periodic words are rational lower or upper mechanical words. They are called *Christoffel words* and can be seen as the finite variant of Sturmian words.

In Theorem 4 we observe that in (3) the slope is given explicitly (see Definition 6), while in the first two characterizations of Sturmian words the existence of the slope follows from the definitions of the balance property in (2) and of Sturmian words by the complexity function in (1). Given Sturmian word $\mathbf{x} = x_1x_2\cdots$ we can find its slope by analyzing the densities $h(X_n)/n$, where $X_n = x_1x_2\cdots x_n$ is the prefix with length n and h is the height-function from Definition 7. Theorem 4 guarantees that the limit of $h(X_n)/n$ (with $n \to \infty$) exists and is finite.

Morse and Hedlund (1940) [65] formulated a definition of natural codings of irrational rotations. The modern version of this definition which we present now, comes from Berthé (2009) [11]. An example is given in Figure 1.10.

Definition 8 *Let $R_a\colon \mathbf{R}/\mathbf{Z} \to \mathbf{R}/\mathbf{Z}$, $R_a(x) = x + a \pmod 1$ be the rotation of angle a of the 1-torus $\mathbf{T} = \mathbf{R}/\mathbf{Z}$.*
A sequence $u = (u_n)_{n\in\mathbf{N}} \in \{0,1\}^\omega$ is a natural coding of an irrational rotation *iff there exist a positive irrational $a < 1$ and $x \in \mathbf{R}$ such that*

$$\forall\, n \in \mathbf{N}, \quad u_n = i \quad \Leftrightarrow \quad R_a^n(x) = na + x \in I_i \pmod 1,$$

with $I_0 = [0, 1-a[$ and $I_1 = [1-a, 1[$, or $I_0 =]0, 1-a]$ and $I_1 =]1-a, 1]$.

The following theorem (the formulation from [12, Th. 2.6]), which is the basis for the interest in Sturmian sequences, combines two results, one from Morse and Hedlund (1940) [65, p. 22, Th. 7.1] characterizing the natural coding by the already mentioned *Condition S*, a property similar to the balance, and one in Coven and Hedlund (1973) [27] establishing the equivalence between balanced and Sturmian sequences.

Theorem 5 *A sequence is a natural coding of an irrational rotation if and only if it is Sturmian.*

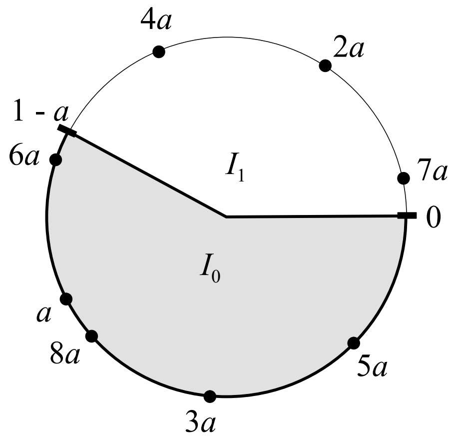

Figure 1.10: Trajectory of rotation of angle a (with starting point $x = 0$). Because $a \in I_0, 2a \in I_1, 3a \in I_0, 4a \in I_1, 5a \in I_0, 6a \in I_0, 7a \in I_1$ and $8a \in I_0$, the trajectory of this rotation is, according to Definition 8, $01010010\cdots$.

Sturmian trajectory in rotation on a torus gives an upper or lower mechanical word, by Theorems 5 and 4.

As we have seen, Sturmian words have quite a lot of equivalent definitions with diverse characteristics—some of them are combinatorial, other geometrical or connected with symbolic dynamics.

1.3.2 CF-based descriptions of Sturmian words

In Section 1.2.2 we listed researchers who presented CF-based descriptions of digital straight lines. Now we will do the same for Sturmian words.

It began with astronomer Johann III Bernoulli (1744–1807) in 1772 and his analysis of a table of *proportional parts*. He noted some interesting rules for evaluation of the tables. His problem consisted of finding the nearest integers $(d_k(a))_{k\in\mathbf{N}^+}$ to a sequence of multiples of a given number a which is the same as asking about the sequence

$$d_k(a) = \lfloor ak + 0.5 \rfloor \quad \text{for} \quad k = 1, 2, 3, \ldots \tag{1.9}$$

(compare this with (1.5)!), because we want $-0.5 \leq d_k(a) - ak < 0.5$. In each step we increase the value of the element of the sequence by $\lfloor a \rfloor$ or by $\lfloor a \rfloor + 1$. Bernoulli showed (but he did not give a proof) in which way the CF-expansion of a can be used to determine in which step we get an increase by $\lfloor a \rfloor$, and in which by $\lfloor a \rfloor + 1$. He described these rules in [7]. Bernoulli's claim was proved by Markov in 1882 [61] and described in English in Venkov (1970) [87, pp. 65–68]. This description has been widely used by researchers in combinatorics on words; see for example Stolarsky

(1976) [80], Ito and Yasutomi (1990) [45]. It is clear that it is equivalent with a description of the characteristic word $c(a)$. The description is as follows.

Theorem 6 (Bernoulli, Markov, Venkov) *For each positive irrational a less than 1 with CF-expansion $a = [0; a_1, a_2, a_3, \ldots]$, the characteristic word is $c(a) = C_1 C_2 C_3 \cdots$, where*

$$\begin{cases} C_1 = 0^{a_1 - 1}1 \\ D_1 = 0^{a_1}1 \end{cases}, \quad \begin{cases} C_2 = C_1^{a_2 - 1} D_1 \\ D_2 = C_1^{a_2} D_1 \end{cases}, \quad \ldots, \quad \begin{cases} C_n = C_{n-1}^{a_n - 1} D_{n-1} \\ D_n = C_{n-1}^{a_n} D_{n-1}. \end{cases}$$

The following description is by Shallit (1991) [75].

Theorem 7 *Let $a = [0; a_1, a_2, \ldots]$ be irrational and $(X_n)_{n \in \mathbf{N}}$ be such that $X_0 = 0$ and $(X_i)_{i \geq 1}$ is the sequence of prefixes of $c = c(a)$, the characteristic word with the slope a (thus c is the* 0-1-*word defined by*

$$c_n = \lfloor (n+1)a \rfloor - \lfloor na \rfloor$$

for $n \geq 1$) with length defined by consecutive denominators q_i of the convergents in the CF-expansion of a, i.e., $X_i = c_1 c_2 c_3 \cdots c_{q_i}$ for $i \geq 1$. Then $X_n = X_{n-1}^{a_n} X_{n-2}$ for $n \geq 2$.

The same result can be found in Lothaire (2002) [60, p. 76] as Proposition 2.2.24. In [60, pp. 75, 76, 104, 105] we can find both methods quoted above (in Theorems 6 and 7) and some relationships between them.

There is an intimate connection between the approach used by Shallit and the notion of *standard words* and *standard pairs*. The following description is heavily based on Lothaire (2002) [60, pp. 63–64]. Let us define two functions Γ and Δ from $\{0,1\}^\star \times \{0,1\}^\star$ into itself:

$$\Gamma(u, v) = (u, uv), \qquad \Delta(u, v) = (vu, v).$$

The set of *standard pairs* is the smallest set of pairs of words containing the pair $(0, 1)$ and closed under Γ and Δ. A *standard word* is any component of a standard pair. The beginning of the tree of standard pairs is presented in Figure 1.11. We note a striking similarity between the Stern–Brocot tree illustrated in Figure 1.2 and the tree of standard pairs. (Compare the binary-word length of the words in the latter with the numerators and denominators of the nodes in the former.) The leftmost and rightmost paths are $(0, 0^n 1), (1^n 0, 1)$ $(n \geq 1)$, corresponding to the nodes $1/(n+1)$ and $(n+1)/1$. The rightmost beginning with $(0, 01)$ and the leftmost beginning with $(10, 1)$ are $(0(10)^n, 01), (10, (10)^n 1)$ $(n \geq 1)$, corresponding to the nodes $(2n+1)/2$ and $2/(2n+1)$.

This set-theoretic definition of standard words is based on Rauzy's method of construction of infinite standard Sturmian words as presented

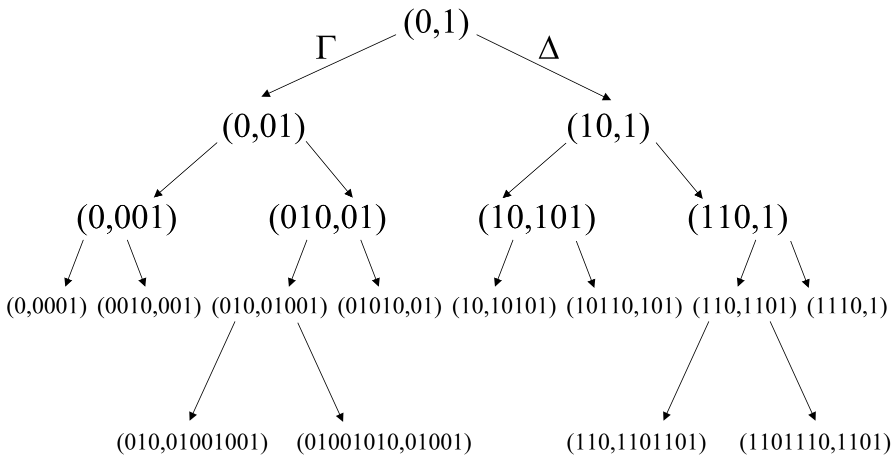

Figure 1.11: The top of the tree of standard pairs.

in Rauzy (1984) [70]; see also Berstel and de Luca (1997) [8], and Lothaire (2002) [60, pp. 63–83].

Shallit (Theorem 7 above) and other researchers working independently in the same period, and earlier, Fraenkel et al. (1978) [37], have thus shown that standard words are basic building blocks for constructing characteristic Sturmian words, in the sense that every characteristic Sturmian word is the limit of a sequence of standard words. We have namely $X_0 = 0$, $X_1 = 0^{a_1-1}1$, and $X_n = X_{n-1}^{a_n} X_{n-2}$ for $n \geq 2$, $(X_n)_{n\in\mathbf{N}}$ is thus a standard sequence. From the definition of Γ and Δ, we get

$$(X_n, X_{n-1}) = \Delta^{a_n}(X_{n-2}, X_{n-1}), \quad (X_{n-1}, X_n) = \Gamma^{a_n}(X_{n-1}, X_{n-2})$$

for $n \geq 2$. Each characteristic word with irrational slope is thus a limit of a sequence of standard words. The sequence $(a_1 - 1, a_2, a_3, a_4, \ldots)$ is called the *directive sequence* of a; cf. Lothaire (2002) [60, p. 75].

In Paper III of this thesis we compare our method with the methods described above and we show that our method is different. There are also other places where we can find similar formulae, see for example Gaujal and Hyon (2004) [40]. The method presented there does resemble ours, but does not give any special attention to the CF-elements equal to 1, which causes it not to reflect the run-hierarchical structure.

1.3.3 Fixed-point theorems for words

There exist different kinds of fixed-point theorems for words. Some of them concern *morphisms* (*substitutions*), others are formulated for *operators*.

1.3.3.1 Fixed-point theorems for morphisms

First we recall the notion of morphism.

Definition 9 *Let $\mathcal{A}$ and $\mathcal{B}$ be finite alphabets. A* morphism *is a map $\varphi\colon \mathcal{A}^\star \to \mathcal{B}^\star$ such that for all $u, v \in \mathcal{A}^\star$ we have $\varphi(uv) = \varphi(u)\varphi(v)$.*

It is clear from the definition that each morphism is uniquely determined by its values over the alphabet $\mathcal{A}$ and that $\varphi(\varepsilon) = \varepsilon$. Further, we can define the n^{th} *iteration* of a morphism $\varphi\colon \mathcal{A}^\star \to \mathcal{A}^\star$, $\varphi^n(a)$, on a letter $a \in \mathcal{A}$, as follows:

$$\varphi^0(a) = a, \quad \varphi^n(a) = \varphi(\varphi^{n-1}(a)), \;\; n \geq 1.$$

Non-erasing morphisms, i.e., such morphisms that $\varphi(a) \neq \varepsilon$ for each $a \in \mathcal{A}$, are called *substitutions*. If $\varphi\colon \mathcal{A}^\star \to \mathcal{A}^\star$ is such a substitution that $\varphi(a) = aw$ for some letter $a \in \mathcal{A}$ and some word $w \in \mathcal{A}^+$ (we say in such cases: φ is *extendable* on a), then we have $\varphi^2(a) = aw\varphi(w)$, $\varphi^3(a) = aw\varphi(w)\varphi^2(w), \ldots$, and so on. This means that, for each $n \in \mathbf{N}$, $\varphi^n(a)$ is a proper prefix of $\varphi^{n+1}(a)$, and the limit of the sequence $(\varphi^n(a))_{n\in\mathbf{N}}$ is a unique infinite word

$$\mathbf{x} = \lim_{n\to\infty} \varphi^n(a) = \varphi^\omega(a) \;\; (= aw\varphi(w)\varphi^2(w)\varphi^3(w)\cdots). \tag{1.10}$$

Such a word is then called a *morphic* (or *substitutive*) *sequence* and we say that it is *generated* by φ. This gives an explanation why φ is called *extendable* on a: we can define $\varphi^\omega(a) = \mathbf{x} \in \mathcal{A}^\omega$ as in (1.10). This allows us to talk about $\varphi(\mathbf{x})$ even though φ is only defined on $\mathcal{A}^\star$. Since $\varphi(\mathbf{x}) = \mathbf{x}$, $\mathbf{x}$ is called a *fixed point* of the extended mapping φ.

Two really old examples we find in Berstel and Perrin (2007) [10, pp. 998–999]. One of them is the first infinite square-free (i.e., not containing two identical factors following immediately after each other) word (defined in 1906) and the second one is the Thue–Morse word (defined in 1912), both introduced and examined by Axel Thue. None of these words is Sturmian. The 1906 word is the unique fixed point u_T of the substitution φ defined on the monoid generated by $\{a, b, c, d\}$ as follows:

$$\varphi(a) = adbcb, \; \varphi(b) = abdcb, \; \varphi(c) = abcdb, \; \varphi(d) = abcbd,$$

i.e., $u_T = \varphi(a)\varphi(d)\varphi(b)\varphi(c)\varphi(b)\cdots = adbcbabcbdabdcbabcdbabdcb\cdots$.

The Thue–Morse word is the binary word w_a which is one of the two fixed points (particularly the one which begins with a) of the substitution ψ defined on the monoid generated by $\{a, b\}$ in the following way:

$$\psi(a) = ab, \; \psi(b) = ba,$$

i.e., $w_a = abbabaabbaababbabaababbaabbabaab\cdots$. This word is *overlap-free* (has no factors of the form $uvuvu$ where u and v are some words and

u is not empty) and can be defined in more than one way; see Berstel and Perrin (2007) [10, pp. 999–1000] and Pytheas Fogg (2002) [69, pp. 35–41].

One of the most well-known examples of a fixed-point theorem for Sturmian words is the result formulated by Shallit in 1991 [75] and, independently, at more or less the same time, by Brown [23], Borel and Laubie [14], and Ito and Yasutomi [45]; see the references in [75]. Shallit's theorem (Theorem 8 below in this thesis) concerns characteristic words of some irrational numbers as fixed points of substitutions. Somewhat later, Crisp et al. (1993) [28] formulated *necessary and sufficient conditions* for the CF-expansion of an irrational number α for the existence of a non-trivial substitution φ which does not change the characteristic word $c(\alpha)$ (then we say that $c(\alpha)$ is a fixed point of φ or that the sequence $c(\alpha)$ is *invariant* under the substitution φ). The results are formulated for both characteristic words and cutting sequences by the lines $y = \alpha x$. Here we recall only the theorem by Shallit, even though it is not in an *if-and-only-if* form as the one in [28] mentioned above, because we use it in Paper III.

Theorem 8 (Shallit, 1991) *Let an irrational number a have a purely periodic CF-expansion, i.e.,*

$$a = [0; a_1, a_2, \ldots, a_r, a_1, a_2, \ldots, a_r, a_1, a_2, \ldots, a_r, \ldots].$$

Define the morphism φ by $\varphi(0) = X_r$, $\varphi(1) = X_r X_{r-1}$, where $(X_n)_{n\in\mathbf{N}}$ are the prefixes of the characteristic word $c(a)$ as defined in Theorem 7. Then the infinite word $c = c(a)$ is a fixed point of φ.

As an example we can take the *Fibonacci word*, the characteristic word with the slope $\theta = [0; 1, 1, 1, \ldots]$. This word is the fixed point of $\varphi(0) = 1$, $\varphi(1) = 10$, beginning with 1 (see Figure 1.12).

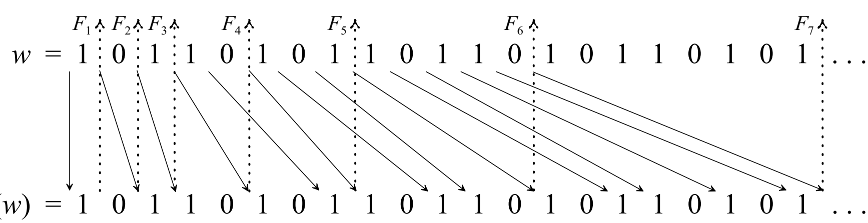

Figure 1.12: The Fibonacci sequence w as the unique fixed point of the substitution $\varphi(0) = 1$, $\varphi(1) = 10$, beginning with 1. For $i = 1, \ldots, 7$, we put 'F_i' above the last letter of the prefix $F_i = \varphi^{i-1}(1)$ as described in the text.

If we denote $F_n = \varphi^{n-1}(1)$, so that $F_1 = 1$ and $F_2 = 10$ then, by definition, the sequence of binary-word length $(|F_n|)_{n\in\mathbf{N}^+}$ and $|F_0| = 1$ forms the well-known *Fibonacci sequence of numbers*. This is where the name of Fibonacci word (w—the infinite limit word $\lim_{n\to\infty} \varphi^{n-1}(1)$ as described by (1.10) or, equivalently, obtained from the prefixes F_n as $\lim_{n\to\infty} F_n$) comes from. We have clearly

$$F_{i+1} = F_i F_{i-1} \quad \text{for} \quad i \geq 1,$$

which is illustrated in Figure 1.12. The 0-1-sequence has thus the same property as the number sequence, but the operation we use instead of addition (as for numbers) is concatenation; see also Karhumäki (2004) [48, p. 8], Pytheas Fogg (2002) [69, p. 7 and pp. 51–52]. In Lagarias (1992) [59, pp. 63–66] one-dimensional quasicrystals (sequences with quasi-periodic structure) are discussed and the Fibonacci word in that context is called a *Fibonacci quasicrystal*. Following Lagarias (1992) [59, p. 64] we can call the process of forming F_n from F_{n-1} *inflation*. The Fibonacci word w is a fixed point of the substitution rules φ, i.e., it is *self-similar* under inflation. We also observe that finite Fibonacci words are standard, since $(F_2, F_1) = (10, 1)$ and, for $n \geq 1$, $(F_{2n+2}, F_{2n+1}) = \Delta\Gamma(F_{2n}, F_{2n-1})$; see Lothaire (2002) [60, p. 64].

1.3.3.2 Fixed-point theorems for operators

The most well-known example of a word which is a fixed point of an operator is the *Kolakoski word*; see Kolakoski (1965) [56] and Pytheas Fogg (2002) [69, p. 93]. The Kolakoski word is defined as one of the two fixed points of the *run-length encoding operator* $\Delta_l \colon \{1,2\}^\omega \to \mathbf{N}^\omega$. The value of this operator on

$$w = \begin{cases} 1^{k_1} 2^{k_2} 1^{k_3} 2^{k_4} \cdots, & \text{if } w \in 1 \cdot \{1,2\}^\omega \\ 2^{k_1} 1^{k_2} 2^{k_3} 1^{k_4} \cdots, & \text{if } w \in 2 \cdot \{1,2\}^\omega \end{cases}$$

is $\Delta_l(w) = k_1 k_2 k_3 \cdots$; see Brlek (1989) [20] and Brlek et al. (2008) [21]. Both fixed points of Δ_l are identical with their own run-length encoding sequences. The one beginning with 2 looks like this:

$$K = 2211212212211211221211212211211212212211212212\cdots.$$

This word is obviously not Sturmian. Brlek et al. have studied some generalizations of the Kolakoski word to an arbitrary alphabet, which were named *smooth words*; see [21] and references there.

Our new CF-description of upper mechanical words with positive irrational slopes less than 1 allows us to formulate an analogous fixed-point theorem, for the *run-construction encoding operator* defined in Paper VI. This theorem concerns only Sturmian words.

2. Summary of papers

As we have seen, the problem of describing the sequence $(\lfloor an \rfloor)_{n\in\mathbf{N}^+}$ for a positive irrational a less than 1 has appeared in many different contexts. The main aim of this thesis is to introduce a new way of classifying and examining upper mechanical words (or, equivalently, digital straight lines) with positive irrational slopes less than 1, according to their run-hierarchical construction. We present our continued-fraction-based method, compare it with some well-known methods of the same kind, and show some ways in which our method can be used in order to reach our aim. We define and examine two equivalence relations on the set of slopes. We formulate and prove a new fixed-point theorem for Sturmian words and describe the set of all fixed points in terms of one of our equivalence relations.

Our journey begins in digital geometry. Papers I and II are about digital lines. Then, in Paper III, we proceed to the combinatorics on words. Papers IV and V are about both lines and words, with a slight domination of lines. And, Paper VI is about Sturmian words.

The presentation of this thesis is scheduled for 25 September 2009, several days after the end of *Words 2009* (the 7th International Conference on Words) and several days before the beginning of *DGCI 2009* (the 15th International Conference on Discrete Geometry for Computer Imagery). That symbolically categorizes the topics covered in this thesis, somewhere between digital geometry and combinatorics on words.

2.1 Paper I

In Paper I we deal with digital half-lines with positive irrational slopes less than 1. For each $a \in]0,1[\setminus \mathbf{Q}$ we introduce *digitization parameters*, which determine the construction of a digital line $y = ax$ in terms of digitization *runs* on all levels.

The concept of *run* was already introduced and explored by Azriel Rosenfeld in 1974 [72, p. 1265]. We call $\mathrm{run}_k(j)$ for $k, j \in \mathbf{N}^+$ a *run of digitization level k*. Each $\mathrm{run}_1(j)$ can be identified with a subset of $\mathbf{Z}^2$

$$\{(i_0 + 1, j), (i_0 + 2, j), \ldots, (i_0 + m, j)\}, \tag{2.1}$$

where m is the length $\|\mathrm{run}_1(j)\|$ of the run; see Figure 2.1.

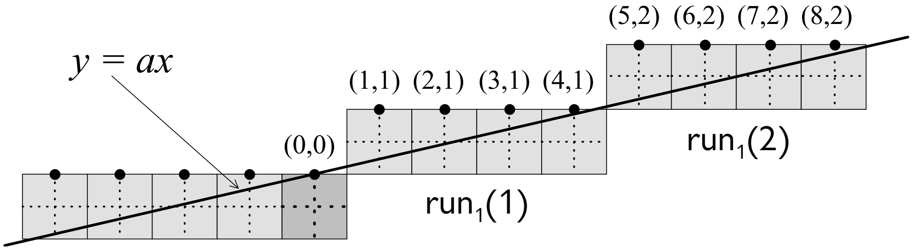

Figure 2.1: Digitization runs of level 1 in the R′-digitization of the line $y = ax$: $\text{run}_1(1)$ and $\text{run}_1(2)$.

For each $a \in \,]0,1[\,\setminus \mathbf{Q}$ we have only two possible run_1 lengths: $\left\lfloor \frac{1}{a} \right\rfloor$ and $\left\lfloor \frac{1}{a} \right\rfloor + 1$. All the runs with one of those lengths always occur alone, i.e., do not have any neighbors of the same length in the sequence $(\text{run}_1(j))_{j\in\mathbf{N}^+}$, while the runs of the other length can appear in sequences. The same holds for the sequences $(\text{run}_k(j))_{j\in\mathbf{N}^+}$ on each level $k \geq 2$. We use the notation $S_k^m L_k$, $L_k S_k^m$, $L_k^m S_k$ and $S_k L_k^m$, when describing the form of digitization runs_{k+1} (where the index $k+1$ refers to the digitization level number $k+1$). For example, $S_k^m L_k$ means that the run_{k+1} (short S_{k+1} or long L_{k+1}) consists of m short runs_k (S_k) and one long run_k (L_k) in that order; see Figure 2.2 and the four-cases formula for P_k in Theorem 9 (p. 41 in this thesis).

At the end of Paper I we can find two pointers (Lemma 3.15 and Theorem 3.16) in the direction of continued fractions (CF).

2.2 Paper II

We continue our work concerning digital lines in Uscka-Wehlou (2008) [83], where we introduce the *index jump function*, which makes it possible to express the results from Paper I in terms of CFs.

Definition 10 *For each positive irrational* a *less than* 1*, the* index jump function $i_a\colon \mathbf{N}^+ \to \mathbf{N}^+$ *is defined by* $i_a(1) = 1,\ i_a(2) = 2$ *and*

$$i_a(k+1) = i_a(k) + 1 + \delta_1(a_{i_a(k)}) \ \text{ for } k \geq 2, \text{ where } \delta_1(x) = \begin{cases} 1, & x = 1 \\ 0, & x \neq 1 \end{cases},$$

and a_j *for* $j \in \mathbf{N}^+$ *are the CF-elements of* a.

In Paper II we show how CF-elements of the slope describe the run-hierarchical structure of a digital line as defined by Azriel Rosenfeld.

In Theorem 9 (p. 2250 in Paper II) we express the digitization parameters defined in Paper I in terms of CF-elements of the slope a of the line and the index jump function corresponding to a.

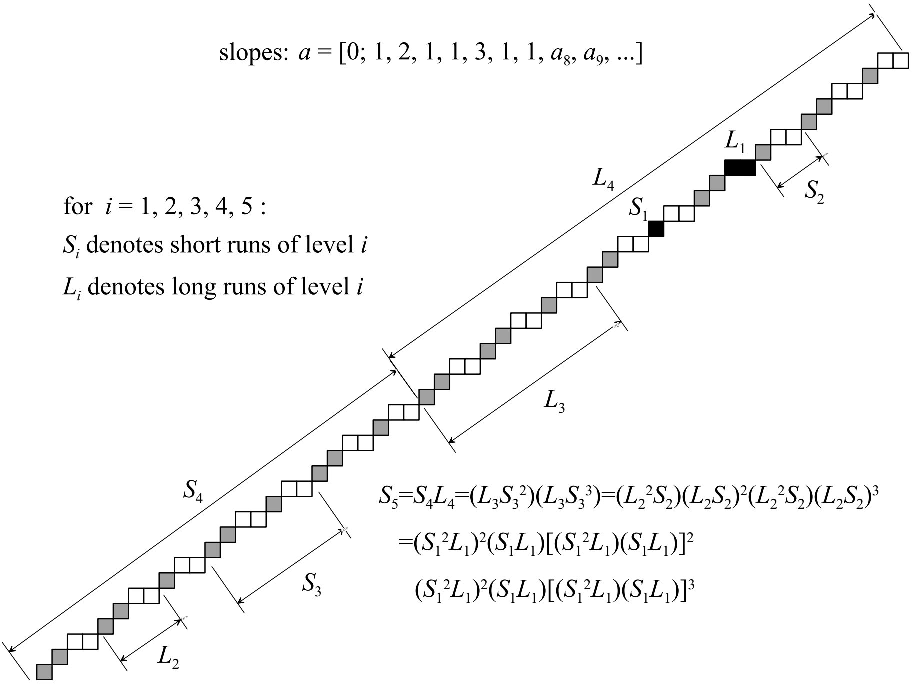

Figure 2.2: Runs of runs of runs of . . . .

In Theorem 10 (p. 2250 in Paper II) we take one more step towards the translation of the description by digitization parameters into a CF based description of runs, formulated in Theorem 11.

Further in Paper II we give some examples of irrational slopes with a periodical CF-expansion (quadratic surds) and irrational slopes with a periodical pattern in their CF-expansions. We describe with exact formulae the digitization runs for lines with such slopes. We also present the link between the digitization parameters and the Gauss map.

2.3 Paper III

Paper III revolves around the run-hierarchical structure of upper mechanical words with irrational positive slopes less than 1. For upper mechanical words, the counterpart of the $\text{run}_1(j) \subset \mathbf{Z}^2$ as described by (2.1) is run 10^{m-1}, when $m-1$ is the number of recurring letter 0 between the letter 1 in the beginning of the run and the next occurring letter 1 in the word. We formulate the following theorem (Theorem 3 in Paper III).

Theorem 9 (a run-hierarchical CF description of $s'(a)$) *Let $a \in]0,1[\setminus \mathbf{Q}$ and $a = [0; a_1, a_2, \ldots]$. For the upper mechanical word $s'(a)$*

we have $s'(a) = \lim_{k\to\infty} P_k$, *where* $P_1 = S_1 = 10^{a_1 - 1}$, $L_1 = 10^{a_1}$, *and, for* $k \geq 2$,

$$P_k = \begin{cases} L_k = S_{k-1}^{a_{i_a(k)}} L_{k-1} & \text{if} \quad a_{i_a(k)} \neq 1 \quad \text{and} \quad i_a(k) \quad \text{is even} \\ S_k = S_{k-1} L_{k-1}^{a_{i_a(k)+1}} & \text{if} \quad a_{i_a(k)} = 1 \quad \text{and} \quad i_a(k) \quad \text{is even} \\ S_k = L_{k-1} S_{k-1}^{-1+a_{i_a(k)}} & \text{if} \quad a_{i_a(k)} \neq 1 \quad \text{and} \quad i_a(k) \quad \text{is odd} \\ L_k = L_{k-1}^{1+a_{i_a(k)+1}} S_{k-1} & \text{if} \quad a_{i_a(k)} = 1 \quad \text{and} \quad i_a(k) \quad \text{is odd,} \end{cases}$$

where i_a *is the index jump function corresponding to* a. *The meaning of the symbols is the following: for* $k \geq 1$, P_k *means* ***P****refix number* k, S_k *means* ***S****hort* run_k *and* L_k *means* ***L****ong* run_k. *To make the recursive formula complete, we add that for each* $k \geq 2$, *if* $P_k = S_k$, *then* L_k *is defined in the same way as* S_k, *with the only difference being that the exponent defined by* $a_{i_a(k)}$ *(or by* $a_{i_a(k)+1}$*) is increased by* 1. *If* $P_k = L_k$, *then* S_k *is defined in the same way as* L_k, *with the only difference being that the exponent defined by* $a_{i_a(k)}$ *(or by* $a_{i_a(k)+1}$*) is decreased by* 1.

The second main result of Paper III (Theorem 6 there) is the following.

Theorem 10 (a quantitative description of runs) *Let* $a \in \,]0,1[\setminus \mathbf{Q}$ *and* $a = [0; a_1, a_2, \ldots]$. *For the word* $s'(a)$ *we have for all* $k \in \mathbf{N}^+$*:*

$$|S_k| = q_{i_a(k+1)-1} \quad \text{and} \quad |L_k| = q_{i_a(k+1)-1} + q_{i_a(k+1)-2},$$

where i_a *is the index jump function,* $|S_k|$ *and* $|L_k|$ *for* $k \in \mathbf{N}^+$ *denote the binary-word length of short, respectively long runs of level* k *as in Theorem 9, and* q_k *are the denominators of the convergents in the CF-expansion of* a.

This theorem allowed us to present a comparison between our method of producing prefixes and two other well-known methods, namely, the method using standard sequences and the method formulated by Bernoulli, proven by Markov, and described in Venkov (1970) [87]. The comparison contained in Paper III shows clearly that our method is different from the two other methods and that the other methods do not reflect the hierarchy of runs as defined by Rosenfeld.

2.4 Papers IV and V

The first three papers are actually a prelude to the real work, which begins with Paper IV. The strength and the originality of our method using CFs lies in the special treatment which we give to some CF-elements equal to 1. This allows us to fully explore the run-hierarchical structure of digital lines or, equivalently, of upper mechanical words. Probably the clearest formulation of this coding of the run-hierarchical structure by CFs can be found in Paper VI. There we find the following corollary which states that the value of the index jump function for each natural $k \geq 2$ describes the index of the CF-element which selects the most frequent run on level $k-1$ (denoted main_{k-1}). The corollary also describes the cardinality-wise run length on each digitization level and shows how to draw conclusion, for each $k \geq 2$, from the parity of $i_a(k)$ about what kind (long L_{k-1} or short S_{k-1}) of prefix P_{k-1} (as obtained in Theorem 9) we get.

Corollary 1 *Let $a \in \,]0,1[\, \setminus \mathbf{Q}$ and $a = [0; a_1, a_2, a_3, \ldots]$. If $s'(a)$ is the upper mechanical word with slope a and intercept 0 then, in the run-hierarchic structure of $s'(a)$, we have for each $k \geq 2$*

- $a_{i_a(k)} \geq 2 \quad \Rightarrow \quad \mathrm{main}_{k-1} = S_{k-1}$,
- $a_{i_a(k)} = 1 \quad \Rightarrow \quad \mathrm{main}_{k-1} = L_{k-1}$,
- $i_a(k)$ *is odd* $\quad \Rightarrow \quad P_{k-1} = L_{k-1}$,
- $i_a(k)$ *is even* $\quad \Rightarrow \quad P_{k-1} = S_{k-1}$,

where i_a is the corresponding index jump function. Moreover, the cardinality-wise run length on each level is the following: $\|S_n\| = b_n$, $\|L_n\| = b_n + 1$, where

$$b_1 = a_1 \ \textit{ and, for } \ n \geq 2, \quad b_n = \begin{cases} a_{i_a(n)}, & a_{i_a(n)} \neq 1 \\ 1 + a_{i_a(n)+1}, & a_{i_a(n)} = 1 \ . \end{cases}$$

This corollary explains the influence of those elements in the CF-expansion $a = [0; a_1, a_2, a_3, \ldots]$ which are indexed by the values of the index jump function corresponding to a on the run-hierarchical structure of the word $s'(a)$ (the line $y = ax$). For each $k \geq 2$, the value of $a_{i_a(k)}$ determines how the runs on level k (runs_k) are constructed of runs on level $k-1$ (runs_{k-1}). As an example, look to the line found in Figure 2.2. Both for the digital lines $y = ax$ and, equivalently, for the words $s'(a)$ (described according to their run-hierarchical construction) with $a = [0; 1, 2, 1, 1, 3, 1, 1, a_8, a_9, \ldots]$ where $a_8, a_9 \geq 2$, we have:

k	$a_{i_a(k+1)} = 1?$	main_k	$i_a(k+1)$	b_k	prefix P_k of $s'(a)$
1	$a_{i_a(2)} = a_2 = 2 \geq 2$	S_1	even	1	$S_1 = 1$
2	$a_{i_a(3)} = a_3 = 1$	L_2	odd	2	$L_2 = S_1^2 L_1 = 1110$
3	$a_{i_a(4)} = a_5 = 3 \geq 2$	S_3	odd	2	$L_3 = L_2^2 S_2$
4	$a_{i_a(5)} = a_6 = 1$	L_4	even	3	$S_4 = L_3 S_3^2$
5	$a_{i_a(6)} = a_8 \geq 2$	S_5	even	2	$S_5 = S_4 L_4$
6	$a_{i_a(7)} = a_9 \geq 2$	S_6	odd	a_8	$L_6 = S_5^{a_8} L_5$

In Paper IV we define two equivalence relations. Equivalence relation $\sim_{\mathrm{len}}$ identifies all the slopes $a \in \,]0,1[\, \setminus \mathbf{Q}$ with the same sequence of length specifications. In the equivalence class under $\sim_{\mathrm{len}}$ defined by

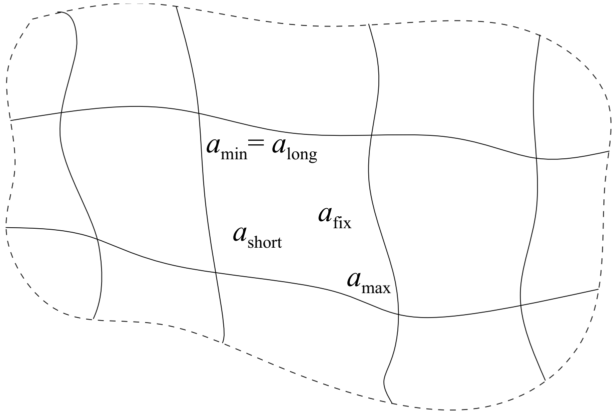

Figure 2.3: Equivalence classes under $\sim_{\text{len}}$; in each class we can distinguish four particularly significant members: a_{max}, $a_{\text{min}} = a_{\text{long}}$, a_{short}, and a_{fix}.

$(b_1, b_2, b_3, \ldots)$ we can distinguish four special slopes, which is illustrated in Figure 2.3. These are:

- $a_{\text{max}} = [0; b_1, b_2, 1, b_3 - 1, 1, b_4 - 1, 1, b_5 - 1, \ldots]$ which is the largest slope in the class; in the run-hierarchical structure of $s'(a_{\text{max}})$ the short run is dominating only on level 1,
- $a_{\text{min}} = a_{\text{long}} = [0; b_1, 1, b_2 - 1, 1, b_3 - 1, 1, b_4 - 1, \ldots]$ which is the smallest slope in the class; on all levels the long run dominates,
- $a_{\text{short}} = [0; b_1, b_2, b_3, b_4, \ldots]$ with the short run dominating on all levels in the run-hierarchical structure,
- a_{fix} such that $\gamma(a_{\text{fix}}) = c(a_{\text{fix}})$, where $\gamma(a_{\text{fix}})$ is the *constructional word* (which will be discussed in Section 2.5 of this thesis) associated with $s'(a_{\text{fix}}) = 1c(a_{\text{fix}})$.

If $(b_n)_{n \in \mathbf{N}^+} = (1, 2, 2, 2, 2, \ldots)$, as in Example 6 in Paper VI, we get:

- $a_{\text{max}} = [0; 1, 2, 1, 1, 1, 1, 1, \ldots] = \frac{\sqrt{5}+3}{\sqrt{5}+5} \approx 0.7236$,
- $a_{\text{min}} = a_{\text{long}} = [0; 1, 1, 1, 1, \ldots] = \frac{\sqrt{5}-1}{2} \approx 0.618$, the slope of the Fibonacci word,
- $a_{\text{short}} = [0; 1, 2, 2, 2, 2, \ldots] = \frac{\sqrt{2}}{2} \approx 0.7071$,
- $a_{\text{fix}} = [0; 1, 1, 1, 2, 1, 1, 1, 1, 2, 1, 1, 1, 1, 2, 1, 1, 2, \ldots] \approx 0.6327822\ldots$.

One can work further with these classes and try to find some relations between the words a_{max}, a_{short}, a_{fix} and $a_{\text{min}} = a_{\text{long}}$. Let us for example calculate how a_{max} and a_{min} depend on each other in the general case. If we denote $A = [0; b_2, 1, b_3 - 1, 1, b_4 - 1, 1, b_5 - 1, \ldots]$, then (by Lemma 5 from Paper II) $[0; 1, b_2 - 1, 1, b_3 - 1, 1, b_4 - 1, 1, b_5 - 1, \ldots] = 1 - A$, thus

$$a_{\text{max}} = \frac{1}{b_1 + A}, \quad a_{\text{min}} = \frac{1}{b_1 + 1 - A}. \tag{2.2}$$

This gives

$$\frac{1}{a_{\min}} + \frac{1}{a_{\max}} = 2b_1 + 1, \quad \text{thus} \quad a_{\min} = \frac{a_{\max}}{(2b_1 + 1)a_{\max} - 1}. \tag{2.3}$$

The second equivalence relation defined in Paper IV is $\sim_{\text{con}}$, based on a run construction in terms of long and short runs on all levels. We group in classes the lines with the same type (short or long) of runs dominating on corresponding levels.

No equivalence class under $\sim_{\text{con}}$ has a least element. The infimum in each class is equal to zero. The answer related to the greatest elements is much more interesting. The partition of all the irrational numbers from the interval $]0, 1[$ into equivalence classes under $\sim_{\text{con}}$ gives sets with suprema equal to the odd-numbered convergents of the Golden Section, thus with no largest element belonging to the class (which is a set of irrational numbers). The only exception is the class generated by $(s_n)_{n \in \mathbf{N}^+} = (2n)_{n \in \mathbf{N}^+}$, which has a greatest element and it is equal to the Golden Section.

In Paper V we take another look at the equivalence relation defined by run construction, this time independently of the context. This gives a general result for CF-expansions of $a \in]0, 1[\setminus \mathbf{Q}$.

2.5 Paper VI

In Paper VI, the last paper in this thesis, we define the *run-construction encoding operator* Δ_c (Definition 6 in Paper VI), analogous to the well-known *run-length encoding operator* Δ_l mentioned on p. 38 in this thesis. The definition of the former is slightly more complicated than the definition of the latter. In order to introduce it, we have to define a constructional word $\gamma(a)$, a new binary word associated with a positive irrational a less than 1. The n^{th} letter of the constructional word is:

- the letter 1 if the dominating run in the run-hierarchical construction of $s'(a)$ on level n is the long run,
- the letter 0 if the dominating run in the run-hierarchical construction of $s'(a)$ on level n is the short one.

The Sturmian word $s'(a)$ is a fixed point of the run-construction encoding operator if the corresponding characteristic word $c(a)$ is equal to the constructional word $\gamma(a)$. Such a word is then called a *Sturmian word with self-balanced construction*. In Paper VI we show that in each equivalence class under $\sim_{\text{len}}$ there exists exactly one fixed point of the run-construction encoding operator. Theorem 4 in Paper VI is a new fixed-point theorem for Sturmian words.

3. Sammanfattning på svenska

Digitala linjer, sturmianska ord och kedjebråk

I centrum av denna avhandling står vissa heltalsföljder. Dessa bildas på följande sätt: vi tar ett positivt irrationellt tal a mindre än 1 och letar efter de heltal $d_m(a)$ för $m = 1, 2, 3, \ldots$ som bäst approximerar motsvarande multipel am för $m = 1, 2, 3, \ldots$. Det betyder att vi söker en följd $(d_m(a))_{m \in \mathbf{N}^+}$ sådan att för varje positivt heltal m

$$-\tfrac{1}{2} \leq d_m(a) - am < \tfrac{1}{2}, \quad \text{alltså} \quad d_m(a) = \lfloor am + \tfrac{1}{2} \rfloor, \tag{3.1}$$

där $\lfloor x \rfloor$ för $x \in \mathbf{R}$ betyder avrundningen nedåt till närmaste heltal. Denna fråga ställdes först 1772, då astronomen Johann III Bernoulli analyserade en tabell av proportionella delar. Det är tydligt att vi, medan vi bygger upp följden $(d_m(a))_{m \in \mathbf{N}^+}$, i varje steg adderar antingen 0 (som är lika med $\lfloor a \rfloor$) eller 1 till det föregående elementet i följden. Bernoulli formulerade vissa regler som gäller för att man skall veta i vilket steg elementet ökar och i vilket det förblir oförändrat. Dessa regler hänger nära ihop med kedjebråkutvecklingen av talet a, vilket, av vissa skäl som blir tydliga snart, kallas för *lutningen*. Bernoulli bevisade inte formellt sina påståenden, men det gjorde A. Markov 1882. Han undersökte en homogen talföljd

$$h_m(a) = \lfloor am \rfloor \tag{3.2}$$

i stället (vilket vi också gör i denna avhandling). Beskrivningen av problemet och dess lösning publicerades på engelska 1970, i översättningen av Venkov's *Elementary number theory*. Där, på sidan 67, hittar vi följande exempel. Om vi tar $a = \sqrt{2} - 1$, så får vi följande tabell:

m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\lfloor am \rfloor$	0	0	1	1	2	2	2	3	3	4	4	4	5	5	6

m	16	17	18	19	20	21	22	23	24	25	26	$\cdots$
$\lfloor am \rfloor$	6	7	7	7	8	8	9	9	9	10	10	$\cdots$

Först får vi alltså ett *block* av nollor (blockets längd är två), efteråt ett block av ettor (två), tvåor (tre), treor (två), fyror (tre), femmor (två), sexor (två), sjuor (tre), åttor (två), nior (tre), tior (två), o.s.v. Redan

Bernoulli upptäckte att hur blocken för ett visst a ser ut beror på kedjebråkutvecklingen av a. I detta fall har vi

$$a = \sqrt{2} - 1 = \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\cdots}}}} = [0; 2, 2, 2, \ldots] = [0; a_1, a_2, a_3, \ldots]$$

och därför, eftersom $a_1 = 2$, får vi blocklängden 2 eller 3. Inte nog med det, blocken med längd 3 är alltid ensamma i längdföljden $(2, 2, 3, 2, 3, 2, 2, 3, 2, 3, \ldots)$. Grupperar vi nu alla tvåor som kommer efter varann i följden av längder tillsammans med den trea som följer efter dem i ett block av block, så får vi en ny följd,

$$(\{2, 2, 3\}, \{2, 3\}, \{2, 2, 3\}, \{2, 3\}, \{2, 2, 3\}, \{2, 3\}, \{2, 3\}, \{2, 2, 3\}, \{2, 3\}, \ldots),$$

följden av blockens block, vilken har samma egenskaper som den ursprungliga följden! Detta heter på engelska *self-similarity*—följden är konstruerad enligt regler som liknar följden själv. Även i följden av blockens block har vi två olika block, i vårt exempel $\{2, 2, 3\}$ som är alltid ensam i följden, och $\{2, 3\}$ som kan komma med flera efter varann. I denna andra, deriverade följd har blocken av block längder definierade av $a_2 = 2$ och $a_2 + 1 = 3$. Med *längden* menar man här antalet block i blockens block, alltså kardinaliteten av mängden av block (så har vi till exempel att längden av $\{2, 2, 3\}$ är tre). Undersöker man en större del av följden, märker man att detsamma som gäller block, tillämpas även på blockens block (på engelska, *runs of runs*), block av blockens block, o.s.v. Vi kan skriva block$_1$, block$_2$, block$_3$, o.s.v. i stället, för att antyda med siffran på vilken nivå i hierarkin av block vi befinner oss. Grupperar vi block$_2$ som maximala delar $\{\{2, 2, 3\}, \{2, 3\}\}$ och $\{\{2, 2, 3\}, \{2, 3\}, \{2, 3\}\}$, får vi en följd av block$_3$, igen med samma egenskaper (två olika block$_3$-längder definierade av $a_3 = 2$ och $a_3 + 1 = 3$, långa block$_3$ alltid ensamma i följden, medan korta block$_3$ kan komma med flera efter varann). I vissa fall är det långa block$_k$ som kommer med flera efter varann. Detta händer om kedjebråkelementet som bär information om nivå k är lika med 1. Detta analyserar vi noggrant i våra artiklar.

Följderna som vi just har beskrivit har åtnjutit enormt stor uppmärksamhet av forskare på många olika områden de senaste sjuttio åren. Ett av dessa områden är digital geometri och objekt som beskrivs med hjälp av sådana följder kallas digitala linjer. På bild 3.1 ser vi ett exempel på den digitaliserade räta linjen $y = ax$, där $a = \sqrt{2} - 1 = [0; 2, 2, 2, \ldots]$. Vi observerar likheten med tabellen ovan i denna sammanfattning. På bilden kan man även se hierarkin av korta och långa block (*run hierarchy*) på alla nivåer.

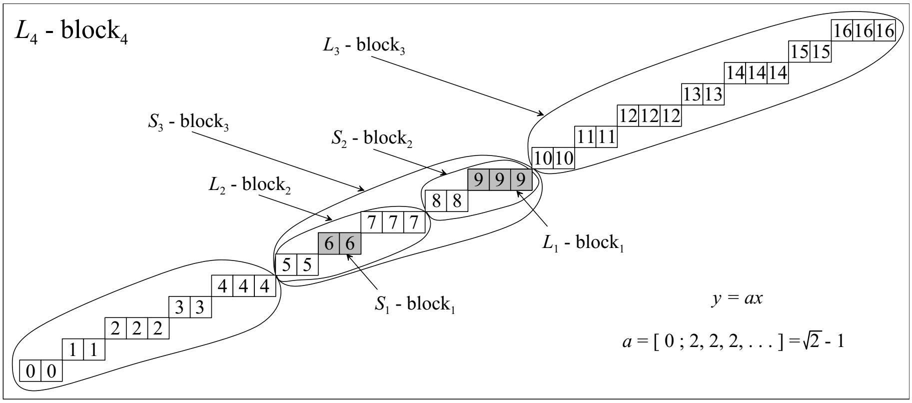

Figure 3.1: Digitaliserad linje $y = ax$ där $a = \sqrt{2} - 1 = [0; 2, 2, 2, \ldots]$. På bilden kan man se hierarkin av block. Block på nivå k (för $k = 1, 2, 3, 4$) markeras med block$_k$; L_k står för ett *långt block*, S_k betyder ett *kort block* (från engelska L—long och S—short).

Begreppet *run* i konstruktion av digitala linjer introducerades av Azriel Rosenfeld 1974. Sambandet mellan block-hierarkin och kedjebråk uppmärksammades så gott som omedelbart och påpekades av Brons redan samma år. Sedan dess har det gjorts många olika både algoritmiska och teoretiska beskrivningar av digitala linjer med hjälp av kedjebråk.

Formler som beskriver den block-hierarkiska strukturen med hjälp av kedjebråk är användbara inom datagrafik. De tillåter oss att producera snabba och effektiva algoritmer för att rita räta linjer på skärmen.

Om vi nu undersöker följden av skillnader i (3.2)

$$c_m(a) = h_{m+1}(a) - h_m(a) = \lfloor a(m+1) \rfloor - \lfloor am \rfloor \tag{3.3}$$

i stället för själva $(h_m(a))_{m \in \mathbf{N}^+}$, så får vi det *karakteristiska ordet* för a i stället för digitala linjer. *Ordteorin* är det andra området där man arbetade med de följder som vi beskriver i denna avhandling. Följder i (3.3) för irrationella lutningar a formar så kallade *sturmianska ord*, som har sina rötter i symbolisk dynamik och som har många ekvivalenta definitioner av olika karaktär (geometriska, kombinatoriska, analytiska).

I den engelskspråkiga delen av kappan har vi förklarat i detalj varför det är viktigt att undersöka följderna i (3.1)–(3.3). I artiklarna som följer (Papers I–VI) presenterar vi våra egna resultat som alla är relaterade till sådana följder. Vi presenterar en ny beskrivning av digitala linjer och sturmianska ord som är baserad på kedjebråkutvecklingen av lutningen och som reflekterar den hierarkiska strukturen av blocken som definierades av Rosenfeld. Det som gör vår lösning unik är den speciella roll som vissa kedjebråkelement lika med 1 i kedjebråkutvecklingen får i formeln. Allt styrs av den så kallade *index-hopp-funktionen*, på engelska *index jump function.*

Definition *För varje positivt irrationellt tal a mindre än* 1*, definieras* index-hopp-funktionen $i_a\colon \mathbf{N}^+ \to \mathbf{N}^+$ *på följande sätt:*
$i_a(1) = 1,\ i_a(2) = 2,$ *och*

$$i_a(k+1) = i_a(k) + 1 + \delta_1(a_{i_a(k)}) \text{ för } k \geq 2, \text{ där } \delta_1(x) = \begin{cases} 1, & x = 1 \\ 0, & x \neq 1 \end{cases},$$

och a_j *för* $j \in \mathbf{N}^+$ *är kedjebråkselementen för* a.

Vi visar på vilket sätt man kan avläsa den hierarkiska strukturen av räta digitala linjer (sturmianska ord) från kedjebråkselement för lutningen och med hjälp av index-hopp-funktionen. Vi definierar två ekvivalensrelationer som delar upp mängden av lutningar $]0,1[\setminus \mathbf{Q}$ i ekvivalensklasser. En uppdelning är definierad med hjälp av blocklängd på varje nivå (med detta menas längden av block_k för varje $k \in \mathbf{N}^+$). Den andra uppdelningen grupperar ihop alla lutningar som leder till samma konstruktion av digitala linjer (sturmianska ord). Vi presenterar några satser om minimala och maximala element i alla klasser för båda ekvivalensrelationerna.

I den sista artikeln formulerar vi och bevisar en ny fixpunktssats för sturmianska ord. Fixpunkter är sådana ord som själva innehåller kodningen för sin hierarkiska struktur. Vi visar att det finns exakt en fixpunkt i varje klass under ekvivalensrelationen definierad av blocklängden på alla nivåer.

4. Acknowledgments

I would like to express my deepest gratitude towards my main advisor, Professor Christer Kiselman, for generously sharing his knowledge, wisdom, and time with me and other members of our research group, Shiva Samieinia, Erik Melin, Robin Strand, and Ola Weistrand. Christer's attitude for learning in all situations of life and his open mind towards new people and new domains of science have been very inspiring for me. It has been a real privilege to work together with you, Christer.

My thanks also go to Professor Maciej Klimek, who took over the function of my main supervisor after the official retirement of Christer. He has always been there for me when I asked him for help and always found the time in his busy schedule for a short, but efficient and constructive conversation.

To be able to begin my Ph.D. studies, I first of all needed to find someone at the university who would believe in me. The first one was Christer; then I also got strong support from Professor Gunilla Borgefors (CBA, Centre for Image Analysis in Uppsala) and from Professor Mikael Passare (Stockholm University). They also agreed to be my advisors. I would like to thank them for their confidence in my ability to succeed.

My Ph.D. studies were financially supported by the Graduate School in Mathematics and Computing (FMB). FMB not only sponsored the purchasing of my books and participation in conferences, but also gave me and other students the opportunity of following courses especially designed for us.

I am grateful to Christer Kiselman, Martin Wehlou, David Schnur, Kim Nevelsteen, Edward Ochmański, and Valérie Berthé for proofreading and commenting on drafts of this thesis or its constituent papers. My thanks to Professor Mariusz Lemańczyk from the Nicolaus Copernicus University in Toruń for our e-mail discussions about my scientific future after my Ph.D. studies. Thanks to all of you for your time and the attention you gave me!

This thesis is also (indirectly) a fruit of the work of the teachers I had in Poland. I would especially like to mention Mirosław Uscki (my father and math teacher), Zbigniew Bobiński, and Paweł Jarek from the Nicolaus Copernicus University of Toruń. They provided me with a strong base of mathematical knowledge, taught me to respect the strictness of mathematics, and to enjoy solving mathematical problems.

Professor Andrzej Granas from the Nicolaus Copernicus University supported me in the most difficult moment of my studies when I left Poland and wrote my master thesis in Belgium. He used to tell us stories about great Polish mathematicians from the celebrated Polish School of Topology and Analysis and tried to convince us that we are not only able to understand English, but that even French is not a problem.

If it hadn't been for Lieve Derycke, this thesis would not have been written. She chased me out of the carpet factory in Belgium where I had been working for more than two years, to go get a life. I only wish more people had a Lieve to make them leave their carpet factories.

I would like to thank Wanda Uscka (my mother), Helena Kanthak (my aunt), and Ilona Cichoń (a.k.a. *Pani Marciniak*, my history teacher), for showing their belief in my abilities already a long, long time ago.

Thanks to my daughter Milena and my son Julian, for distracting me from my work and reminding me that there is a life outside of mathematics. To our friend Kim Nevelsteen and to my sister Maria for coming to Sweden and continuing their education here at Uppsala University. It gave us an opportunity to spend a lot of time during my Ph.D. studies in their company, having fun together and supporting each other. Special thanks to Kim for introducing psychedelic-trance music to us. This thesis is entirely (both the introductory part and all the papers) written with it continuously playing in the background. Even now, when I am writing these words, I am listening to this music.

Last, but not least, my gratitude goes to Martin Wehlou, my husband. He is the permanent source of my inspiration and joy. With his love and acceptance anything is possible. His mere presence would force stones to creativity and, in this way, without actually trying to make it so, he is the main responsible for the existence of this thesis.

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Paper I

PAPER I, ERRATA

1. p. 168, line 21
   is: twice
   should be: and $n_{i+1} - 1$

2. p. 169, line 4
   is: $a = [0, k_1, \ldots, k_i, 1, 1, k_{i+2}, \ldots, k_n]$
   should be: $a = [0, k_1, \ldots, k_i, 1, k_{i+1} - 1, k_{i+2}, \ldots, k_n]$

3. p. 169, item [7] in References
   is: calculus
   should be: calcul

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Theoretical Computer Science
www.elsevier.com/locate/tcs

Digital lines with irrational slopes

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Received 28 March 2005; received in revised form 29 January 2007; accepted 18 February 2007

Communicated by E. Pergola

Abstract

How to construct a digitization of a straight line and how to be able to recognize a straight line in a set of pixels are very important topics in computer graphics. The aim of the present paper is to give a mathematically exact and consistent description of digital straight lines according to Rosenfeld's definition. The digitizations of lines with slopes $0 < a < 1$, where a is irrational, are considered. We formulate a definition of digitization runs, and formulate and prove theorems containing necessary and sufficient conditions for digital straightness. The proof was successfully constructed using only methods of elementary mathematics. The developed and proved theory can be used in research into the theory of digital lines, their symmetries, translations, etc.

Keywords: Digital geometry; Theory of digital lines; Irrational slope; Continued fractions

1. Introduction

Our aim here is to give a mathematically exact and consistent description of digital straight lines according to Rosenfeld's definition [8]. We will consider the digitizations of lines with slopes $0 < a < 1$ where a is irrational. The theory for such lines appears to be very elegant and simple. When treating rational slopes together with irrational, however, we are forced to deal with special cases and exceptions which would make the theory less clear.

A detailed review on digital straightness can be found in Rosenfeld and Klette [9]. Necessary and sufficient conditions for digital straightness are formulated there; see for example Wu's theorem from 1982 (Theorem 3.5 in Rosenfeld and Klette [9]). Different approaches and kinds of proofs (algorithms, using word theory, etc.) are also discussed there.

There has been done a lot of research concerning digital straightness lately; see for example Reveillès [7], Debled [2] and Vittone [12]. They describe digital lines with rational slopes. Lines with irrational slopes, however, have not got enough attention in scientific papers. There are very few researchers dealing with this subject. Some of them have used the link between combinatorics on words and digital lines and planes; see Arnoux et al. [1] and Jamet [4]. We present a description of digital lines with irrational slopes without using any advanced theories.

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doi:10.1016/j.tcs.2007.02.037

Stephenson and Litow [10,11] have described fast algorithms for drawing digital lines with rational slopes. Although the present paper covers the theory for lines with irrational slopes, one can easily use it as a basis for the formal proof of the results for lines with rational slopes presented by them.

The central role in the construction of the theory presented here is played by Lemma 3.6. The most important definitions are Definitions 3.4 and 3.7. The main results are formulated in Theorem 3.13 and Corollary 3.14 (necessary conditions to be a digital line with irrational slope). The corollary is more practically useful than the theorem itself. A sufficient condition to be a digital line with irrational slope is formulated in Theorem 3.17.

The proof of the necessary condition for digital straightness is based on as elementary mathematics as possible, without resorting to algorithms.

2. Rosenfeld's digitization

Rosenfeld's definition of the digitization of a straight line can be presented as follows. See also Rosenfeld [8] and Melin [6].

Rosenfeld's plane can be identified with $\mathbf{Z}^2$. With each point (k, n) of this plane we can associate the following two subsets of $\mathbf{R}^2$:

$$S_R(k,n) = \left]k - \frac{1}{2}, k + \frac{1}{2}\right] \times \left]n - \frac{1}{2}, n + \frac{1}{2}\right]$$

and

$$C_R(k,n) = \left(\{k\} \times \left]n - \frac{1}{2}, n + \frac{1}{2}\right]\right) \cup \left(\left]k - \frac{1}{2}, k + \frac{1}{2}\right] \times \{n\}\right).$$

We will call them *R-squares* and *R-crosses* in (k, n) respectively. One can easily see that the R-squares form a partition of $\mathbf{R}^2$, i.e.:

$$\mathbf{R}^2 = \bigcup_{(k,n)\in\mathbf{Z}^2} S_R(k,n), \quad \text{and}$$
$$(k_1, n_1) \neq (k_2, n_2) \;\Rightarrow\; S_R(k_1, n_1) \cap S_R(k_2, n_2) = \emptyset.$$

Rosenfeld's digitization of a straight line l (which we will denote by $D_R(l)$) is the set of all (k, n) in $\mathbf{Z}^2$ for which the intersection of l and $C_R(k, n)$ is not empty:

$$D_R(l) = \{(k,n) \in \mathbf{Z}^2;\;\; l \cap C_R(k,n) \neq \emptyset\}.$$

For some lines, such as $y = x + \frac{1}{2}$, we obtain thick digitizations which can be adjusted to one pixel thin lines (*naive lines* according to Reveillès [7]) by elimination of some pixels; see Melin [6, Section 1] and Kiselman [5, Theorem 6.1].

We will discuss the digitization of the positive half line only, i.e., the digitization of $y = ax$ where $x > 0$ (*rays* in Rosenfeld and Klette [9]), since the digitization of the negative half line can be derived by symmetries.

It is worth mentioning that the slope is the most important feature characterizing a digital line:

- Two lines $y = a_1x + b_1$ and $y = a_2x + b_2$ where $a_1 \neq a_2$ cannot have the same digitization, because $|a_1x + b_1 - (a_2x + b_2)| \to \infty$ when $x \to \infty$. The slope is thus determined by the digitization and this is why we can say that a digital line has a slope.
- Two lines $y = ax + b_1$ and $y = ax + b_2$, where $b_1 \neq b_2$, can have the same digitization, like for example lines $y = \frac{2}{5}x$ and $y = \frac{2}{5}x + \frac{1}{40}$. Parallel translated lines cannot always be distinguished in their digitized form.

Exact description of those two items can be found in Rosenfeld and Klette [9], formulated in Theorem 1.2 (theorem of Bruckstein).

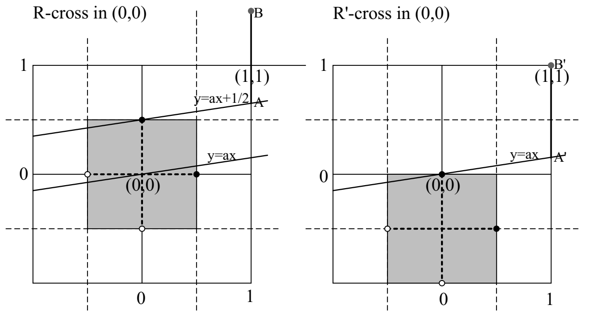


Fig. 1. R-cross and R′-cross in (0, 0). $AB = A'B'$, so $D_{R'}(y = ax) = D_R(y = ax + \frac{1}{2})$.

3. The necessary and sufficient conditions

We are mainly interested in straight lines with an irrational slope between 0 and 1 which pass through the origin, i.e., lines $y = ax$ where $0 < a < 1$ and a is irrational. Digitizations of lines with irrational slopes $a < 0$ and $a > 1$ can be obtained by a change of coordinates; see Rosenfeld [8].

In order to make it easier to handle descriptions and equations, we will modify the definition of the R-digitization by changing the definitions of R-squares and R-crosses in the following way:

$$S_{R'}(k,n) = \left]k - \frac{1}{2}, k + \frac{1}{2}\right] \times]n-1, n] = S_R\left(k, n - \frac{1}{2}\right)$$

and

$$C_{R'}(k,n) = (\{k\} \times]n-1, n]) \cup \left(\left]k - \frac{1}{2}, k + \frac{1}{2}\right] \times \left\{n - \frac{1}{2}\right\}\right) = C_R\left(k, n - \frac{1}{2}\right).$$

We call these *R′-squares* and *R′-crosses* respectively. Then we define the R′-digitization of line l as follows:

$$D_{R'}(l) = \{(k,n) \in \mathbf{Z}^2;\ l \cap C_{R'}(k,n) \neq \emptyset\} = \{(k, \lceil ak \rceil);\ k \in \mathbf{Z}\}.$$

Fig. 1 shows a comparison of the two digitizations.
The R′-digitization of the line with equation $y = ax$ is equal to the R-digitization of $y = ax + \frac{1}{2}$:

$$(k,n) \in D_R\left(y = ax + \frac{1}{2}\right) \Leftrightarrow n - \frac{1}{2} < ak + \frac{1}{2} \leqslant n + \frac{1}{2} \quad \Leftrightarrow \quad n - 1 < ak \leqslant n$$
$$\Leftrightarrow (k,n) \in D_{R'}(y = ax).$$

This is also illustrated in Fig. 1.

If $0 < a < 1$, then $f(x) = ax$ is a function and it is increasing, so the R′-digitization of line l with equation $y = ax$ consists of horizontal *runs*:

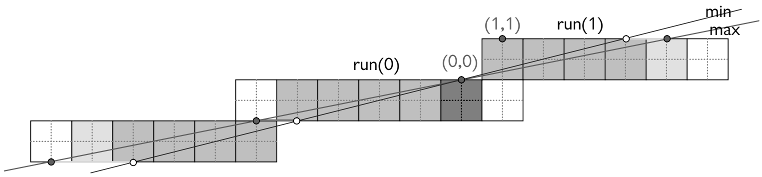
$$\text{run}(n) = \{(k,n) \in D_{R'}(l)\} = \{(k,n) \in \mathbf{Z}^2;\ n - 1 < f(k) \leqslant n\}$$

(hence $\lceil f(k) \rceil = n$), where the second coordinate gives an enumeration of R′-digitization runs. We can also talk about the first, second, ..., last element of a run, using the order in $\mathbf{Z}$ on the first coordinate. For example, the last element of run(0) is (0, 0), the first element of run(1) is (1, 1), since $a \in]0, 1[$ (see Fig. 2).

We define the *length of a run* as the number of its elements, thus its cardinality card(run(n)).

First we will describe the R′-digitization on the level of runs as defined above. From now on, when we write *digitization*, we refer to the R′-digitization. Because we only analyze straight lines $y = ax$ (where $0 < a < 1$, and a is irrational) for $x > 0$, we begin the description of the digitization with run(1). We use the notation $\mathbf{N}^+ = \mathbf{N} \setminus \{0\}$.

The following lemma is useful for further calculations:



Run(0) with length 5 belongs to the R'-digitizations of the lines y=ax lying between the two lines as on the picture.
The only possible lengths of run(1) in the digitizations containing run(0) defined above are 4 or 5.

Fig. 2. Digitization runs.

Lemma 3.1. *If $\sigma \neq 0$, then for every number $i \in \mathbf{N}^+$ the value of $\lfloor \frac{i}{\sigma} \rfloor - \lfloor \frac{i-1}{\sigma} \rfloor$ is one of two consecutive natural numbers $\lfloor \frac{1}{\sigma} \rfloor$ and $\lfloor \frac{1}{\sigma} \rfloor + 1$.*

We observe that $\lfloor \frac{i}{\sigma} \rfloor$ is increasing (or decreasing, if $\sigma < 0$) on average like $\frac{i}{\sigma}$ (i.e., we have $\lfloor \frac{i}{\sigma} \rfloor / \frac{i}{\sigma} \to 1$ when $i \to \infty$), thus the average of $\lfloor \frac{i}{\sigma} \rfloor - \lfloor \frac{i-1}{\sigma} \rfloor$ over intervals $[1, k]_{\mathbf{Z}}$ with $k \to \infty$ is $\frac{1}{\sigma}$, meaning

$$\lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} \left(\left\lfloor \frac{i}{\sigma} \right\rfloor - \left\lfloor \frac{i-1}{\sigma} \right\rfloor \right) = \frac{1}{\sigma}.$$

Lemma 3.1 says that $\lfloor \frac{i}{\sigma} \rfloor - \lfloor \frac{i-1}{\sigma} \rfloor$ for $i \in \mathbf{N}^+$ can have only one of the two possible values: $\lfloor \frac{1}{\sigma} \rfloor$ and $\lfloor \frac{1}{\sigma} \rfloor + 1$. This means that $\lfloor \frac{i}{\sigma} \rfloor - \lfloor \frac{i-1}{\sigma} \rfloor$ takes the value of $\lfloor \frac{1}{\sigma} \rfloor$ and $\lfloor \frac{1}{\sigma} \rfloor + 1$ with such frequencies that the average is $\frac{1}{\sigma}$. This implies that $\lfloor \frac{1}{\sigma} \rfloor$ must appear with frequency $1 - \mathrm{frac}(\frac{1}{\sigma})$ and $\lfloor \frac{1}{\sigma} \rfloor + 1$ with frequency $\mathrm{frac}(\frac{1}{\sigma})$, because

$$\left(1 - \mathrm{frac}\left(\frac{1}{\sigma}\right)\right) \left\lfloor \frac{1}{\sigma} \right\rfloor + \mathrm{frac}\left(\frac{1}{\sigma}\right) \left(\left\lfloor \frac{1}{\sigma} \right\rfloor + 1 \right) = \frac{1}{\sigma}.$$

By frequency of value $\lfloor \frac{1}{\sigma} \rfloor$ (resp. $\lfloor \frac{1}{\sigma} \rfloor + 1$) we mean the limit (when $k \to \infty$) of the number of these $i \in [1, k]_{\mathbf{Z}}$ for which $\lfloor \frac{i}{\sigma} \rfloor - \lfloor \frac{i-1}{\sigma} \rfloor = \lfloor \frac{1}{\sigma} \rfloor$ (resp. $\lfloor \frac{1}{\sigma} \rfloor + 1$) divided by k. Expressed symbolically, the frequency of value $\lfloor \frac{1}{\sigma} \rfloor$ is

$$\lim_{k \to \infty} \frac{1}{k} \mathrm{card}(S(k)), \quad \text{where} \quad S(k) = \left\{ i \in [1, k]_{\mathbf{Z}};\ \left\lfloor \frac{i}{\sigma} \right\rfloor - \left\lfloor \frac{i-1}{\sigma} \right\rfloor = \left\lfloor \frac{1}{\sigma} \right\rfloor \right\}.$$

Later (in Lemma 3.6) we will indicate in detail for which i we get which values of $\lfloor \frac{i}{\sigma} \rfloor - \lfloor \frac{i-1}{\sigma} \rfloor$.

Proof. For each $x, y \in \mathbf{R}$ we have:

$$\lfloor x + y \rfloor = \begin{cases} \lfloor x \rfloor + \lfloor y \rfloor & \text{if } \mathrm{frac}(x) + \mathrm{frac}(y) < 1 \\ \lfloor x \rfloor + \lfloor y \rfloor + 1 & \text{if } \mathrm{frac}(x) + \mathrm{frac}(y) \geqslant 1. \end{cases}$$

Taking $x = \frac{i-1}{\sigma}$ and $y = \frac{1}{\sigma}$, we get the assertion of the lemma. □

We can use Lemma 3.1 for the proof of the following proposition about the digitization runs:

Proposition 3.2. *For the digitization of the half line $y = ax$ (where $x > 0$, a is irrational and $0 < a < 1$) we have:*

1. *The length of the* $\mathrm{run}(j)$ *for $j \in \mathbf{N}^+$ is equal to $\lfloor \frac{j}{a} \rfloor - \lfloor \frac{j-1}{a} \rfloor$.*
2. *There are exactly two run lengths in the digitization: $\lfloor \frac{1}{a} \rfloor$ (short runs) and $\lfloor \frac{1}{a} \rfloor + 1$ (long runs).*
3. *The first run is short.*

Proof. In this proof, i counts the elements within runs, j counts the runs. Let $j \in \mathbf{N}^+$ be given. We examine the function $f(x) = ax$ for all integer arguments greater than or equal to 1, which we will call i (thus $i \in \mathbf{N}^+$). According to the definition of the R′-digitization, we have:

$$(i, j) \in \mathrm{run}(j) \quad \Leftrightarrow \quad j - 1 < f(i) \leqslant j \quad \Leftrightarrow \quad \frac{j-1}{a} < i \leqslant \frac{j}{a} \quad \Leftrightarrow \quad \left\lfloor \frac{j-1}{a} \right\rfloor < i \leqslant \left\lfloor \frac{j}{a} \right\rfloor$$

(the second equivalence we get because $a > 0$, the third one because $i \in \mathbf{Z}$). This means that the run(j) for $j \in \mathbf{N}^+$ begins in $(\lfloor \frac{j-1}{a} \rfloor + 1, j)$ and ends in $(\lfloor \frac{j}{a} \rfloor, j)$, and this means that the length of run(j) for $j \in \mathbf{N}^+$ is equal to $\lfloor \frac{j}{a} \rfloor - \lfloor \frac{j-1}{a} \rfloor$, which proves assertion 1. In particular, for $j = 1$: the first run begins in $(1, 1)$ and ends in $(\lfloor \frac{1}{a} \rfloor, 1)$, so its length is $\lfloor \frac{1}{a} \rfloor$. This means that the first run is short for all a, which proves assertion 3. Assertion 2 of the proposition follows now from Lemma 3.1, by replacing σ with a. □

Our aim in this paper is a full description of the digitization of a given straight half line l $(x > 0)$ with equation $y = ax$, where $0 < a < 1$ and a is irrational. The first level of digitization has already been discussed. The notion of digitization level k will be formulated later. The *digitization parameters*, which will be defined now, are sufficient to derive a complete description of the digitization of the line they come from. In the definition of the digitization parameters we will use the following *modification operation* $\cdot^\wedge : [0, 1] \to [0, \frac{1}{2}]$:

Definition 3.3. For $t \in [0, 1]$ we define $t^\wedge = \min(t, 1 - t)$.

Definition 3.4. For $y = ax$ where $0 < a < 1$ and a is irrational, the *digitization parameters* are:

$\sigma_1 = \operatorname{frac}(\frac{1}{a})$,
$\sigma_k = \operatorname{frac}\left(1/\sigma_{k-1}^\wedge\right)$ for all natural $k > 1$.

For $j \in \mathbf{N}^+$, σ_j and $\sigma_j^\wedge$ are the *digitization parameters* and *modified digitization parameters* of the *digitization level* j respectively.

For an irrational slope a there exist parameters σ_j for all $j \in \mathbf{N}^+$. We have $0 < \sigma_j < 1$ and σ_j is irrational for all $j \in \mathbf{N}^+$. The definition of σ_1 differs from the definition of σ_j for natural $j \geqslant 2$, since digitization runs of the first level as described in Proposition 3.2 are built of elements of *one* kind (elements of $\mathbf{Z}^2$) while the runs on all the following digitization levels will be composed of *two* kinds of element (short and long). We will use the digitization parameters to compute the length of the runs on all the levels. To compute it correctly, it is important to know which kind of element is the most frequent on each level and how to use the digitization parameters in both cases, i.e., depending on whether the short element or the long element is the most frequently occurring. It is obvious that $0 < \sigma_k^\wedge < \frac{1}{2}$ for each $k \in \mathbf{N}^+$.

We introduce an auxiliary function which counts for each digitization level k where $k \in \mathbf{N}^+$ all the previous levels (i.e., levels with numbers $1 \leqslant i \leqslant k - 1$) with digitization parameters fulfilling the condition $\sigma_i < \frac{1}{2}$:

Definition 3.5. For a given straight line l with equation $y = ax$, where $0 < a < 1$ and a is irrational, we define function $\operatorname{Reg} : \mathbf{N}^+ \longrightarrow \mathbf{N}$ as follows:

$$\operatorname{Reg}(k) = \begin{cases} 0 & \text{if } k = 1 \\ \displaystyle\sum_{i=1}^{k-1} \chi_{]0, \frac{1}{2}[}(\sigma_i) & \text{if } k \in \mathbf{N}^+ \setminus \{1\}, \end{cases}$$

where $\chi_{]0, \frac{1}{2}[}$ is the characteristic function of the interval $]0, \frac{1}{2}[$.

In order to make our central Definition 3.7 easier to construct and understand, we also formulate the following lemma. The σ in the lemma works as a placeholder for the modified digitization parameters $\sigma_k^\wedge$.

Lemma 3.6. *If an irrational number σ fulfills $0 < \sigma < 1$ and $\delta = \operatorname{frac}(\frac{1}{\sigma})$, then the value of $\lfloor \frac{i}{\sigma} \rfloor - \lfloor \frac{i-1}{\sigma} \rfloor$ for natural $i \geqslant 2$ is the following:*

$$\left\lfloor \frac{i}{\sigma} \right\rfloor - \left\lfloor \frac{i-1}{\sigma} \right\rfloor = \begin{cases} \left\lfloor \dfrac{1}{\sigma} \right\rfloor + \chi_{]0, \frac{1}{2}[}(1 - \delta) & \textit{iff } \exists j \in \mathbf{N}^+ : \ \left\lfloor \frac{j-1}{\delta^\wedge} \right\rfloor + 2 \leqslant i \leqslant \left\lfloor \frac{j}{\delta^\wedge} \right\rfloor \\ \left\lfloor \dfrac{1}{\sigma} \right\rfloor + \chi_{]0, \frac{1}{2}[}(\delta) & \textit{iff } \exists j \in \mathbf{N}^+ : \ i = \left\lfloor \frac{j}{\delta^\wedge} \right\rfloor + 1. \end{cases}$$

This lemma introduces a recursive definition of digitization runs on all the digitization levels (Definition 3.7).

It is worth mentioning that the lemma determines the values of $\lfloor \frac{i}{\sigma} \rfloor - \lfloor \frac{i-1}{\sigma} \rfloor$ for all natural $i \geqslant 2$. For example we get the value of $\lfloor \frac{2}{\sigma} \rfloor - \lfloor \frac{1}{\sigma} \rfloor$ (i.e., $i = 2$) by taking $j = 1$. Then, for each $j \geqslant 1$, number $\lfloor \frac{j}{\delta^\wedge} \rfloor + 1$ comes directly after $\lfloor \frac{j}{\delta^\wedge} \rfloor$ (the last value of i in the first line), while the next one, $\lfloor \frac{j}{\delta^\wedge} \rfloor + 2$, we get for $j+1$ as the first value of i in the first line. The lemma above thus implies that the values of $\lfloor \frac{i}{\sigma} \rfloor - \lfloor \frac{i-1}{\sigma} \rfloor$ for $i = 2, 3, \dots$ in this order are, if $\delta < \frac{1}{2}$: $\lfloor \frac{1}{\delta^\wedge} \rfloor - 1$ times $\lfloor \frac{1}{\sigma} \rfloor$, then one time $\lfloor \frac{1}{\sigma} \rfloor + 1$, then $\lfloor \frac{2}{\delta^\wedge} \rfloor - \lfloor \frac{1}{\delta^\wedge} \rfloor - 1$ times $\lfloor \frac{1}{\sigma} \rfloor$, then one time $\lfloor \frac{1}{\sigma} \rfloor + 1, \dots, \lfloor \frac{j}{\delta^\wedge} \rfloor - \lfloor \frac{j-1}{\delta^\wedge} \rfloor - 1$ times $\lfloor \frac{1}{\sigma} \rfloor$, then one time $\lfloor \frac{1}{\sigma} \rfloor + 1$, and so on. If $\delta > \frac{1}{2}$, we only have to replace $\lfloor \frac{1}{\sigma} \rfloor + 1$ by $\lfloor \frac{1}{\sigma} \rfloor$ and $\lfloor \frac{1}{\sigma} \rfloor$ by $\lfloor \frac{1}{\sigma} \rfloor + 1$ in the above text.

Lemma 3.6 is a continuation of Lemma 3.1. Lemma 3.1 states that $\lfloor \frac{i}{\sigma} \rfloor - \lfloor \frac{i-1}{\sigma} \rfloor$ for natural $i \geqslant 2$ can have one of two values $\lfloor \frac{1}{\sigma} \rfloor$ and $\lfloor \frac{1}{\sigma} \rfloor + 1$. Lemma 3.6 indicates exactly for which i we get each of the two values. It also shows with which frequencies both values appear. The frequencies are δ for the value $\lfloor \frac{1}{\sigma} \rfloor + 1$ and $1 - \delta$ for $\lfloor \frac{1}{\sigma} \rfloor$, where $\delta = \mathrm{frac}(\frac{1}{\sigma})$ (see the discussion of Lemma 3.1). If $\delta < \frac{1}{2}$, the value $\lfloor \frac{1}{\sigma} \rfloor$ is the most frequent one; when $\delta > \frac{1}{2}$ the most frequent one is $\lfloor \frac{1}{\sigma} \rfloor + 1$.

Because the phrase "the most frequent one" will become very important later in the text (see Proposition 3.12), we will discuss this in depth now. First, the sets of indices in the first line of the formula in Lemma 3.6 are nonempty for all $j \in \mathbf{N}^+$. More precisely, the sets of all consecutive indices $i \geqslant 2$ for which $\lfloor \frac{i}{\sigma} \rfloor - \lfloor \frac{i-1}{\sigma} \rfloor = \lfloor \frac{1}{\sigma} \rfloor + \chi_{]0,\frac{1}{2}[}(1-\delta)$ has the cardinality $\lfloor \frac{j}{\delta^\wedge} \rfloor - \lfloor \frac{j-1}{\delta^\wedge} \rfloor - 1$ for $j \in \mathbf{N}^+$; thus, because $0 < \delta^\wedge < 1$ is irrational, Lemma 3.6 can also be used for the calculation of these cardinalities and we get $\lfloor \frac{1}{\delta^\wedge} \rfloor$ or $\lfloor \frac{1}{\delta^\wedge} \rfloor - 1$ consecutive indices $i \geqslant 2$ for which $\lfloor \frac{i}{\sigma} \rfloor - \lfloor \frac{i-1}{\sigma} \rfloor = \lfloor \frac{1}{\sigma} \rfloor + \chi_{]0,\frac{1}{2}[}(1-\delta)$ for $j \in \mathbf{N}^+$. Because $\delta^\wedge < \frac{1}{2}$, so $\lfloor \frac{1}{\delta^\wedge} \rfloor \geqslant 2$ and $\lfloor \frac{1}{\delta^\wedge} \rfloor - 1 \geqslant 1$ which gives the nonemptiness. The formula also ensures that we get the value $\lfloor \frac{1}{\delta^\wedge} \rfloor \geqslant 2$ for some $j \geqslant 2$, namely for those $j \geqslant 2$ which are equal to $\lfloor \frac{k}{\theta^\wedge} \rfloor + 1$ for some $k \in \mathbf{N}^+$ if $\theta < \frac{1}{2}$ and for those $j \geqslant 2$ which are not equal to $\lfloor \frac{k}{\theta^\wedge} \rfloor + 1$ for any $k \in \mathbf{N}^+$ if $\theta > \frac{1}{2}$, where $\theta = \mathrm{frac}(\frac{1}{\delta^\wedge})$. The phrase "the most frequent one" is thus well motivated.

Proof. Let $0 < \sigma < 1$ be any irrational number. For any natural number $i \geqslant 2$ we have:

$$\left\lfloor \frac{i}{\sigma} \right\rfloor - \left\lfloor \frac{i-1}{\sigma} \right\rfloor = \frac{1}{\sigma} + \mathrm{frac}\left(\frac{i-1}{\sigma}\right) - \mathrm{frac}\left(\frac{i}{\sigma}\right).$$

As $\delta = \mathrm{frac}(\frac{1}{\sigma})$ and σ is irrational, so also δ is irrational and $0 < \delta < 1$. Because $\mathrm{frac}(\frac{i}{\sigma}) = \mathrm{frac}(i \cdot \mathrm{frac}(\frac{1}{\sigma})) = \mathrm{frac}(i\delta)$, we can proceed, using δ. Let us take any number $j \in \mathbf{N}^+$ and consider the following two cases:

(c.1.) a natural number $i \geqslant 2$ is such that $(i-1)\delta$ and $i\delta$ have the same value of the floor function, equal to $j - 1$. For those i we have:

$$\mathrm{frac}\left(\frac{i-1}{\sigma}\right) = (i-1)\delta - (j-1) \quad \text{and} \quad \mathrm{frac}\left(\frac{i}{\sigma}\right) = i\delta - (j-1),$$

so we get

$$\left\lfloor \frac{i}{\sigma} \right\rfloor - \left\lfloor \frac{i-1}{\sigma} \right\rfloor = \frac{1}{\sigma} - \mathrm{frac}\left(\frac{1}{\sigma}\right) = \left\lfloor \frac{1}{\sigma} \right\rfloor.$$

(c.2.) a natural number $i \geqslant 2$ is such that $(i-1)\delta$ and $i\delta$ have different values of the floor functions, equal to $j - 1$ and j respectively (because $0 < \delta < 1$ and the integer parts of $(i-1)\delta$ and $i\delta$ are different in this case, they can only differ by 1). For those i we have:

$$\mathrm{frac}\left(\frac{i-1}{\sigma}\right) = (i-1)\delta - (j-1) \quad \text{and} \quad \mathrm{frac}\left(\frac{i}{\sigma}\right) = i\delta - j,$$

so we get

$$\left\lfloor \frac{i}{\sigma} \right\rfloor - \left\lfloor \frac{i-1}{\sigma} \right\rfloor = \frac{1}{\sigma} + 1 - \mathrm{frac}\left(\frac{1}{\sigma}\right) = \left\lfloor \frac{1}{\sigma} \right\rfloor + 1.$$

In order to prove the lemma for $\delta > \frac{1}{2}$, we observe the following:

Remark. Let $\delta \in]\frac{1}{2}, 1[$. For all $j \in \mathbf{N}^+$ and natural $i > j$:

$$[i\delta < i - j \Leftrightarrow i\delta^\wedge > j] \quad \text{and} \quad [i\delta > i - j \Leftrightarrow i\delta^\wedge < j].$$

To prove this it is enough to notice that $\delta^\wedge = 1 - \delta$ for $\delta \in]\frac{1}{2}, 1[$.

Because $\delta^\wedge$ is irrational for all δ as described in the lemma, we have:

$$\exists j \in \mathbf{N}^+ : \left\lfloor \frac{j-1}{\delta^\wedge} \right\rfloor + 2 \leqslant i \leqslant \left\lfloor \frac{j}{\delta^\wedge} \right\rfloor$$

$$\Leftrightarrow \quad \exists j \in \mathbf{N}^+ : \quad j - 1 < (i-1)\delta^\wedge < i\delta^\wedge < j$$

$$\overset{(1)}{\Leftrightarrow} \quad \begin{cases} \exists j \in \mathbf{N}^+ : \quad j - 1 < (i-1)\delta < i\delta < j & \text{if } \delta < \frac{1}{2} \\ \exists j \in \mathbf{N}^+ : \quad [i - j - 1 < (i-1)\delta < i - j & \\ \wedge \quad i - j < i\delta < i - j + 1] & \text{if } \delta > \frac{1}{2} \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} \lfloor i\delta \rfloor = \lfloor (i-1)\delta \rfloor & \text{if } \delta < \frac{1}{2} \\ \lfloor i\delta \rfloor = \lfloor (i-1)\delta \rfloor + 1 & \text{if } \delta > \frac{1}{2} \end{cases}$$

$$\overset{(2)}{\Leftrightarrow} \quad \begin{cases} \left\lfloor \frac{i}{\sigma} \right\rfloor - \left\lfloor \frac{i-1}{\sigma} \right\rfloor = \left\lfloor \frac{1}{\sigma} \right\rfloor & \text{if } \delta < \frac{1}{2} \\ \left\lfloor \frac{i}{\sigma} \right\rfloor - \left\lfloor \frac{i-1}{\sigma} \right\rfloor = \left\lfloor \frac{1}{\sigma} \right\rfloor + 1 & \text{if } \delta > \frac{1}{2} \end{cases}$$

$$\Leftrightarrow \quad \left\lfloor \frac{i}{\sigma} \right\rfloor - \left\lfloor \frac{i-1}{\sigma} \right\rfloor = \left\lfloor \frac{1}{\sigma} \right\rfloor + \chi_{]0, \frac{1}{2}[}(1 - \delta),$$

which proves the first statement in the lemma. Equivalence (1) we get using the above remark for $\delta > \frac{1}{2}$, equivalence (2) using (c.1.) and (c.2.). The fact that $0 < \delta^\wedge < \frac{1}{2}$ (which means that $\frac{1}{\delta^\wedge} > 2$) ensures that the set of such i that $\lfloor \frac{j-1}{\delta^\wedge} \rfloor + 2 \leqslant i \leqslant \lfloor \frac{j}{\delta^\wedge} \rfloor$ is not empty for all $j \in \mathbf{N}^+$.

An analogous reasoning can be made for the second statement in the lemma:

$$\exists j \in \mathbf{N}^+ : i = \left\lfloor \frac{j}{\delta^\wedge} \right\rfloor + 1$$

$$\Leftrightarrow \quad \exists j \in \mathbf{N}^+ : \quad [j - 1 < (i-1)\delta^\wedge < j \quad \wedge \quad j < i\delta^\wedge < j + 1]$$

$$\Leftrightarrow \quad \begin{cases} \exists j \in \mathbf{N}^+ : \quad [j - 1 < (i-1)\delta < j & \\ \wedge \quad j < i\delta < j + 1] & \text{if } \delta < \frac{1}{2} \\ \exists j \in \mathbf{N}^+ : \quad i - j - 1 < (i-1)\delta < i\delta < i - j & \text{if } \delta > \frac{1}{2} \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} \lfloor i\delta \rfloor = \lfloor (i-1)\delta \rfloor + 1 & \text{if } \delta < \frac{1}{2} \\ \lfloor i\delta \rfloor = \lfloor (i-1)\delta \rfloor & \text{if } \delta > \frac{1}{2} \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} \left\lfloor \frac{i}{\sigma} \right\rfloor - \left\lfloor \frac{i-1}{\sigma} \right\rfloor = \left\lfloor \frac{1}{\sigma} \right\rfloor + 1 & \text{if } \delta < \frac{1}{2} \\ \left\lfloor \frac{i}{\sigma} \right\rfloor - \left\lfloor \frac{i-1}{\sigma} \right\rfloor = \left\lfloor \frac{1}{\sigma} \right\rfloor & \text{if } \delta > \frac{1}{2} \end{cases}$$

$$\Leftrightarrow \quad \left\lfloor \frac{i}{\sigma} \right\rfloor - \left\lfloor \frac{i-1}{\sigma} \right\rfloor = \left\lfloor \frac{1}{\sigma} \right\rfloor + \chi_{]0, \frac{1}{2}[}(\delta).$$

The lemma is proved. $\square$

The next definition is the basis for the theorem describing digital straight lines with irrational slope:

Definition 3.7. For a given straight line l with equation $y = ax$, where $0 < a < 1$ and a is irrational, we define the following functions:

- $\mathrm{run}_1 : \mathbf{N}^+ \to \mathcal{P}(\mathbf{N}^+)$, defined as follows:

  $\mathrm{run}_1(j) = \{i;\ \ \lfloor \frac{j-1}{a} \rfloor + 1 \leqslant i \leqslant \lfloor \frac{j}{a} \rfloor\}$ for $j \in \mathbf{N}^+$.
- For $k \in \mathbf{N}^+ \setminus \{1\}$: $\mathrm{run}_k : \mathbf{N}^+ \to \mathcal{P}(\mathrm{run}_{k-1}(\mathbf{N}^+))$ defined as follows:

  $\mathrm{run}_k(1) = \{\mathrm{run}_{k-1}(i);\ \ 1 \leqslant i \leqslant \lfloor \frac{1}{\sigma^\wedge_{k-1}} \rfloor + \mathrm{Rmod}_2(k)\}$, and for natural $j \geqslant 2$:

  $\mathrm{run}_k(j) = \{\mathrm{run}_{k-1}(i);\ \ \lfloor \frac{j-1}{\sigma^\wedge_{k-1}} \rfloor + \mathrm{Rmod}_2(k) + 1 \leqslant i \leqslant \lfloor \frac{j}{\sigma^\wedge_{k-1}} \rfloor + \mathrm{Rmod}_2(k)\}$, where

  $$\mathrm{Rmod}_2(k) = \begin{cases} 0 & \text{if } \mathrm{Reg}(k) \text{ is even} \\ 1 & \text{if } \mathrm{Reg}(k) \text{ is odd,} \end{cases}$$

  $\sigma^\wedge_k$ are the modified digitization parameters defined in Definition 3.4, the function Reg is defined in Definition 3.5, and $\mathcal{P}(A)$ denotes the power set of a set A.

We shall say that $\mathrm{run}_k(j)$ for $k, j \in \mathbf{N}^+$ is a *run of digitization level k*. We will also write run_k or in plural runs_k, meaning $\mathrm{run}_k(j)$ *for some* $j \in \mathbf{N}^+$, or, respectively, $\{\mathrm{run}_k(i);\ \ i \in I\}$ *where* $I \in \mathcal{P}(\mathbf{N}^+)$. Also here we define the *length of a digitization run* as its cardinality.

From the definition of run_1 it is clear that runs_1 can be identified with digitization runs described in the beginning of this section, because for $j \in \mathbf{N}^+$ (according to Proposition 3.2):

$$\begin{aligned} \mathrm{run}_1(j) &= \left\{ i \in \mathbf{N}^+;\ \ \left\lfloor \frac{j-1}{a} \right\rfloor + 1 \leqslant i \leqslant \left\lfloor \frac{j}{a} \right\rfloor \right\} \\ &= \{i \in \mathbf{N}^+;\ \ j-1 < ai \leqslant j\} \\ &= \{i \in \mathbf{N}^+;\ \ (i, j) \in D_{R'}(l)\}, \end{aligned}$$

while

$$\mathrm{run}(j) = \{(i, j) \in (\mathbf{N}^+)^2;\ \ (i, j) \in D_{R'}(l)\}.$$

Proposition 3.8. *Let l be given by the equation $y = ax$ where $0 < a < 1$ and a is irrational. For each $k \in \mathbf{N}^+ \setminus \{1\}$, the runs of the level k can have one of the two lengths: $\lfloor \frac{1}{\sigma^\wedge_{k-1}} \rfloor$ (short runs) or $\lfloor \frac{1}{\sigma^\wedge_{k-1}} \rfloor + 1$ (long runs). The runs of level 1 can have lengths $\lfloor \frac{1}{a} \rfloor$ or $\lfloor \frac{1}{a} \rfloor + 1$.*

Proof. For level k where $k \in \mathbf{N}^+ \setminus \{1\}$ the length of $\mathrm{run}_k(1)$ is equal to $\lfloor \frac{1}{\sigma^\wedge_{k-1}} \rfloor$ if $\mathrm{Rmod}_2(k) = 0$ and $\lfloor \frac{1}{\sigma^\wedge_{k-1}} \rfloor + 1$ if $\mathrm{Rmod}_2(k) = 1$. If $j \geqslant 2$ is a natural number, then the length of $\mathrm{run}_k(j)$ is equal to $\lfloor \frac{j}{\sigma^\wedge_{k-1}} \rfloor - \lfloor \frac{j-1}{\sigma^\wedge_{k-1}} \rfloor$ and, because $0 < \sigma^\wedge_i < 1$ for $i \in \mathbf{N}^+$, we can apply Lemma 3.6 with $\sigma = \sigma^\wedge_{k-1}$ for $k = 2, 3, \ldots$. For $k = 1$ we apply Lemma 3.6 with $\sigma = a$. □

Lemma 3.6 and Proposition 3.8 allow us to formulate the following definitions:

Definition 3.9. For a given straight line l with equation $y = ax$, where $0 < a < 1$ and a is irrational, we define the following functions for $k \in \mathbf{N}^+$:

$$\mathrm{kind\_run}_k : \mathbf{N}^+ \to \{S, L\},$$

where 'S' and 'L' are abbreviations for *short* and *long* respectively. For $j \in \mathbf{N}^+$:

$$\mathrm{kind\_run}_1(j) = \begin{cases} S & \text{if } \mathrm{card}(\mathrm{run}_1(j)) = \lfloor \frac{1}{a} \rfloor \\ L & \text{if } \mathrm{card}(\mathrm{run}_1(j)) = \lfloor \frac{1}{a} \rfloor + 1 \end{cases}$$

$$\mathrm{kind\_run}_k(j) = \begin{cases} S & \text{if } \mathrm{card}(\mathrm{run}_k(j)) = \lfloor \frac{1}{\sigma^{\wedge}_{k-1}} \rfloor \\ L & \text{if } \mathrm{card}(\mathrm{run}_k(j)) = \lfloor \frac{1}{\sigma^{\wedge}_{k-1}} \rfloor + 1 \end{cases} \quad \text{for } k \in \mathbf{N}^+ \setminus \{1\},$$

where $\mathrm{card}(\mathrm{run}_k(j))$ denotes the number of elements in $\mathrm{run}_k(j)$ (the *length* of $\mathrm{run}_k(j)$).

Definition 3.10. For a given straight line l with equation $y = ax$, where $0 < a < 1$ and a is irrational, we define the *alternation-function*

$$\mathrm{alt} : \{S, L\} \to \{S, L\}$$

as follows:

$$\mathrm{alt}(S) = L, \qquad \mathrm{alt}(L) = S.$$

We define three functions with level numbers as arguments:

Definition 3.11. For a given straight line l with equation $y = ax$, where $0 < a < 1$ is irrational, we define three functions:

$$\mathrm{single}_{(.)},\ \mathrm{main}_{(.)},\ \mathrm{first}_{(.)} : \mathbf{N}^+ \to \{S, L\}.$$

For $k \in \mathbf{N}^+$:

$$\mathrm{single}_k = \begin{cases} S & \text{if } \{j \in \mathbf{N}^+;\ \mathrm{kind\_run}_k(j) = \mathrm{kind\_run}_k(j+1) = S\} = \emptyset \\ L & \text{if } \{j \in \mathbf{N}^+;\ \mathrm{kind\_run}_k(j) = \mathrm{kind\_run}_k(j+1) = L\} = \emptyset \end{cases}$$
$$\mathrm{main}_k = \mathrm{alt} \circ \mathrm{single}_k$$
$$\mathrm{first}_k = \mathrm{kind\_run}_k(1).$$

We remark that the kth digitization parameter defined in Definition 3.4 has the following influence on the most frequent (main) run length on level k:

Proposition 3.12. *For a digital line $y = ax$, where $0 < a < 1$ and a is irrational, we have on level k where $k \in \mathbf{N}^+$:*

- $\sigma_k < \frac{1}{2} \ \Rightarrow\ \mathrm{main}_k = S,$
- $\sigma_k > \frac{1}{2} \ \Rightarrow\ \mathrm{main}_k = L.$

Proof. Combine Proposition 3.8 with the discussion after the statement of Lemma 3.6. □

This brings us to the following theorem:

Theorem 3.13 (*Necessary Condition to be a Digital Line with Irrational Slope*). *For a given straight line l with equation $y = ax$, where $0 < a < 1$ and a is irrational, the R'-digitization of the positive half line of l is the following subset of $\mathbf{Z}^2$:*

$$D_{R'}(l) = \bigcup_{j \in \mathbf{N}^+} \{\mathrm{run}_1(j) \times \{j\}\}.$$

For each $k \in \mathbf{N}^+$ runs of level k defined in Definition 3.7 fulfill the following conditions:

[N1]: *There are only two possible run-lengths on level k. They are expressed by two consecutive natural numbers. The length of* $\mathrm{run}_k(j)$ *for $j \in \mathbf{N}^+ \setminus \{1\}$ is namely $\lfloor \frac{1}{\sigma^{\wedge}_{k-1}} \rfloor$ ($\lfloor \frac{1}{a} \rfloor$ if $k = 1$) or $\lfloor \frac{1}{\sigma^{\wedge}_{k-1}} \rfloor + 1$ ($\lfloor \frac{1}{a} \rfloor + 1$ if $k = 1$), where $\sigma^{\wedge}_k$ is the modified digitization parameter defined in Definition* 3.4. *We write* $\mathrm{kind\_run}_k(j) = S$ *or* $\mathrm{kind\_run}_k(j) = L$ *respectively. S and L are abbreviations of* short *and* long *respectively.*

[N2]: $\mathrm{kind\_run}_k(\lfloor \frac{j}{\sigma^{\wedge}_k} \rfloor + 1) = \mathrm{single}_k$ *for all $j \in \mathbf{N}^+$ and* $\mathrm{kind\_run}_k(i) = \mathrm{main}_k$ *for all natural $i \geqslant 2$ such that $i \neq \lfloor \frac{j}{\sigma^{\wedge}_k} \rfloor + 1$ for all $j \in \mathbf{N}^+$.* single_k *means the kind of* run_k *which can never appear more than once in a sequence and* main_k *means the kind of* run_k *which comes in multiples.*

[N3]: *The kind of the first run of level k is determined by the following formula:*

$$\text{first}_k = \text{kind\_run}_k(1) = \begin{cases} S & \textit{if } \operatorname{Reg}(k) \textit{ is even} \\ L & \textit{if } \operatorname{Reg}(k) \textit{ is odd}, \end{cases}$$

where the function Reg *is defined in Definition* 3.5.

Proof. Let us first consider the case $k = 1$. Because runs_1 can be identified with digitization runs described in the beginning of this section, Proposition 3.2 and Lemma 3.6 with $\sigma = a$ prove the conditions [N1], [N2] and [N3] for level 1.

The case $k > 1$ remains to be considered. From Definition 3.4 follows that we can apply Lemma 3.6 to $\sigma = \sigma^{\wedge}_{k-1}$ (so $\delta = \sigma_k$) for $k = 2, 3, \ldots$. This lemma proves by simple induction the conditions [N1] and [N2], because runs_{k-1} are the elements of runs_k.

It remains to prove the condition [N3]. First we assume that for the digitization parameters of the line to digitize the following holds: $\sigma_k < \frac{1}{2}$ for all $k \in \mathbf{N}^+$ and we prove the condition [N3] for lines like this. If $j \in \mathbf{N}^+$, $\text{runs}_k(i)$ ($i \geqslant 2$) belonging to the $\text{run}_{k+1}(j)$ are short (i.e., have length $\lfloor \frac{1}{\sigma_{k-1}} \rfloor$) if and only if

$$\left\lfloor \frac{j-1}{\sigma_k} \right\rfloor + 2 \leqslant i \leqslant \left\lfloor \frac{j}{\sigma_k} \right\rfloor.$$

(Lemma 3.6 with $\sigma = \sigma_{k-1}$), so the $\text{run}_{k+1}(j)$ consists of $\lfloor \frac{j}{\sigma_k} \rfloor - \lfloor \frac{j-1}{\sigma_k} \rfloor - 1$ short runs_k and one long, $\text{run}_k(\lfloor \frac{j-1}{\sigma_k} \rfloor + 1)$ or $\text{run}_k(\lfloor \frac{j}{\sigma_k} \rfloor + 1)$. In particular, for $j = 1$ we get that $\text{run}_{k+1}(1)$ consists of $\lfloor \frac{1}{\sigma_k} \rfloor - 1$ short runs_k (numbers $2, \ldots, \lfloor \frac{1}{\sigma_k} \rfloor$) and we know (Lemma 3.6) that $\text{run}_k(\lfloor \frac{1}{\sigma_k} \rfloor + 1)$ is long, so $\text{run}_{k+1}(1)$ is:

- **short** if $\text{run}_k(1)$ is long,
- **long** if $\text{run}_k(1)$ is short.

Because the first run of level 1 (first_1) is always short, we get by simple induction the following statement for lines with all digitization parameters less than $\frac{1}{2}$. For $k \in \mathbf{N}^+$:

$$\text{first}_k = \begin{cases} S & \text{if } k \text{ is odd} \\ L & \text{if } k \text{ is even.} \end{cases}$$

We can also say that for the lines as described above: *the kind of the first run is alternating for consecutive levels*. From Definition 3.5 it follows that for lines with all the digitization parameters $\sigma_1, \sigma_2, \ldots$ less than $\frac{1}{2}$ we have $\operatorname{Reg}(k) = k-1$ for $k \in \mathbf{N}^+$, thus its value is odd for even k and even for odd k. This shows that the statement above is equivalent to the condition [N3] for lines with $\sigma_k < \frac{1}{2}$ for all $k \in \mathbf{N}^+$ and the proof of the theorem for this type of line is complete. If $\sigma_k > \frac{1}{2}$ for some $k \in \mathbf{N}^+$ then we get by the same reasoning as above (Lemma 3.6 with $\sigma = a$ if $k = 1$ and $\sigma = \sigma^{\wedge}_{k-1}$ if $k > 1$, thus $\delta = \sigma_k$ and $1 - \delta = \sigma^{\wedge}_k$) that $\text{run}_{k+1}(1)$ consists of $\lfloor \frac{1}{\sigma^{\wedge}_k} \rfloor - 1$ long runs_k (numbers $2, \ldots, \lfloor \frac{1}{\sigma^{\wedge}_k} \rfloor$) and we know (also Lemma 3.6) that $\text{run}_k(\lfloor \frac{1}{\sigma^{\wedge}_k} \rfloor + 1)$ is short, so $\text{run}_{k+1}(1)$ is:

- **long** if $\text{run}_k(1)$ is long,
- **short** if $\text{run}_k(1)$ is short

and the alternation pattern breaks. We get no alternation of the kind of the first run from level k to level $k+1$ if $\sigma_k > \frac{1}{2}$, and a simple induction proof gives us the following recurrent description of the kind of the first run on each level:

- $\text{first}_1 = S$,
- For each natural $k \geqslant 2$: if $\sigma_{k-1} < \frac{1}{2}$, then $\text{first}_k = \text{alt} \circ \text{first}_{k-1}$ (where $\text{alt}(S) = L$ and $\text{alt}(L) = S$ according to Definition 3.10),
- For each natural $k \geqslant 2$: if $\sigma_{k-1} > \frac{1}{2}$, then $\text{first}_k = \text{first}_{k-1}$,

which, according to Definition 3.5, leads to the condition [N3] in Theorem 3.13. The proof is now complete. □

Generally speaking, we have two important questions in connection with digital lines:

- how to find the digitization of a given real line (necessary condition to be a digital line)
- how to recognize a digital line in a subset of $\mathbf{Z}^2$ (sufficient condition to be a digital line).

To give a simple answer to the first question, we will reformulate the results from Theorem 3.13 in a more practically useful way. To do this, we will use function Reg to describe the form of runs on each digitization level. The form of runs on level $k+1$ depends on both main (thus on σ_k in a very explicit way) and first on level k (the first on level k for $k \geqslant 2$ is fully determined only by the digitization parameters $\sigma_1, \ldots, \sigma_{k-1}$. They show where the kind of the first run alternates from one level to the next level and where not).
It can be convenient to use the symbols $S \cdots SL$, $LS \cdots S$, $L \cdots LS$ and $SL \cdots L$ when describing the form of digitization runs. For example $S \cdots SL$ will mean that the run_k we are talking about consists of $\lfloor 1/\sigma^{\wedge}_{k-1} \rfloor - 1$ or $\lfloor 1/\sigma^{\wedge}_{k-1} \rfloor$ short runs_{k-1} (abbrev. S) and one long run_{k-1} (abbrev. L) in this order, so it is a run with main element short.

Corollary 3.14 (*Necessary Condition to be a Digital Line with Irrational Slope*). *For a straight line l with equation $y = ax$, where $0 < a < 1$ and a is irrational, we have: for each $j \in \mathbf{N}^+$, $\text{run}_1(j)$ can have two possible lengths: $\lfloor \frac{1}{a} \rfloor$ (S — short) and $\lfloor \frac{1}{a} \rfloor + 1$ (L — long) and the forms of* runs_{k+1} (form_run_{k+1}) *for $k \in \mathbf{N}^+$ are as follows:*

$$\text{form\_run}_{k+1} = \begin{cases} S \cdots SL & \textit{iff } \operatorname{Reg}(k+1) = \operatorname{Reg}(k) + 1, & \operatorname{Reg}(k) \textit{ is even} \\ SL \cdots L & \textit{iff } \operatorname{Reg}(k+1) = \operatorname{Reg}(k), & \operatorname{Reg}(k) \textit{ is even} \\ LS \cdots S & \textit{iff } \operatorname{Reg}(k+1) = \operatorname{Reg}(k) + 1, & \operatorname{Reg}(k) \textit{ is odd} \\ L \cdots LS & \textit{iff } \operatorname{Reg}(k+1) = \operatorname{Reg}(k), & \operatorname{Reg}(k) \textit{ is odd}\,, \end{cases}$$

where S means run_k *with length $\lfloor \frac{1}{\sigma^{\wedge}_{k-1}} \rfloor$ and L means* run_k *with length $\lfloor \frac{1}{\sigma^{\wedge}_{k-1}} \rfloor + 1$ and the function* Reg *is defined in Definition* 3.5.

Proof. This corollary follows from Definition 3.7 and Theorem 3.13. We have two implications: $\sigma_k < \frac{1}{2} \Rightarrow \text{main}_k = S$ and $\sigma_k > \frac{1}{2} \Rightarrow \text{main}_k = L$ (Proposition 3.12). The parity of $\operatorname{Reg}(k)$ determines the first run of level k (first_k is short if $\operatorname{Reg}(k)$ is even and long if $\operatorname{Reg}(k)$ odd — Condition [N3]).

The reasoning of the proof is illustrated in the following table; the assumptions are in the first two columns, and the conclusions, which are based on the above statements, are in the three last columns:

σ_k	$\operatorname{Reg}(k)$	main_k	first_k	form of run_{k+1}
$< \frac{1}{2}$	even	S	S	$S \cdots SL$, $\lfloor \frac{1}{\sigma_k} \rfloor - 1$ or $\lfloor \frac{1}{\sigma_k} \rfloor$ times 'S'
$> \frac{1}{2}$	even	L	S	$SL \cdots L$, $\lfloor \frac{1}{1-\sigma_k} \rfloor - 1$ or $\lfloor \frac{1}{1-\sigma_k} \rfloor$ times 'L'
$< \frac{1}{2}$	odd	S	L	$LS \cdots S$, $\lfloor \frac{1}{\sigma_k} \rfloor - 1$ or $\lfloor \frac{1}{\sigma_k} \rfloor$ times 'S'
$> \frac{1}{2}$	odd	L	L	$L \cdots LS$, $\lfloor \frac{1}{1-\sigma_k} \rfloor - 1$ or $\lfloor \frac{1}{1-\sigma_k} \rfloor$ times 'L'

The relation of the parities of $\operatorname{Reg}(k)$ and $\operatorname{Reg}(k+1)$ determines the main of level k:

- if $\operatorname{Reg}(k+1)$ and $\operatorname{Reg}(k)$ have the same parities, then $\chi_{]0,\frac{1}{2}[}(\sigma_k) = 0$, so $\sigma_k > \frac{1}{2}$ and main of level k is long.
- if $\operatorname{Reg}(k+1)$ and $\operatorname{Reg}(k)$ have different parities, then $\chi_{]0,\frac{1}{2}[}(\sigma_k) = 1$, so $\sigma_k < \frac{1}{2}$ and main of level k is short.

Because runs_k are elements of the runs_{k+1}, the conclusion about the form of the runs of level $k+1$ follows from the information above. □

The corollary is constructive. It shows exactly how to find the R′-digitization of the positive half line $y = ax$ (where $0 < a < 1$ and a is irrational). We get the digitization by calculating the digitization parameters and proceeding step by step, following the recursive description. The knowledge about the kind of the first run on each level allows us go as far as we want in the digitization.

Corollary 3.14 shows a necessary condition for a subset of $(\mathbf{N}^+)^2$ to be a digital (half) line. Now the question remains whether the condition is also sufficient. We can ask ourselves whether all the subsets of $(\mathbf{N}^+)^2$ fulfilling on all levels the three conditions named in Theorem 3.13 and with the short run length on level k equal to $n_k \geqslant 2$ are digitizations of some (half) lines with irrational slope. In other words: can all the sequences of natural numbers greater or equal to 2 be the short run lengths for some line? Run length 1 on level with number greater than 1 is only possible for lines with rational slope, where we get periodical digitization, so there is only one kind of run on some level k, where $k \in \mathbf{N}^+$ depends on the slope. If the slope is irrational, we can only have short run length 1 on level 1, i.e., only short run$_1$ can have length 1.

Lemma 3.15. *For each $k \in \mathbf{N}^+$:*

- *For each $0 < r < 1$ it is possible to find a real straight line with level k parameter $\sigma_k = r$.*
- *If $k \geqslant 2$: for each $0 < r < 1$ and each set $\{i_1, \dots, i_l\} \subset \{1, \dots, k-1\}$ with cardinality $1 \leqslant l \leqslant k-1$ it is possible to find a real straight line with the level k parameter $\sigma_k = r$ and such that $\sigma_i > \frac{1}{2}$ for all $i \in \{i_1, \dots, i_l\}$.*

Proof. We construct the slope of the line $y = ax$ fulfilling this condition as follows:

- In the first case we take $a = [0, n_1, \dots, n_{k-1}, [n_k, r]]$, where $n_1 \geqslant 1$ and $n_i \geqslant 2$ for $i \geqslant 2$ are natural numbers. $[0, n_1, \dots, n_{k-1}, [n_k, r]]$ is a compact abbreviated form of the continued fraction (see Hardy and Wright [3, p. 130].):

  $$\frac{1}{n_1 + \frac{1}{n_2 + \cdots + \frac{1}{n_{k-1} + \frac{1}{n_k + r}}}}.$$

  Then $n_i = \lfloor \frac{1}{\sigma_{i-1}} \rfloor$ for $i = 2, \dots, k$ is the length of short run$_i$ and $n_1 = \lfloor \frac{1}{a} \rfloor$ is the length of short run$_1$. All the straight lines with the slopes a like above fulfill the imposed condition. In each case we have $\sigma_k = \mathrm{frac}(n_k + r) = r$. The restriction $n_i \geqslant 2$ for $i \geqslant 2$ ensures that all the σ_i for $i = 1, \dots, k-1$ are less than $\frac{1}{2}$, so we never have to modify the digitization parameters according to Definition 3.4, and we really get $\sigma_k = \mathrm{frac}(n_k + r) = r$.
- In the second case we do similarly as in the proof of the first part of the lemma. If we wish to have $\sigma_i > \frac{1}{2}$, then we put 1 twice in place of n_{i+1} in the continued fraction, i.e., we replace

  $$[0, n_1, \dots, n_i, n_{i+1}, n_{i+2}, \dots, n_{k-1}, [n_k, r]]$$

  by

  $$[0, n_1, \dots, n_i, 1, n_{i+1} - 1, n_{i+2}, \dots, n_{k-1}, [n_k, r]].$$

  In other words, we put

  $$1 + \frac{1}{n_{i+1} - 1 + \cdots}$$

  in the continued fraction in place of '$n_{i+1}+$'. We can repeat this on each of the levels with numbers $i \in \{i_1, \dots, i_l\} \subset \{1, \dots, k-1\}$. Each digitization level i with $\sigma_i > \frac{1}{2}$ causes increasing (by one) of the number of levels (literally) in the continued fraction which is going to be the slope. The construction of the slope is based purely on Definition 3.4.

The proof is now complete. □

This leads to the following theorem:

Theorem 3.16. *Let $n \in \mathbf{N}^+$. For each sequence of natural numbers $(k_1, k_2, \dots, k_n)$ such that $k_1 \geqslant 1$ and $k_i > 1$ for $1 < i \leqslant n$ there exist m lines $y = ax$ with rational slopes, where*

$$m = \begin{cases} 2^{n-1} & \text{if } k_n \neq 2 \\ 2^{n-2} & \text{if } k_n = 2 \end{cases}$$

and their digitization fulfills the following conditions: for $i = 1, \dots, n$ the short run's length on digitization level i is k_i.

Proof. For a sequence $(k_1, k_2, \dots, k_n)$ fulfilling the assumptions named in the theorem, we define the slopes of the lines as follows: $a = [0, k_1, \dots, k_n]$ (continued fraction $[0, k_1, \dots, k_{n-1}, [k_n, 0]]$ as defined in the proof of Lemma 3.15) if we want all the $\sigma_j < \frac{1}{2}$ for $j = 1, \dots, n-1$. If we want $\sigma_i > \frac{1}{2}$ for some $1 \leqslant i \leqslant n-1$, we take $a = [0, k_1, \dots, k_i, 1, 1, k_{i+2}, \dots, k_n]$. We have to make a decision about $\sigma_i < \frac{1}{2}$ or $\sigma_i > \frac{1}{2}$ for $i = 1, \dots, n-1$, which means in $n-1$ places. This gives us 2^{n-1} possibilities. We have $\sigma_{n-1} = \frac{1}{k_n}$, so, if $k_n = 2$, then $\sigma_{n-1} = \frac{1}{2}$ and we have one place less to make a choice, so we have only 2^{n-2} possibilities. It follows from Theorem 3.13 that lines with those slopes fulfill the desired condition about the short runs' lengths. □

Theorem 3.16 states that all sequences of natural numbers greater or equal to 2 (and the first element possibly equal to 1) generate the digitization of some lines with short runs' lengths on each level defined by the elements of the sequence. This means that each construction of pixels as described in Theorem 3.13, with infinitely many (*n* was arbitrary!) digitization levels is the R′-digitization of the positive half line of some line $y = ax$, where $0 < a < 1$ is irrational. This gives the following theorem, which states that the necessary condition for being a digital line with irrational slope $0 < a < 1$ is also sufficient:

Theorem 3.17 (*Sufficient Condition to be a Digital Line with Irrational Slope*). *Each subset of* $(\mathbf{N}^+)^2$ *containing* $(1, 1)$ *and fulfilling the conditions* [N1], [N2] *and* [N3] *on all the levels is the R′-digitization of the positive half line of some line* $y = ax$, *where* $0 < a < 1$ *and a is irrational.*

Continued fractions have already been used in this context; Rosenfeld and Klette indicate in their paper two independent publications from 1991: one by M. Bruckstein and another one by K. Voss; see Rosenfeld and Klette [9].

4. Conclusions

We have formulated a formal definition of digitization runs and theorems containing necessary and sufficient conditions for subsets of $(\mathbf{N}^+)^2$ being the digitization of a straight (half) line with irrational slope passing through the origin. Only methods of elementary mathematics have been applied. The main topic of interest was Theorem 3.13 with the necessary condition. The restrictions put on the line (irrational slope $0 < a < 1$ and digitization of the positive half line only) are not severe restrictions. It is not difficult to expand the theory to the cases not explicitly covered in this paper. The developed and proved theory can be used in the research into the theory of digital lines, their symmetries, translations, etc.

Acknowledgments

I am grateful to Christer Kiselman, Damien Jamet and Erik Melin for comments on earlier versions of the manuscript.

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Paper II

Pattern Recognition 42 (2009) 2247 – 2254

Contents lists available at ScienceDirect

Pattern Recognition

journal homepage: www.elsevier.com/locate/pr

Run-hierarchical structure of digital lines with irrational slopes in terms of continued fractions and the Gauss map

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ARTICLE INFO

Article history:
Received 25 June 2008
Received in revised form 5 November 2008
Accepted 10 November 2008

Keywords:
Digital geometry
Digital line
Irrational slope
Continued fraction
Quadratic surd
Gauss map

ABSTRACT

We study relations between digital lines and continued fractions. The main result is a parsimonious description of the construction of the digital line based only on the elements of the continued fraction representing its slope and containing only simple integer computations. The description reflects the hierarchy of digitization runs, which raises the possibility of dividing digital lines into equivalence classes depending on the continued fraction expansions of their slopes. Our work is confined to irrational slopes since, to our knowledge, there exists no run-hierarchical and continued fraction based description for these, in contrast to rational slopes which have been extensively examined. The description is exact (it does not use approximations by rationals). Examples of lines with irrational slopes and with very simple digitization patterns are presented. These include both slopes with periodic and non-periodic continued fraction expansions, i.e. both quadratic surds and other irrationals. We also derive the connection between the Gauss map and the digitization parameters introduced by the author in 2007.

1. Introduction

The aim of the present paper is to solve the following problem: given the continued fraction (CF) expansion of a positive irrational number a less than 1, how is the digitization of the line $y = ax$ constructed? The description uses only the elements of the CF and is exact, i.e. does not use the commonly applied approximations by rationals. The method is based on simple integer computations that can be easily applied to computer programming.

This description forms the main result (Theorem 11; *description by CFs*). The theoretical basis for this article is [1] by Uscka-Wehlou. The main result there is recalled in Section 2 of the present paper (Theorem 4; *description by the σ-parameters*). It gives a description of digitization runs on all digitization levels for lines $y = ax$ where $a \in]0,1[\setminus\mathbf{Q}$, which is based on digitization parameters defined in Definition 1 and the function Reg_a defined in Definition 2.

Although Theorem 4 looks similar to Theorem 11, the former involves computations on irrational numbers, which is not the case in the latter.

In our CF description, like in all the other CF descriptions, we replace the heavy computations (involved, in our case, in the method by digitization parameters) by simple computations on integers. In order to do that, the digitization parameters and the function Reg_a for each $a \in]0,1[\setminus\mathbf{Q}$ were expressed by the elements of the CF expansion of a. The key role in this transform is played by the *index jump function* (Definition 7).

The computations on irrationals did not disappear during the translation of Theorem 4 into the CF version (Theorem 11). They were moved into the process of finding the CF expansion of the slope. For some slopes we are able to compute the CF expansions exactly, using mathematical methods; some examples will be shown in Section 4, for both algebraic and transcendental numbers. For other slopes we can use algorithms for finding CF expansions. In Section 4.2.2 we show some possibilities of applying our method for digital rotations.

The main work leading to the successful translation of Theorem 4 into the CF description (Theorem 11) has been done in Theorems 9 and 10. The first one expresses the digitization parameters in terms of CFs and the second one does the same with the function Reg_a. These results allowed us to replace the computationally challenging conditions and formulae for run lengths from Theorem 4 by equivalent conditions and formulae based on the elements of the CF expansion of a.

In general, it is hard to perform arithmetical operations on CFs (e.g. addition and multiplication of CFs); see Khinchin [2, p. 20]. However, Definition 1 and Theorem 4 involve only the operations which form an exception to this rule. These operations are finding the integer (fractional) part of the inverse to a CF and subtracting a CF

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doi:10.1016/j.patcog.2008.11.005

from 1. The formula for the last operation is described in Lemmas 5 and 6, the others are clearly easy to perform. This made it possible to find the simple description formulated in Theorem 11.

The present CF description of digital lines is similar to the formula of Markov and Venkov [3, p. 67], but since their method was not meant for descriptions of digital lines, it does not reflect the hierarchical structure of digitization runs on all levels, which our method does. This permits the grouping of digital lines into classes according to properties defined by the elements of the CF expansions of the slopes, which has been presented in the author's submitted manuscript [4]. In Section 5 we derive the connection between the iterates of the Gauss map for a given $a \in]0,1[\setminus\mathbf{Q}$ and the digitization parameters associated with a as introduced by the author in [1].

The method presented here is computationally simple, involving only easy computations with integers, excepting the algorithm for determining the CF expansion of the slope. The method applies to irrational slopes and gives the exact results instead of approximations by rationals. To the author's knowledge, there are no previous descriptions of digital lines with irrational slopes fulfilling all the criteria just mentioned, and reflecting the hierarchy of runs.

Earlier developments: The use of CFs in modelling digital lines was discussed by Brons [5] as early as in 1974. Already then it was clear that the patterns generated in the digitization process of straight lines were related to the CF expansions of the slopes. However, the algorithm provided by Brons is only valid for rational slopes.

Some other researchers describing the construction of digital lines with rational slopes in terms of CFs were Reveillès [6], Voss [7, pp. 153–157]—the *splitting formula*, Troesch [8]—Euclid's algorithm and digitization runs, Debled [9, pp. 59–66]—description by the Stern-Brocot tree, Stephenson [10]—an algorithmic solution, de Vieilleville and Lachaud [11]—a combinatoric approach. See also the review of Klette and Rosenfeld from 2004 [12].

Irrational numbers have been less central in research on digital line construction, possibly because irrational slopes must appear not to have direct applications for computer graphics. A CF description of digital lines was presented by Dorst and Duin [13]. Although their solution can be applied to irrational slopes, it is formulated as an algorithm. Since it is not a mathematical theorem, it will not result in descriptions of digital lines as mathematical objects, or help research on their abstract properties.

L.D. Wu formulated in 1982 a theorem describing digital straightness. Proofs of this theorem based on CFs were published in 1991 independently by Bruckstein and Voss; see Klette and Rosenfeld [12, pp. 208–209]. Bruckstein [14] described digital straightness by a number of transformations preserving it. Some of these transformations were defined by means of CFs.

Some work on the subject has also been done outside digital geometry and computer graphics, however, the solutions obtained in other fields do not reflect the hierarchical structure of digitization runs, which is an important feature of digital lines as mathematical objects.

For example, as far back as in 1772, astronomer Johan III Bernoulli applied the CF expansion of a to the solution of the problem of describing the sequence $(\lfloor na\rfloor)_{n\in\mathbf{N}^+}$ for an irrational a. The problem is clearly equivalent to finding the digitization of $y = ax$. Bernoulli failed to provide any proofs. Venkov catalogued the entire history of the problem and its solution (including the solution by Markov from 1882) in [3, pp. 65–71].

Stolarsky described in [15] applications of CFs to *Beatty sequences*.

Last but not least, we have to mention the research on *Sturmian words*, because this is very closely related to the research on digital lines with irrational slopes; see chapter 2 in Lothaire [16] (by J. Berstel and P. Séébold).

2. Description of digital lines by the digitization parameters

To give the necessary background to the present results, we recall that arithmetical description of the modified Rosenfeld digitization (R′-digitization) of the positive half line $y = ax$ for $a \in]0,1[\setminus\mathbf{Q}$ as a subset of $\mathbf{Z}^2$ is the following: $D_{R'}(y = ax, x > 0) = \{(k, \lceil ak\rceil); k \in \mathbf{N}^+\}$. The R′-digitization of $y = ax$ was obtained in [1] using the following *digitization parameters*.

Definition 1. For $y=ax$, where $a \in]0,1[\setminus\mathbf{Q}$, the *digitization parameters* are $\sigma_1 = \mathrm{frac}(1/a)$, and, for all natural numbers $k > 1$ $\sigma_k = \mathrm{frac}(1/\sigma^\wedge_{k-1})$, where $\sigma^\wedge_{k-1} = \min(\sigma_{k-1}, 1 - \sigma_{k-1}) \in]0, \frac{1}{2}[\setminus\mathbf{Q}$.

For $j \in \mathbf{N}^+$, σ_j and $\sigma^\wedge_j$ are the *digitization parameters* and *modified digitization parameters* of the *digitization level j*, respectively.

For each $a \in]0,1[\setminus\mathbf{Q}$, an auxiliary function Reg_a was introduced. This function gives for each $k \geqslant 2$ the number of all the digitization levels i, where $1 \leqslant i \leqslant k-1$, with digitization parameters fulfilling the condition $\sigma_i < \frac{1}{2}$.

Definition 2. For a given line with equation $y=ax$, where $a \in]0,1[\setminus\mathbf{Q}$, we define a function $\mathrm{Reg}_a : \mathbf{N}^+ \longrightarrow \mathbf{N}$ as follows: $\mathrm{Reg}_a(1) = 0$ and $\mathrm{Reg}_a(k) = \sum_{i=1}^{k-1} \chi_{]0,1/2[}(\sigma_i)$ if $k \in \mathbf{N}^+\setminus\{1\}$, where $\chi_{]0,1/2[}$ is the characteristic function of the interval $]0, \frac{1}{2}[$.

The digitization runs of level k for $k \in \mathbf{N}^+$ were defined recursively as sets of runs of level $k-1$ (if we define integer numbers as runs of level 0).

Definition 3 (*Definition 3.7 from [1]*). For a given straight line with equation $y=ax$, where $a \in]0,1[\setminus\mathbf{Q}$, we define the following functions:

- $\mathrm{run}_1 : \mathbf{N}^+ \to \mathscr{P}(\mathbf{N}^+)$, defined as follows: $\mathrm{run}_1(j) = \{i; \lfloor (j-1)/a\rfloor + 1 \leqslant i \leqslant \lfloor j/a\rfloor\}$ for $j \in \mathbf{N}^+$.
- For $k \in \mathbf{N}^+\setminus\{1\}$: $\mathrm{run}_k : \mathbf{N}^+ \to \mathscr{P}(\mathrm{run}_{k-1}(\mathbf{N}^+))$ defined as follows: $\mathrm{run}_k(1) = \{\mathrm{run}_{k-1}(i); 1 \leqslant i \leqslant \lfloor 1/\sigma^\wedge_{k-1}\rfloor + \mathrm{Rmod}_2(k)\}$, and for natural $j \geqslant 2$: $\mathrm{run}_k(j) = \{\mathrm{run}_{k-1}(i);$ $\lfloor (j-1)/\sigma^\wedge_{k-1}\rfloor + \mathrm{Rmod}_2(k) + 1 \leqslant i \leqslant \lfloor j/\sigma^\wedge_{k-1}\rfloor + \mathrm{Rmod}_2(k)\}$, where $\mathrm{Rmod}_2(k) = 0$ if $\mathrm{Reg}_a(k)$ is even and $\mathrm{Rmod}_2(k) = 1$ if $\mathrm{Reg}_a(k)$ is odd; $\sigma^\wedge_k$ are the modified digitization parameters defined in Definition 1, Reg_a is defined in Definition 2 and $\mathscr{P}(A)$ denotes the power set of a set A.

We call $\mathrm{run}_k(j)$ for $k, j \in \mathbf{N}^+$ a *run of digitization level k*. We use notation run_k or in plural runs_k, meaning $\mathrm{run}_k(j)$ for some $j \in \mathbf{N}^+$, or, respectively, $\{\mathrm{run}_k(i); i \in I\}$ *where* $I \in \mathscr{P}(\mathbf{N}^+)$. The *length of a digitization run* is defined as its cardinality.

Function Reg_a defined in Definition 2 was very important in the description of the form of runs. It helped to recognize which kind of runs was the most frequent (also called *main*) on each level and which kind of runs was first, i.e., beginning in (1,1). The digitization runs were defined in Definition 3.7 of [1] such that a run of digitization level k is a set of consecutive runs of level $k-1$ composed of one single digitization run_{k-1} and a sequence of main digitization runs_{k-1} and which is maximal for inclusion.

We showed that for a given straight line l with equation $y = ax$, where $a \in]0,1[\setminus\mathbf{Q}$, the R′-digitization of the positive half line of l is the following subset of $\mathbf{Z}^2$: $D_{R'}(l) = \bigcup_{j\in\mathbf{N}^+}\{\mathrm{run}_1(j) \times \{j\}\}$.

The main theorem of [1] was a formalization of the well-known conditions the digitization runs fulfil. On each level k for $k \geqslant 1$ we have short runs S_k and long runs L_k, which are composed of the runs of level $k-1$. Only one type of the runs (short or long) on each level can appear in sequences, the second type always occurs alone. In the present paper we will use the notation $S_k^m L_k$, $L_k S_k^m$, $L_k^m S_k$ and $S_k L_k^m$, where $m = \lfloor 1/\sigma_k^\wedge \rfloor - 1$ or $m = \lfloor 1/\sigma_k^\wedge \rfloor$, when describing the form of digitization runs_{k+1}. For example, $S_k^m L_k$ means that the run_{k+1} we are talking about consists of m short runs_k (abbreviated S_k) and one long run_k (abbreviated L_k) in this order, so it is a run_{k+1} with the most frequent element short. The length of such a run_{k+1}, being its cardinality, i.e., the number of runs_k contained in it, is then equal to $m+1$. We will also use the notation $\|S_{k+1}\|$ and $\|L_{k+1}\|$ for the length of the short resp. long runs_{k+1}.

We will use the following reformulation of the main result from [1]

Theorem 4 (*Main Result in [1]; description by* $(\sigma_k)_{k\in\mathbf{N}^+}$). *For a straight line with equation* $y=ax$, *where* $a \in]0,1[\backslash\mathbf{Q}$, *we have* $\|S_1\| = \lfloor 1/a \rfloor$, $\|L_1\| = \lfloor 1/a \rfloor + 1$, *and the forms of* runs_{k+1} (form_run_{k+1}) *for* $k \in \mathbf{N}^+$ *are as follows*:

$$\text{form\_run}_{k+1} = \begin{cases} S_k^m L_k & \text{if } \mathrm{Reg}_a(k+1) = \mathrm{Reg}_a(k)+1 \text{ and } \mathrm{Reg}_a(k) \text{ is even,} \\ S_k L_k^m & \text{if } \mathrm{Reg}_a(k+1) = \mathrm{Reg}_a(k) \text{ and } \mathrm{Reg}_a(k) \text{ is even,} \\ L_k S_k^m & \text{if } \mathrm{Reg}_a(k+1) = \mathrm{Reg}_a(k)+1 \text{ and } \mathrm{Reg}_a(k) \text{ is odd,} \\ L_k^m S_k & \text{if } \mathrm{Reg}_a(k+1) = \mathrm{Reg}_a(k) \text{ and } \mathrm{Reg}_a(k) \text{ is odd,} \end{cases}$$

where $m = \lfloor 1/\sigma_k^\wedge \rfloor - 1$ *if the* run_{k+1} *is short and* $m = \lfloor 1/\sigma_k^\wedge \rfloor$ *if the* run_{k+1} *is long. The function* Reg_a *is defined in Definition* 2, *and* σ_k *for* $k \in \mathbf{N}^+$ *in Definition* 1.

Theorem 4 shows exactly how to find the R′-digitization of the positive half line $y = ax$ for $a \in]0,1[\backslash\mathbf{Q}$. We get the digitization by calculating the digitization parameters and proceeding step by step, recursively. The knowledge about the kind of the first run on each level allows us go as far as we want in the digitization. The only problem was in the heavy computation of the digitization parameters, but this will be solved now, in Section 3.

3. Main result: description of digital lines by CFs

Before presenting the description of the digitization, we provide a brief introduction on CFs. The following algorithm gives the regular (or simple) CF for $a \in \mathbf{R}\backslash\mathbf{Q}$, which we denote by $[a_0; a_1, a_2, a_3, \ldots]$. We define a sequence of integers (a_n) and a sequence of real numbers (α_n) by $\alpha_0 = a$; $a_n = \lfloor \alpha_n \rfloor$ and $\alpha_n = a_n + 1/(\alpha_{n+1})$ for $n \geqslant 0$. Then $a_n \geqslant 1$ and $\alpha_n > 1$ for $n \geqslant 1$. The integers $a_0, a_1, a_2, \ldots$ are called the *elements* of the CF (or *terms*, or *partial quotients*). We use the word *elements*, following Khinchin [2]. Because a is irrational, so is each α_n, and the sequences (a_n) and (α_n) are infinite. A CF expansion exists and is unique for all $a \in \mathbf{R}\backslash\mathbf{Q}$; see [2, p. 16]. The following lemmas concern subtracting CFs from 1.

Lemma 5. *Let* $b_i \in \mathbf{N}^+$ *for all* $i \in \mathbf{N}^+$ *and* $b_1 \geqslant 2$. *Then*

$$1 - [0; b_1, b_2, b_3, \ldots] = [0; 1, b_1 - 1, b_2, b_3, \ldots]. \tag{1}$$

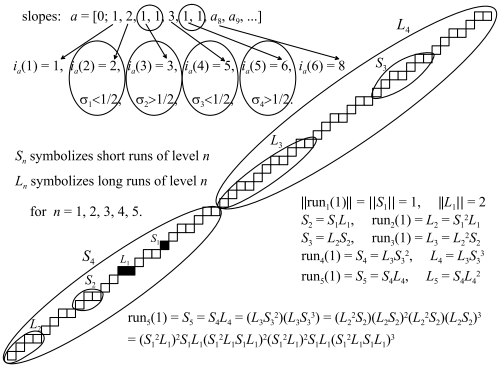

Fig. 1. The index jump function, digitization parameters and hierarchy of runs.

Proof. Let $b = [0; b_1, b_2, \ldots]$ and $b_1 \geqslant 2$. Then $1/b = [b_1; b_2, \ldots]$ and we get

$$1 - b = \frac{1}{\dfrac{1}{1-b}} = \frac{1}{1 + \dfrac{b}{1-b}} = \frac{1}{1 + \dfrac{1}{\dfrac{1}{b} - 1}} = \frac{1}{1 + \dfrac{1}{[b_1; b_2, b_3, \ldots] - 1}}$$

$$= [0; 1, b_1 - 1, b_2, \ldots]. \qquad \square$$

Lemma 6. *If* $a_i \in \mathbf{N}^+$ *for all* $i > 1$, *then we have* $1 - [0; 1, a_2, a_3, \ldots] = [0; a_2 + 1, a_3, \ldots]$.

Proof. Put $b_1 - 1 = a_2, b_2 = a_3, b_3 = a_4, \ldots, b_i = a_{i+1}, \ldots$ in Lemma 5. $\square$

Because clearly $[0; 1, a_2, a_3, \ldots] > \frac{1}{2}$ for all sequences $(a_2, a_3, \ldots)$ of positive integers, Lemma 6 illustrates the modification operation for the σ-parameters according to Definition 1. This leads us to define the following *index jump function*, which will allow us to describe the digitization in terms of CFs.

Definition 7. For each $a \in]0,1[\backslash\mathbf{Q}$, the index jump function $i_a : \mathbf{N}^+ \to \mathbf{N}^+$ is defined by $i_a(1) = 1, i_a(2) = 2$ and $i_a(k+1) = i_a(k) + 1 + \delta_1(a_{i_a(k)})$ for $k \geqslant 2$, where $\delta_1(x) = 1$ for $x = 1$, $\delta_1(x) = 0$ for $x \neq 1$, and $a_1, a_2, \ldots \in \mathbf{N}^+$ are the CF elements of a.

The index jump function is a renumbering which avoids elements following directly after some 1s in the CF expansion (in particular, it avoids every second element in the sequences of consecutive 1s with index greater than 1). We will illustrate how it works with the following example.

Example 8. Let us consider the slopes with the first CF elements as in Fig. 1. We present an illustration of the forming of the index jump function for those slopes. If $a = [0; 1, 2, 1, 1, 3, 1, 1, a_8, a_9, 1, a_{11}, a_{12}, 1, 1, 1, a_{16}, a_{17}, \ldots]$, where a_k is greater than 1 for $k = 8, 9, 11, 12, 16, 17$, then the index jump function i_a is formed as follows:

$$\begin{array}{rl} a = & [0; \overset{b_1}{1}, \overset{b_2}{2}, \overbrace{\underline{1}, 1}^{b_3}, \overset{b_4}{3}, \overbrace{\underline{1}, 1}^{b_5}, \overset{b_6}{a_8}, \overset{b_7}{a_9}, \overbrace{\underline{1}, a_{11}}^{b_8}, \overset{b_9}{a_{12}}, \overbrace{\underline{1}, 1}^{b_{10}}, \overbrace{\underline{1}, a_{16}}^{b_{11}}, \overset{b_{12}}{a_{17}}, \ldots] \\ & \quad \downarrow \; \downarrow \; \downarrow \quad \downarrow \; \downarrow \quad \downarrow \; \downarrow \; \downarrow \quad \downarrow \; \downarrow \; \downarrow \quad \downarrow \quad \cdots \\ (i_a(k))_{k \in \mathbf{N}^+} = & (\; 1, 2, 3, \; 5, 6, \; 8, 9, 10, \; 12, 13, 15, \; 17, \ldots). \end{array}$$

In the last row we presented the first 12 elements of the sequence of the values of the index jump function for these a, that is $(i_a(k))_{1\leqslant k\leqslant 12}$. The underlined 1s are essential for the construction of the digital line $y=ax$ (this will be explained later). The sequence of the run length on all the digitization levels (as will be defined in Theorem 11) for these slopes is $(b_n)_{n\in\mathbf{N}^+}=(1,2,1+1,3,1+1,a_8,a_9,1+a_{11},a_{12},1+1,1+a_{16},a_{17},\ldots)$.

The following theorem translates Definition 1 into the language of CFs. It is a very important step on the way of translating our earlier results into a simple CF description. This will also allow us to examine the connection between the sequence of the consecutive digitization parameters for given $a\in]0,1[\setminus\mathbf{Q}$ and the iterates of the Gauss map $G(a)=\mathrm{frac}(1/a)$, as will be shown in Section 5.

Theorem 9. *Let $a\in]0,1[\setminus\mathbf{Q}$ and $a=[0;a_1,a_2,\ldots]$. For the digital straight line with equation $y=ax$, the digitization parameters as defined in Definition 1 are*

$$\sigma_k=[0;a_{i_a(k+1)},a_{i_a(k+1)+1},\ldots] \quad \text{for } k\geqslant 1, \tag{2}$$

where i_a is the index jump function defined in Definition 7.

Proof. By induction. For $k=1$, the statement is $\sigma_1=[0;a_2,a_3,\ldots]$, because $i_a(2)=2$. From Definition 1 and because $a=[0;a_1,a_2,\ldots]$, we have $\sigma_1=\mathrm{frac}(1/a)=[0;a_2,a_3,\ldots]$, so the induction hypothesis for $k=1$ is true. Let us now suppose that $\sigma_k=[0;a_{i_a(k+1)},a_{i_a(k+1)+1},\ldots]$ for some $k\geqslant 1$. We will show that this implies that $\sigma_{k+1}=[0;a_{i_a(k+2)},a_{i_a(k+2)+1},\ldots]$. From Definition 7 we have $i_a(k+2)=i_a(k+1)+1+\delta_1(a_{i_a(k+1)})$. According to Definition 1, $\sigma_{k+1}=\mathrm{frac}(1/\sigma_k^{\wedge})$. We get two cases:

- $a_{i_a(k+1)}\neq 1$ (thus $\delta_1(a_{i_a(k+1)})=0$). This means that $\sigma_k<\frac{1}{2}$, so $\sigma_k^{\wedge}=\sigma_k$. We get the statement, because $\sigma_{k+1}=\mathrm{frac}(1/\sigma_k)=[0;a_{i_a(k+1)+1},\ldots]=[0;a_{i_a(k+2)},\ldots]$.
- $a_{i_a(k+1)}=1$ (thus $\delta_1(a_{i_a(k+1)})=1$). This means that $\sigma_k>\frac{1}{2}$, so $\sigma_k^{\wedge}=1-\sigma_k$. Lemma 6 and Definition 7 give us the statement, because $\sigma_{k+1}=\mathrm{frac}\,(1/(1-\sigma_k))=[0;a_{i_a(k+1)+2},\ldots]=[0;a_{i_a(k+2)},\ldots]$.

This completes the proof. □

In order to get a CF description of the digitization, we will express the function Reg_a (determining the form of the digitization runs on all the levels) using the function i_a defined in Definition 7. The translation of Definition 2 into the following CF version results in a very simple relationship between the complicated Reg_a and the simple i_a. It is a very important step in translating Theorem 4 into a CF version.

Theorem 10. *For a given $a\in]0,1[\setminus\mathbf{Q}$, there is the following connection between the corresponding functions Reg_a and i_a. For each $k\in\mathbf{N}^+$*

$$\mathrm{Reg}_a(k)=2k-i_a(k+1). \tag{3}$$

Proof. For $k=1$ a direct check gives the equality. Let us assume that $\mathrm{Reg}_a(k)=2k-i_a(k+1)$ for some $k\geqslant 1$. We will show that this implies $\mathrm{Reg}_a(k+1)=2k+2-i_a(k+2)$, which will, by induction, prove our statement.

It follows from Definition 2, that for $k\geqslant 1$

$$\mathrm{Reg}_a(k+1)=\mathrm{Reg}_a(k)+\chi_{]0,1/2[}(\sigma_k). \tag{4}$$

Moreover, according to Definition 7, for $k\geqslant 1$

$$i_a(k+2)=i_a(k+1)+1+\delta_1(a_{i_a(k+1)}). \tag{5}$$

Putting (4) and (5) in the induction hypothesis for $k+1$, we see that we have to show the following:

$$\mathrm{Reg}_a(k)+\chi_{]0,1/2[}(\sigma_k)=2k+2-(i_a(k+1)+1+\delta_1(a_{i_a(k+1)})). \tag{6}$$

Due to the induction hypothesis for k, it is enough to show that for all $k\geqslant 1$

$$\chi_{]0,1/2[}(\sigma_k)=1-\delta_1(a_{i_a(k+1)}). \tag{7}$$

To prove this, we use Theorem 9, which says that $\sigma_k=[0;a_{i_a(k+1)},\ldots]$:

$$\chi_{]0,1/2[}(\sigma_k)=1 \Leftrightarrow [0;a_{i_a(k+1)},a_{i_a(k+1)+1},\ldots]<\tfrac{1}{2}$$
$$\Leftrightarrow a_{i_a(k+1)}\neq 1 \Leftrightarrow 1-\delta_1(a_{i_a(k+1)})=1.$$

This completes the proof. □

Now we are ready to formulate our main theorem. The theorem is more parsimonious from a computational standpoint than Theorem 4, because the function i_a is very simple and contains only computations with integers. This is an important advantage for efficient computer program development. The entire description uses only one function: the index jump function.

Theorem 11 (*Main result; description by CFs*). *Let $a\in]0,1[\setminus\mathbf{Q}$ and $a=[0;a_1,a_2,\ldots]$. For the digital straight line with equation $y=ax$, we have $\|S_1\|=a_1$, $\|L_1\|=a_1+1$, and the forms of* runs_{k+1} ($\mathrm{form\_run}_{k+1}$) *for $k\in\mathbf{N}^+$ are as follows*:

$$\mathrm{form\_run}_{k+1}=\begin{cases} S_k^m L_k & \text{if } a_{i_a(k+1)}\neq 1 \text{ and } i_a(k+1) \text{ is even},\\ S_k L_k^m & \text{if } a_{i_a(k+1)}=1 \text{ and } i_a(k+1) \text{ is even},\\ L_k S_k^m & \text{if } a_{i_a(k+1)}\neq 1 \text{ and } i_a(k+1) \text{ is odd},\\ L_k^m S_k & \text{if } a_{i_a(k+1)}=1 \text{ and } i_a(k+1) \text{ is odd},\end{cases} \tag{8}$$

where $m=b_{k+1}-1$ if the run_{k+1} is short and $m=b_{k+1}$ if the run_{k+1} is long. The function i_a is defined in Definition 7 and $b_{k+1}=a_{i_a(k+1)}+\delta_1(a_{i_a(k+1)})a_{i_a(k+1)+1}$.

Proof. Theorem 11 follows from Theorems 4, 9 and 10. From Theorem 4 we know that the length of the short runs_{k+1} is $\lfloor 1/\sigma_k^{\wedge}\rfloor$. According to Theorem 9, $\sigma_k=[0;a_{i_a(k+1)},a_{i_a(k+1)+1},\ldots]$. We have to consider two cases:

- $a_{i_a(k+1)}>1$. This means that $\sigma_k<\frac{1}{2}$ and $\sigma_k^{\wedge}=\sigma_k$, so the length of the short runs on the level $k+1$ is $a_{i_a(k+1)}$. Because $\delta_1(a_{i_a(k+1)})=0$, we get the statement about the run lengths.
- $a_{i_a(k+1)}=1$. This means that $\sigma_k>\frac{1}{2}$ and (from Definition 1) $\sigma_k^{\wedge}=1-\sigma_k=[0;1+a_{i_a(k+1)+1},a_{i_a(k+1)+2},\ldots]$, so the length of the short runs on the level $k+1$ is $1+a_{i_a(k+1)+1}=a_{i_a(k+1)}+\delta_1(a_{i_a(k+1)})a_{i_a(k+1)+1}$.

Theorem 10 gives the statement concerning the form of runs on all levels. It says that $\mathrm{Reg}_a(k)=2k-i_a(k+1)$ and $\mathrm{Reg}_a(k+1)=2k+2-i_a(k+2)$, so the condition $\mathrm{Reg}_a(k+1)=\mathrm{Reg}_a(k)$ is equivalent to $i_a(k+2)=i_a(k+1)+2$, thus, according to Definition 7, to $\delta_1(a_{i_a(k+1)})=1$, so $a_{i_a(k+1)}=1$. In the same way we show that the condition $\mathrm{Reg}_a(k+1)=\mathrm{Reg}_a(k)+1$ is equivalent to $a_{i_a(k+1)}\neq 1$. Moreover, because $\mathrm{Reg}_a(k)=2k-i_a(k+1)$, the parity of $\mathrm{Reg}_a(k)$ and $i_a(k+1)$ is the same for all k, so we can replace "$\mathrm{Reg}_a(k)$ is even" from Theorem 4 by "$i_a(k+1)$ is even" in the CF description. □

Fig. 1 illustrates the connection between the hierarchy of runs (the first 5 levels), the index jump function and the digitization parameters for $y=ax$, where $a=[0;1,2,1,1,3,1,1,a_8,a_9,\ldots]$ for some $a_8,a_9,\ldots \in \mathbf{N}^+$, as described in Example 8.

Formula (8) shows that the value of the index jump function for each natural $k+1\geqslant 2$ describes the index of the CF element which determines the construction of runs on level $k+1$ in terms of runs of level k. If this CF element ($a_{i_a(k+1)}$) is equal to 1, the most frequent run on level k is the long one (L_k). In all the other cases, i.e., if $a_{i_a(k+1)}>1$, the most frequently appearing run on level k is the short one (S_k). This means that the CF elements equal to 1 which are indexed by the values of the index jump function (greater than 1) play a very special role in the run hierarchical construction of digitized $y=ax$. In the author's paper [4] (submitted manuscript), elements like this are called *essential* 1s. All the essential 1s in Example 8 are underlined. The non-essential 1s there are a_1,a_4,a_7,a_{14}.

Definition 12. Let $a=[0;a_1,a_2,\ldots]$ be irrational. Let $J=\emptyset$ if there are no 1s in the CF expansion of a (excepted maybe for a_1), $J=\mathbf{N}^+$ is there are infinitely many 1s in the CF expansion of a and $J=[1,M]_{\mathbf{Z}}$ for some $M\in\mathbf{N}^+$ if there are M essential 1s in the CF expansion of a. The following sequence $(s_j)_{j\in J}$: $s_1=\min(k\in\mathbf{N}^+; a_k$ is essential), and, for $n\in J\backslash\{1\}$, $s_n=\min(k>s_{n-1}; a_k$ is essential) (and $(s_i)_{i\in\emptyset}=\emptyset$ in case $J=\emptyset$) we will call the sequence of the places of essential 1s in the CF expansion of a.

In Example 8, the sequence of the places of essential 1s is $(3,6,10,13,15,\ldots)$. Essential 1s have been used in the submitted manuscript by the author [4] for a partition of all digital lines with slopes $a\in]0,1[\backslash\mathbf{Q}$ into equivalence classes. The equivalence relation is defined by the essential 1s of the CF expansions of the slopes; all the lines with identical sequences of the places of essential 1s (Definition 12) are joint in the same equivalence class. They have the same construction in terms of the forms of digitization runs. This partition was possible because of the description contained in Theorem 11.

4. Some applications of the main result

4.1. Slopes with periodic CF expansions

4.1.1. Period length 1

We are looking for numbers $a\in]0,1[\backslash\mathbf{Q}$ having periodic CF expansion with period length 1, i.e., $a=[0;n,n,\ldots]=[0;\overline{n}]$. This means that a is a root of equation $a=1/(n+a)$, thus of $a^2+na-1=0$, which gives $a=\frac{1}{2}(\sqrt{n^2+4}-n)$, because $a\in]0,1[$. We have two groups of lines:

- When $n=1$, we get a one-element group, containing the line $y=\frac{1}{2}(\sqrt{5}-1)x$ with the Golden Section as slope. Here we have $i_a(1)=1$ and, for $k\geqslant 2$, $i_a(k)=2k-2$, which is always even. Moreover, $b_1=1$ and for $k\geqslant 1$ we have $b_{k+1}=a_{2k}+a_{2k+1}=2$. According to Theorem 11, we get the following digitization: $\|S_1\|=1$, $\|L_1\|=2$; for $k\in\mathbf{N}^+$: $S_{k+1}=S_kL_k$, $L_{k+1}=S_kL_k^2$.
- When $n\geqslant 2$, we have $i_a(k)=k$ for each $k\in\mathbf{N}^+$ and $b_{k+1}=a_{k+1}=n$ for all $k\in\mathbf{N}$. This means, according to Theorem 11, that for all the lines $y=\frac{1}{2}(\sqrt{n^2+4}-n)x$ where $n\in\mathbf{N}^+\backslash\{1\}$, we get the following description of the digitization: $\|S_1\|=n$, $\|L_1\|=n+1$; for $k\in\mathbf{N}^+$: $S_{2k}=S_{2k-1}^{n-1}L_{2k-1}$, $L_{2k}=S_{2k-1}^{n}L_{2k-1}$, $S_{2k+1}=L_{2k}S_{2k}^{n-1}$, $L_{2k+1}=L_{2k}S_{2k}^{n}$.

4.1.2. Period length 2

Now we are looking for numbers $a\in]0,1[\backslash\mathbf{Q}$ having periodic CF expansion with period length 2, i.e., $a=[0;\overline{n,m}]$. This means that a is a root of equation $a=[0;n,m+a]$, thus of $na^2+mna-m=0$, which gives $a=1/(2n)(\sqrt{m^2n^2+4mn}-mn)$, because $a\in]0,1[$. If $m=n$, see the description for period length 1. If $m\neq n$, we get three possible classes of lines:

- When $m,n\geqslant 2$, we have $i_a(k)=k$ for each $k\in\mathbf{N}^+$ and $b_{k+1}=a_{k+1}$ for all $k\in\mathbf{N}$. This means, from Theorem 11, that for all the lines $y=ax$, where $a=1/(2n)(\sqrt{m^2n^2+4mn}-mn)$ for some $n,m\in\mathbf{N}^+\backslash\{1\}$, we get the following description of the digitization: $\|S_1\|=n$, $\|L_1\|=n+1$; for $k\in\mathbf{N}^+$: $S_{2k}=S_{2k-1}^{m-1}L_{2k-1}$, $L_{2k}=S_{2k-1}^{m}L_{2k-1}$, $S_{2k+1}=L_{2k}S_{2k}^{n-1}$, $L_{2k+1}=L_{2k}S_{2k}^{n}$.
- When $m=1$ and $n\geqslant 2$, we have $i_a(1)=1$ and $b_1=n$. For $k\in\mathbf{N}^+$ we have $i_a(k+1)=2k$ and $b_{k+1}=a_{2k}+a_{2k+1}=n+1$. The digitization is thus $\|S_1\|=n$, $\|L_1\|=n+1$; for $k\in\mathbf{N}^+$: $S_{k+1}=S_kL_k^n$, $L_{k+1}=S_kL_k^{n+1}$.
- When $m\geqslant 2$ and $n=1$, we have $i_a(1)=1, i_a(2)=2$ and $i_a(k+1)=2k-1$ for $k\geqslant 2$, which means that $i_a(k)$ is odd for all $k\neq 2$. Moreover, $b_1=1, b_2=m$ and $b_{k+1}=a_{2k-1}+a_{2k}=1+m$ for $k\geqslant 2$. The digitization is thus as follows: $\|S_1\|=1$, $\|L_1\|=2$, $S_2=S_1^{m-1}L_1$, $L_2=S_1^mL_1$, and for $k\geqslant 2$ we have $S_{k+1}=L_k^mS_k$ and $L_{k+1}=L_k^{m+1}S_k$.

4.1.3. Generally—quadratic surds

Let us recall that an *algebraic number of degree n* is a root of an algebraic equation $a_0x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n=0$ of degree n with integer coefficients, but is not a root of any algebraic equation of lower degree with integer coefficients. Algebraic numbers of the second degree are called *quadratic irrationals* or *quadratic surds*. The following theorem is a merge of Lagrange's theorem from 1770 with Euler's theorem from 1737 (see [17, pp. 66–71]). *Quadratic surds, and only they, are represented by periodic CFs,* meaning purely or mixed periodic [17, p. 66]. It follows from this theorem that all the lines with quadratic surds from the interval]0,1[as slopes have simple digitization patterns, which can be described by general formulae for all of the digitization levels. Moreover, in [18] on p. 88, we find the following theorem.

If d is a positive, non-square integer, then we have $\sqrt{d}=[x_0;\overline{x_1,x_2,\ldots,x_2,x_1,2x_0}]$, *where each partial quotient is a positive integer.*

The CFs of pure quadratic irrationals all have the same structure, involving *palindromes*. Sequence **A003285** in [19] shows for each $n\in\mathbf{N}^+$ the length of the period of CF for $\sqrt{n}$ (0 if n is a square). Also in [18], on p. 89, we find some patterns in the CF expansions of quadratic surds, for example $\sqrt{k^2+1}=[k;\overline{2k}]$, $\sqrt{k^2+2}=[k;\overline{k,2k}]$, $\sqrt{k^2+m}=[k;\overline{2k/m,2k}]$. These patterns make it very easy to construct the digitization of the lines with slopes $\sqrt{k^2+1}-k$, $\sqrt{k^2+2}-k$, or, generally, $\sqrt{k^2+m}-k$, using Theorem 11 from the present paper. See pp. 83–91 in [18] for both theory and examples on this subject.

4.2. Slopes with non-periodic CF expansions

Quadratic irrationals are not the only numbers showing simple patterns in their CF expansion. There also exist transcendental numbers with simple patterns. CF sequences for some transcendental number have *periodic forms*.

4.2.1. Examples involving Euler's number

Brezinski [20, p. 97] gives some examples of transcendental numbers with periodic form of CF expansion. The following examples were given by Euler in 1737, but the first of them was, according to Brezinski, already given by R. Cotes in the *philosophical transactions* in 1714.

$$e-2=[0;1,2,1,1,4,1,1,6,1,\ldots,1,2k,1,\ldots]=[0;\overline{1,2k,1}]_{k=1}^{\infty}, \tag{9}$$

$$\frac{e+1}{e-1}-2=[0;\overline{2+4k}]_{k=1}^{\infty},\quad \frac{e-1}{2}=[0;1,\overline{2+4k}]_{k=1}^{\infty}. \tag{10}$$

On p. 124 in [18] we find the following. For $n \geqslant 2$

$$\sqrt[n]{e} - 1 = [0; \overline{(2k-1)n-1, 1, 1}]_{k=1}^{\infty}. \tag{11}$$

On p. 110 in [20] we find the following formula, obtained by Euler in 1737 and Lagrange in 1776, but each using different methods:

$$\frac{e^2-1}{e^2+1} = [0; 1, 3, 5, \ldots, 2k-1, \ldots] = [0; \overline{2k-1}]_{k=1}^{\infty}. \tag{12}$$

This means that we are able to describe exactly, i.e., not by using approximations by rationals, the construction of the digital lines $y = ax$, where a is equal to $e-2$, $(e+1)/(e-1)-2$, $(e-1)/2$, $\sqrt{e}-1$, $\sqrt[3]{e}-1$ or $(e^2-1)/(e^2+1)$, using Theorem 11 from the present paper. Because of the repeating pattern in the CF expansions of the slopes, we are able to obtain general formulae for all of the digitization levels. Let us consider the following examples.

Example 13. If the slope a is equal to one of the following numbers: $(e+1)/(e-1)-2, (e-1)/2, (e^2-1)/(e^2+1)$, then the digitization patterns can be described for all the lines $y = ax$ in the following way. For all $k \in \mathbf{N}^+$ we have $i_a(k) = k$, thus $b_k = a_k$, because there are no elements $a_k = 1$ for $k \geqslant 2$ in the CF expansions. This gives the following digitization pattern for these lines: $\|S_1\| = a_1, \|L_1\| = a_1 + 1$ and for $k \in \mathbf{N}^+$

$$(S_{k+1}, L_{k+1}) = \begin{cases} (S_k^{a_{k+1}-1}L_k,\ S_k^{a_{k+1}}L_k) & \text{if } k \text{ is odd} \\ (L_kS_k^{a_{k+1}-1},\ L_kS_k^{a_{k+1}}) & \text{if } k \text{ is even.} \end{cases} \tag{13}$$

The only difference in the digitization patterns for the three slopes are different run lengths, defined by the elements a_k of the CF expansions (10) and (12). In all the cases the sequences of the places of essential 1s are $(s_n)_{n \in \emptyset} = \emptyset$.

Example 14. Formula (9) gives the digitization of the line $y = ax$ with $a = e-2$. Here $i_a(2k) = 3k-1$ (with $a_{i_a(2k)} = 2k \neq 1$) and $i_a(2k+1) = 3k$ (with $a_{i_a(2k+1)} = 1$) for $k \in \mathbf{N}^+$, so we get $b_1 = 1$, $b_{2k} = 2k$ and $b_{2k+1} = 2$ for $k \in \mathbf{N}^+$, and the digitization pattern is as follows: $\|S_1\| = 1, \|L_1\| = 2$ and for $k \in \mathbf{N}^+$

$$(S_{k+1}, L_{k+1}) = \begin{cases} (S_k^kL_k,\ S_k^{k+1}L_k) & \text{if } k \equiv 1 \pmod 4, \\ (S_kL_k,\ S_kL_k^2) & \text{if } k \equiv 0 \pmod 4, \\ (L_kS_k^k,\ L_kS_k^{k+1}) & \text{if } k \equiv 3 \pmod 4, \\ (L_kS_k,\ L_k^2S_k) & \text{if } k \equiv 2 \pmod 4. \end{cases} \tag{14}$$

For example, $S_5 = S_4L_4 = (L_3S_3^3)(L_3S_3^4) = (L_2^2S_2)(L_2S_2)^3(L_2^2S_2)(L_2S_2)^4 = (S_1^2L_1)^2\,(S_1L_1)\ [(S_1^2L_1)\ (S_1L_1)]^3\ (S_1^2L_1)^2\ (S_1L_1)\ [(S_1^2L_1)\ (S_1L_1)]^4$, where $\|S_1\| = 1$ and $\|L_1\| = 2$. The corresponding sequence of essential 1s is $(3k)_{k \in \mathbf{N}^+}$.

Example 15. Formula (11) gives the digitization of the lines $y = a_nx$ with $a_n = \sqrt[n]{e} - 1$ for $n \geqslant 2$. Here, for each $a = a_n$, $i_a(2k) = 3k-1$ (with $a_{i_a(2k)} = 1$) and $i_a(2k+1) = 3k+1$ (with $a_{i_a(2k+1)} = (2k+1)n-1 \neq 1$) for $k \in \mathbf{N}^+$, so we get $b_1 = n-1, b_{2k} = 2$ and $b_{2k+1} = (2k+1)n-1$ for $k \in \mathbf{N}^+$, and the digitization pattern is as follows: $\|S_1\| = n-1$, $\|L_1\| = n$ and for $k \in \mathbf{N}^+$

$$(S_{k+1}, L_{k+1}) = \begin{cases} (S_k^{(k+1)n-2}L_k,\ S_k^{(k+1)n-1}L_k) & \text{if } k \equiv 2 \pmod 4, \\ (S_kL_k,\ S_kL_k^2) & \text{if } k \equiv 1 \pmod 4, \\ (L_kS_k^{(k+1)n-2},\ L_kS_k^{(k+1)n-1}) & \text{if } k \equiv 0 \pmod 4, \\ (L_kS_k,\ L_k^2S_k) & \text{if } k \equiv 3 \pmod 4. \end{cases} \tag{15}$$

For example, $S_5 = L_4S_4^{5n-2} = (L_3^2S_3)(L_3S_3)^{5n-2} = (S_2^{3n-1}L_2)^2S_2^{3n-2}L_2(S_2^{3n-1}L_2S_2^{3n-2}L_2)^{5n-2} = [(S_1L_1)^{3n-1}S_1L_1^2]^2(S_1L_1)^{3n-2}S_1L_1^2[(S_1L_1)^{3n-1}S_1L_1^2(S_1L_1)^{3n-2}S_1L_1^2]^{5n-2}$, where $\|S_1\| = n-1$ and $\|L_1\| = n$.

The corresponding sequence of essential 1s is $(3k-1)_{k \in \mathbf{N}^+}$. All the lines $y = a_nx$ with $a_n = \sqrt[n]{e} - 1$ for $n \geqslant 2$ have the same construction in terms of long and short digitization runs on all digitization levels. They all belong to the class generated by the sequence of important 1s being $(3k-1)_{k \in \mathbf{N}^+}$.

4.2.2. Tangent function and digital rotations

Lambert proved in 1761 and described in [21] the following formula. For $x \neq \pi/2 + m\pi$, where $m \in \mathbf{Z}$,

$$\tan x = \cfrac{x}{1 - \cfrac{x^2}{3 - \cfrac{x^2}{5 - \cfrac{x^2}{\ddots}}}} \tag{16}$$

We put in (16) $x = 1/k$ for any $k \in \mathbf{N}^+ \setminus \{1\}$, multiply successively the numerators and denominators of the *tails* (the portions of the number $\tan 1/k$ remaining after a given convergent) by k and apply successively Lemma 5 to them (when we are ready with the multiplication) and we get the following formula:

$$\tan\frac{1}{k} = [0; k-1, \overline{1, (2n+1)k-2}]_{n=1}^{\infty} \quad \text{for } k \geqslant 2. \tag{17}$$

This formula can also be found in Michon [22]. Together with (8), formula (17) shows that for each natural $k \geqslant 2$ the lines $y = (\tan 1/k)x$ (i.e., the lines which form the angle of $1/k$ radian with the positive x-axis) have exactly the same construction in terms of long and short digitization runs on all the levels. Only the run lengths on all the levels differ between each two lines from the set $\{y = (\tan 1/k)x; k = 2, 3, \ldots\}$. They all belong to the equivalence class generated by the Golden Section, or, equivalently, by the sequence of essential 1s $(2n)_{n \in \mathbf{N}^+}$ (cf. Section 4.1.1, example with $n = 1$). As shown in the author's paper [4] (submitted manuscript), it is the only class under the relation defined by essential 1s which has a largest element. The largest element of this class is the Golden Section. Let us consider the following example.

Example 16. Let us put $k = 2$ in (17). Then we get

$$\tan\tfrac{1}{2} = [0; 1, 1, 4, 1, 8, 1, 12, 1, 16, \ldots] = [0; 1, \overline{1, 4n}]_{n=1}^{\infty}. \tag{18}$$

To get the digitization of the line $y = ax$ which forms the angle of $\frac{1}{2}$ radian with the positive x-axis, we use the CF elements of the slope ($a_1 = 1$; $a_{2n} = 1$ and $a_{2n+1} = 4n$ for $n \in \mathbf{N}^+$), which gives $i_a(1) = 1$, $b_1 = 1$, and, for $n \in \mathbf{N}^+$, we have $i_a(n+1) = 2n$ and $b_{n+1} = a_{2n} + a_{2n+1} = 4n+1$. From Theorem 11 we obtain the following digitization pattern for the line $y = (\tan\frac{1}{2})x$: $\|S_1\| = 1$, $\|L_1\| = 2$; for $n \in \mathbf{N}^+$: $S_{n+1} = S_nL_n^{4n}$, $L_{n+1} = S_nL_n^{4n+1}$. A part of S_3 for this line is presented on Fig. 2.

Formula (16) together with the formula

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

and under the condition that CF arithmetic is not too hard to implement and can be economically and quickly performed by computers, would give us a pure digital geometrical formula for rotations around the origin about some angles of digital lines $y = ax$. Pure digital geometrical in the sense of only integer calculations on the elements of the CFs describing the slope of the line to rotate (being tangent of

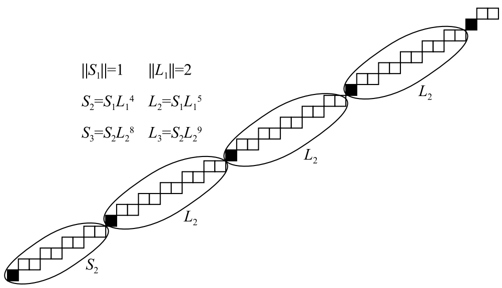

Fig. 2. A part of S_3 for the line $y=(\tan\frac{1}{2})x$.

the angle between the line and the positive x-axis) and the CF representing the tangent of the rotation angle. If $a=[0;a_1,a_2,\ldots]$ and the rotation angle is $\phi=1/k$ (in radians) where $k\in\mathbf{N}^+\backslash\{1\}$, then the CF representing the slope of $\mathrm{R}^{\phi}_{(0,0)}(l)$ being the result of the rotation of l with equation $y=ax$ around the origin with rotation angle ϕ will be the following:

$$\frac{[0;k-1,1,3k-2,1,\ldots,(2n+1)k-2,1,\ldots]+[0;a_1,a_2,\ldots]}{1-[0;k-1,1,3k-2,1,\ldots,(2n+1)k-2,1,\ldots]\cdot[0;a_1,a_2,\ldots]},$$

we thus get the CF expansion of the slope of the rotated line. This would give us rotations about all the rational angles (in radian) because the formula for $\tan(\phi_1+\phi_2)$ gives us the CF expansions of $\tan n/k$ for all $n,k\in\mathbf{N}^+$, where $k\geqslant 2$. While rotating, we would get slopes greater than 1, which we do not handle in Theorem 11. This is however not the major problem—it is easy to adapt our description to slopes greater than 1 (digitizations of lines with irrational slopes $a<0$ and $a>1$ can be obtained by a change of coordinates). A much more serious problem would be to find good and efficient algorithms for performing arithmetical operations on CFs. In the abstract of [23], Gosper states: *Contrary to everybody, this self contained paper will show that continued fractions are not only perfectly amenable to arithmetic, they are amenable to perfect arithmetic.*

The problem of digital rotations has been extensively treated by B. Nouvel and E. Rémila (see [24] and references there), but without the use of CFs.

5. The Gauss map and the digitization parameters

As mentioned in Section 3, Theorem 9 can help us establish the connection between the sequence of the digitization parameters $(\sigma_n)_{n\in\mathbf{N}^+}$ for the line $y=ax$ and the sequence of the iterates of the Gauss map $(G^n(a))_{n\in\mathbf{N}^+}$ for each $a\in]0,1[\backslash\mathbf{Q}$. Let us recall that the Gauss map $G:[0,1]\to[0,1]$ is defined as follows (see also [25]): $G(x)=\mathrm{frac}(1/x)$ if $0<x\leqslant 1$ and $G(0)=0$. For $x=[0;a_1,a_2,\ldots]$ we have obviously

$$G^n(x)=[0;a_{n+1},a_{n+2},\ldots]\quad\text{for } n\in\mathbf{N}^+. \tag{19}$$

Formula (2) from Theorem 9 together with (19) gives us the following:

$$\sigma_k=G^{i_a(k+1)-1}(a)\quad\text{for all } k\in\mathbf{N}^+. \tag{20}$$

Thus, according to Definition 7, the digitization parameters can be expressed in terms of the Gauss map and the index jump function in the following way: $\sigma_1=G(a)$, $\sigma_{k+1}=G^{i_a(k+2)-1}(a)=G^{i_a(k+1)+\delta_1(a_{i_a(k+1)})}(a)$ for $k\in\mathbf{N}^+$.

This means that, for a given $a\in]0,1[\backslash\mathbf{Q}$, the sequence of the corresponding digitization parameters $(\sigma_n)_{n\in\mathbf{N}^+}$ is equal to the sequence of the iterates of the Gauss map $(G^n(a))_{n\in\mathbf{N}^+}$ without some elements. The elements which are absent in the sequence are exactly the iterates which are indexed by the values of the index jump function associated with a (i.e. are equal to $i_a(k+1)$ for some $k\in\mathbf{N}^+$) for which $a_{i_a(k+1)}=1$ (which follows from Definition 7). Going back to Theorem 11 and the comments following its proof we notice that the missing iterates correspond to the digitization levels with runs constructed mainly of the *long* runs of the previous level. Or, following the terminology from the paper [4], the missing elements in the sequence of the iterates of the Gauss map for a are these which are indexed by the place numbers of the essential 1s in the CF expansion of a (Definition 12). We have just proven the following proposition.

Proposition 17. *Let $a\in]0,1[\backslash\mathbf{Q}$ and $a=[0;a_1,a_2,\ldots]$. The relation between the corresponding sequences $(\sigma_n)_{n\in\mathbf{N}^+}$ of the digitization parameters for $y=ax$ and $(G^n(a))_{n\in\mathbf{N}^+}$ of the iterates of the Gauss map is the following*:

$$(\sigma_n)_{n\in\mathbf{N}^+}=(G^n(a))_{n\in I}\quad\text{where } I=\mathbf{N}^+\backslash\{s_j\}_{j\in J},$$

where $\{s_j\}_{j\in J}$ is the sequence of the places of essential 1s in the CF expansion of a, as described in Definition 12.

For each $a\in]0,1[\backslash\mathbf{Q}$, such a subsequence of the sequence of all the iterates of the Gauss map corresponding to a, describes thus the form of digitization runs on all the levels for $y=ax$, according to Theorem 4.

6. Conclusion

We have presented a computationally simple description of the digitization of straight lines $y=ax$ with slopes $a\in]0,1[\backslash\mathbf{Q}$, based on the CF expansions of the slopes. The description reflects the hierarchical structure of digitization runs. Moreover, it is exact, avoiding approximations by rationals.

The theoretical part of the paper was based on [1] and the examples were based on the literature concerning CFs [18,20,21]. The examples show how to use the theory in finding digitization patterns. This description can also be useful in theoretical research on digital lines with irrational slopes. For example, in [4] we have examined some classes of digital lines, defined by the CF expansions of their slopes.

The present method gives a special treatment to the CF elements equal to 1, which makes it very powerful for some slopes with 1s in the CF expansion. To our knowledge, there exist no other methods of describing digital lines with irrational slopes by CFs which give a special treatment to CF elements equal to 1, which makes our method original.

A comparison between the present method and some other CF methods is included in another paper by the author [26]. We show there for example how to construct a slope $a\in]0,1[\backslash\mathbf{Q}$ so that for each $n\geqslant 2$ the difference between the length of the digital straight line segment (its cardinality as a subset of $\mathbf{Z}^2$) of $y=ax$ produced in the nth step of our method and the length of the digital straight line segment of $y=ax$ produced in the nth step of the method described in [3] on p. 67 is as large as we decide in advance. In the same paper we have shown that our method is different from the method by *standard sequences*, as described in Lothaire [16, p. 75, 76, 104, 105].

Acknowledgement

I am grateful to Christer Kiselman for comments on earlier versions of the manuscript.

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Paper III

A Run-hierarchical Description of Upper Mechanical Words with Irrational Slopes Using Continued Fractions

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Abstract. The main result is a run-hierarchical description (by continued fractions) of upper mechanical words with slope $a \in]0,1[\setminus \mathbf{Q}$ and intercept 0. We compare this description with two classical methods of forming of such words. In order to be able to perform the comparison, we present a quantitative analysis of our method. We use the denominators of the convergents of the continued fraction expansion of the slope to compute the length of the prefixes obtained by our method. Due to the special treatment which is given to the elements equal to 1, our method gives in some cases longer prefixes than the two other methods. Our method reflects the hierarchy of runs, by analogy to digital lines, which can give a new understanding of the construction of upper mechanical words.

Keywords: upper mechanical word, characteristic word, digital line, irrational slope, continued fraction, run, hierarchy.

1 Introduction

In the presented paper we have basically two goals. The first one is to create a description of the construction of upper mechanical words (Def. 2) with irrational positive slope $a < 1$ and intercept 0, according to the hierarchy of runs, runs of runs, etc. Such a description can be a useful tool for examining of properties of upper and lower mechanical and characteristic words with irrational slopes, as has been shown in another paper by the author [10]. Our second goal is to show that our method works, in certain cases, faster than two well-known methods of forming of prefixes of characteristic words.

The theoretical base for this article are two earlier papers [8, 9] of the author. The run-hierarchical method is derived from the author's continued fraction (CF) based description of digitization of positive half lines $y = ax$. It is based on simple integer computations, thus can be used with advantage in computer programming. This qualitative description constitutes the first main result of the present paper (Theorem 3).

The second main result is a quantitative description of our method of forming prefixes of upper mechanical words (Theorem 6 and Corollary 1). We show there how to calculate the length of the prefixes of upper mechanical word formed according to our method. The length is expressed in terms of the denominators of the convergents of the CF expansion of the slope. These formulae allow us to compare our method with the classical and most frequently used descriptions by Venkov (1970) [11] and by Shallit (1991) [7]. In both of them one can express the length of the prefixes by these denominators (see Proposition 2 and Theorem 5).

The special treatment the CF elements equal to 1 get in our description (Theorem 3) makes that our process of forming words generates for some slopes longer prefixes than the similar classical recursive formulae presented by Shallit and Venkov.

We show that, for all a, the prefix P_k of the upper mechanical word $s'(a) = 1c(a)$ generated by our method is longer or of the same length compared to the prefix X_k of the corresponding word $c(a)$ generated by Shallit's method for each $k \in \mathbf{N}^+$ (Proposition 3). For some a our method generates *much* longer prefixes (Proposition 4 and Theorem 7).

The comparison with Venkov's method begins with Theorem 8. It depends on the set of 1's in the CF expansion of the slope a. For some a our method gives much longer prefixes than the method of Venkov after the same number of steps and our advantage can be as large as we want. For other slopes the method of Venkov generates longer prefixes (Proposition 5). However, Venkov's advantage in the k^{th} step for each $k \geq 3$ is always bounded by k (Proposition 6), while our advantage in case of slopes containing 1's in their CF expansion can be arbitrarily large. The *advantage* in this paper is expressed by the quotient of the length of the prefixes obtained when using the methods we compare.

The fact that we highlight some CF elements equal to 1 in the expansion of the slopes is not because they give us sometimes an advantage of forming longer prefixes than when using the well-known methods but because they determine the construction of lines (words) in terms of runs. This will be explained in what follows under Theorem 2 and in Section 6.

A list of references to papers concerning CF descriptions of characteristic words with irrational slopes can be found in Lothaire (2002) [4]. The most relevant for the present paper are Bernoulli (1772), Markoff (1882), Stolarsky (1976), Fraenkel et al. (1978) and Brown (1993). The first three papers correspond to the method of Venkov (described already much earlier by Markov), the last two correspond to the method of Shallit. The CF description method presented in Theorem 3 seems to be the only one which gives prefixes constructed according to the run hierarchy. This enables us to analyze the construction of upper mechanical words, which has been presented by the author in [10].

2 Continued Fractions — a Brief Introduction

The following algorithm gives the regular (or simple) CF for $a \in \mathbf{R} \setminus \mathbf{Q}$, which we denote by $[a_0; a_1, a_2, a_3, \ldots]$. We define a sequence of integers (a_n) and a sequence of real numbers (α_n) by: $\alpha_0 = a;\ a_n = \lfloor \alpha_n \rfloor$ and $\alpha_n = a_n + \frac{1}{\alpha_{n+1}}$ for $n \geq 0$. Then $a_n \geq 1$ and $\alpha_n > 1$ for $n \geq 1$. The integers $a_0, a_1, a_2, \ldots$ are called the *elements* of the

CF (or *terms*, or *partial quotients*). We use the word *elements*, following Khinchin (1997:1) [2]. Because a is irrational, so is each α_n, and the sequences (a_n) and (α_n) are infinite. A CF expansion exists and is unique for all $a \in \mathbf{R} \setminus \mathbf{Q}$; see [2], p. 16.

For $k \in \mathbf{N}$, the k^{th} *order convergent* of the CF $a = [a_0; a_1, a_2, \ldots]$ is the canonical representation of the number $s_k = [a_0; a_1, a_2, \ldots, a_k]$. We will denote it by p_k/q_k. The following theorem comes from the definition of CFs and can be found for example in Khinchin (1997:4) [2].

Theorem 1. *For the denominators of the convergents of each* $a = [a_0; a_1, a_2, \ldots]$ *we have* $q_0 = 1,\ q_1 = a_1,$ *and, for* $k \geq 2,\quad q_k = a_k q_{k-1} + q_{k-2}$.

It follows immediately from the recursive formula in Theorem 1 that the sequence $(q_n)_{n \in \mathbf{N}^+}$ for each $a \in \mathbf{R} \setminus \mathbf{Q}$ is a **strictly increasing** sequence of natural numbers. We will exploit this fact heavily in what follows.

3 Earlier Results

In this section we recapitulate some results obtained by the author in [9]. Arithmetical description of the modified Rosenfeld digitization (R′-digitization) of the positive half line $y = ax$ for $a \in \,]0,1[\, \setminus \mathbf{Q}$ as a subset of $\mathbf{Z}^2$ is the following:

$$D_{R'}(y = ax,\ x > 0) = \{(k, \lceil ak \rceil);\quad k \in \mathbf{N}^+\}. \tag{1}$$

Our CF description from [9] was based on the description by digitization parameters from Uscka-Wehlou (2007) [8] and the following *index jump function*.

Definition 1. *For each* $a \in \,]0,1[\, \setminus \mathbf{Q}$, *the* index jump function $i_a : \mathbf{N}^+ \to \mathbf{N}^+$ *is defined by* $i_a(1) = 1,\ i_a(2) = 2$ *and* $i_a(k+1) = i_a(k) + 1 + \delta_1(a_{i_a(k)})$ *for* $k \geq 2$, *where* $\delta_1(x) = \begin{cases} 1, x = 1 \\ 0, x \neq 1 \end{cases}$ *and* $a_1, a_2, \ldots \in \mathbf{N}^+$ *are the CF elements of* a.

The index jump function is a renumbering, which avoids elements following directly after some 1's in the CF expansion (in particular, it avoids every second element in the sequences of consecutive 1's with index greater than 1).

In both papers [8] and [9], digital lines were described according to the hierarchy of runs on all the digitization levels. The term *run* was already introduced by Azriel Rosenfeld (1974:1265) [6]. For the formal definition of runs and the modification of Rosenfeld digitization see [8]. We called $\mathrm{run}_k(j)$ for $k, j \in \mathbf{N}^+$ a *run of digitization level* k. Each $\mathrm{run}_1(j)$ can be identified with a subset of $\mathbf{Z}^2$: $\{(i_0+1, j), (i_0+2, j), \ldots, (i_0+m, j)\}$, where m is the length $|\mathrm{run}_1(j)|$ of this run. For each $a \in \,]0,1[\, \setminus \mathbf{Q}$ we have only two possible run_1 lengths: $\left\lfloor \frac{1}{a} \right\rfloor$ and $\left\lfloor \frac{1}{a} \right\rfloor + 1$. All the runs with one of those lengths always occur alone, i.e., do not have any neighbors of the same length in the sequence $(\mathrm{run}_1(j))_{j \in \mathbf{N}^+}$, while the runs of the other length can appear in sequences. The same holds for the sequences $(\mathrm{run}_k(j))_{j \in \mathbf{N}^+}$ on each level $k \geq 2$, i.e., runs on each level k can have one of two possible lengths (being consecutive natural numbers) and runs with one of these lengths always appear alone in the sequence of runs_k. Runs of level $k+1$ for $k \in \mathbf{N}^+$ are defined recursively, as sets of runs_k and, in this context (but it will no longer be so in Section 5), by the length of run_{k+1} we mean its cardinality. Each run_{k+1} consists of one singly appearing run_k (called *short* run of level k and denoted S_k if its length is expressed by the least of the mentioned consecutive numbers for level k and called *long* run of level k and denoted L_k otherwise) and all the runs_k (L_k or S_k, respectively) which can appear in sequences comming between this single run_k and the next or the previous single run_k, depending on $\text{run}_k(1)$, in the sequence $(\text{run}_k(j))_{j\in\mathbf{N}^+}$. This means that runs_{k+1} for each $k \in \mathbf{N}^+$ can have one of following four shapes: $S_k^m L_k$, $L_k S_k^m$, $L_k^m S_k$ or $S_k L_k^m$, where m can be one of two consecutive positive integers which depend on the slope a and the level number k. For example, $S_k^m L_k$ means that the run_{k+1} consists of m short runs_k (S_k) and one long run_k (L_k) in this order. For the purpose of this paper this description suffices; for the formal definition see [8]. Moreover, Theorem 2, proven in [9], can serve as a definition of runs in the digitizations of straight lines $y = ax$ for $a \in \,]0,1[\, \setminus \mathbf{Q}$, since it presents a complete recurrent description of these. The theorem is completely CF based.

Theorem 2 (Main Result in [9]; description by CFs). *Let* $a \in \,]0,1[\, \setminus \mathbf{Q}$ *and* $a = [0; a_1, a_2, \ldots]$. *For the digital line with equation* $y = ax$, *we have* $|S_1| = a_1,\ |L_1| = a_1 + 1,$ *and the forms of* runs_k (form_run_k) *for* $k \geq 2$ *are as follows:*

$$\text{form\_run}_k = \begin{cases} S_{k-1}^m L_{k-1} & \textit{if} \quad a_{i_a(k)} \neq 1 \quad \textit{and} \quad i_a(k) \quad \textit{is even} \\ S_{k-1} L_{k-1}^m & \textit{if} \quad a_{i_a(k)} = 1 \quad \textit{and} \quad i_a(k) \quad \textit{is even} \\ L_{k-1} S_{k-1}^m & \textit{if} \quad a_{i_a(k)} \neq 1 \quad \textit{and} \quad i_a(k) \quad \textit{is odd} \\ L_{k-1}^m S_{k-1} & \textit{if} \quad a_{i_a(k)} = 1 \quad \textit{and} \quad i_a(k) \quad \textit{is odd,} \end{cases} \tag{2}$$

where $m = b_k - 1$ *if the* run_k *is short* (S_k) *and* $m = b_k$ *if the* run_k *is long* (L_k). *The function* i_a *is defined in Def. 1 and* $b_k = a_{i_a(k)} + \delta_1(a_{i_a(k)}) a_{i_a(k)+1}$.

We remark that the value of the index jump function for each natural $k \geq 2$ describes the index of the CF element which determines the construction of runs on level k in terms of runs of level $k-1$. If this CF element ($a_{i_a(k)}$) is equal to 1, the most frequent run on level $k-1$ is the long one (L_{k-1}). In all the other cases, i.e., if $a_{i_a(k)} > 1$, the most frequently appearing run on level $k-1$ is the short one (S_{k-1}). This means that the CF elements equal to 1 which are indexed by the values of the index jump function (greater than 1) play a very special role in the run hierarchical construction of digitized $y = ax$. In the author's paper [10] elements like this are called *essential* 1's. They have been used in [10] for a partition of all digital lines with slopes $a \in \,]0,1[\, \setminus \mathbf{Q}$ into equivalence classes. The equivalence relation is defined by the essential 1's of the CF expansions of the slopes and all the lines belonging to the same class have the same construction in terms of the forms of digitization runs. This partition was possible because of the description contained in Theorem 2 and can be of interest for combinatorics on words, due to the equivalence between digital lines and mechanical words.

4 Characteristic and (Upper, Lower) Mechanical Words and the Modified Rosenfeld Digitization

First we provide a brief introduction to characteristic and upper and lower mechanical words. The following definition comes from Lothaire (2002:53) [4].

Definition 2. *Given two real numbers α and ρ with $0 \le \alpha \le 1$, we define two infinite words $s_{\alpha,\rho} : \mathbf{N} \to \{0,1\}, \quad s'_{\alpha,\rho} : \mathbf{N} \to \{0,1\}$ by*

$$s_{\alpha,\rho}(n) = \lfloor \alpha(n+1)+\rho \rfloor - \lfloor \alpha n + \rho \rfloor, \quad s'_{\alpha,\rho}(n) = \lceil \alpha(n+1)+\rho \rceil - \lceil \alpha n + \rho \rceil.$$

The word $s_{\alpha,\rho}$ is the lower mechanical word *and $s'_{\alpha,\rho}$ is the* upper mechanical word *with* slope α *and* intercept ρ. *A lower or upper mechanical word is* irrational *or* rational *according as its slope is irrational or rational.*

In the present paper we deal with the special case when $\alpha \in]0,1[$ is irrational and $\rho = 0$. In this case we will denote the lower and upper mechanical words by $s = s(\alpha)$ and $s' = s'(\alpha)$ respectively. We have $s_0 = s_0(\alpha) = \lfloor \alpha \rfloor = 0$ and $s'_0 = s'_0(\alpha) = \lceil \alpha \rceil = 1$ and, because $\lceil x \rceil - \lfloor x \rfloor = 1$ for irrational x, we have

$$s = s(\alpha) = 0c(\alpha), \quad s' = s'(\alpha) = 1c(\alpha) \tag{3}$$

(meaning 0, resp. 1 concatenated to $c(\alpha)$). The word $c(\alpha)$ is called the *characteristic word* of α. For each $\alpha \in]0,1[\setminus \mathbf{Q}$, the characteristic word associated with α is thus the following infinite word $c = c(\alpha) : \mathbf{N}^+ \to \{0,1\}$:

$$c_n = \lfloor \alpha(n+1) \rfloor - \lfloor \alpha n \rfloor = \lceil \alpha(n+1) \rceil - \lceil \alpha n \rceil, \quad n \in \mathbf{N}^+. \tag{4}$$

The connection between characteristic words and digital lines is a well-known fact. See for example Lothaire (2002:53, 2.1.2 *Mechanical words, rotations*) [4], Pytheas Fogg (2002:143, 6. *Sturmian Sequences*) [5] or Klette and Rosenfeld (2004) [3]. In [8] the author remarks that the modified Rosenfeld digitization of the line $y = ax$, where $a \in]0,1[\setminus \mathbf{Q}$, is the subset of $\mathbf{Z}^2$ described by (1). This means, from (3) and (4), that the sequence $s'_0 = 1, \ s'_n = \lfloor (n+1)a \rfloor - \lfloor na \rfloor$ for $n \in \mathbf{N}^+$ describes the R$'$-digitization of $y = ax, x > 0$. So, for any $a \in]0,1[\setminus \mathbf{Q}$, the upper mechanical word $s'(a)$, as defined in Def. 2, describes completely the digitization of the positive half line $y = ax$. We can thus write $s'(a) = 10^{m_1}10^{m_2}10^{m_3}\ldots$, where $m_i \in \mathbf{N}$ for $i \in \mathbf{N}^+$. We have $|\mathrm{run}_1(i)| = 1 + m_i$, each run begins with a 1. Moreover, there exists $d_1 \in \mathbf{N}$ such that for all $i \in \mathbf{N}^+$ we have $m_i = d_1$ or $m_i = d_1 + 1$ and we know from the theory for digital lines that $d_1 = \lfloor \frac{1}{a} \rfloor - 1$. If $\lfloor \frac{1}{a} \rfloor = 1$, then $d_1 = 0$ and $|S_1| = 1$. Because of the correspondence between digital lines $y = ax$ and upper mechanical words $s'(a)$ for $a \in]0,1[\setminus \mathbf{Q}$, we also have the run hierarchical structure of upper mechanical words. Runs of level 1 are $S_1 = 10^{d_1}$ and $L_1 = 10^{d_1+1}$, where $d_1 = \lfloor \frac{1}{a} \rfloor - 1$ and we can defined recursively for each $k \in \mathbf{N}^+$ the runs of level $k+1$ as sets of runs of level k symbolically denoted as $S_k^{d_{k+1}} L_k$, $L_k S_k^{d_{k+1}}$, $L_k^{d_{k+1}} S_k$ or $S_k L_k^{d_{k+1}}$, where

d_{k+1} can be one of two consecutive positive integers which depend on the slope a and the level number $k+1$. We again use the notation of S for *short* and L for *long*, because words also have two possible run lengths (cardinalities) per level, due to the equivalence between digital lines and upper mechanical words.

The upper mechanical words s' for the slopes of all the lines with digitization around the origin as shown in the picture in Fig. 1 begin with 10001000.

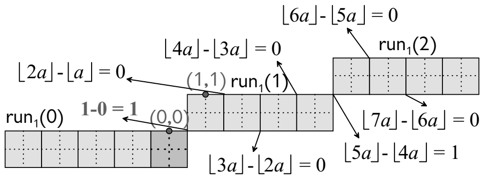

Fig. 1. Upper mechanical words $s'(a)$ and digital lines $y = ax$ for $a \in \left]\frac{1}{5}, \frac{1}{4}\right[\setminus \mathbf{Q}$.

This correspondence between the words and digital lines allows us to derive the following CF description of upper mechanical words from our result for digital lines. Because we have (3), our result will also give a description of lower mechanical words and characteristic words.

Theorem 3 (Main Result 1; a run-hierarchical CF description of upper mechanical words). *Let $a \in]0,1[\setminus \mathbf{Q}$ and $a = [0; a_1, a_2, \ldots]$. For $s'(a)$ as in Def. 2 we have $s'(a) = \lim_{k\to\infty} P_k$, where $P_1 = S_1 = 10^{a_1-1}$, $L_1 = 10^{a_1}$, and, for $k \geq 2$,*

$$P_k = \begin{cases} L_k = S_{k-1}^{a_{i_a(k)}} L_{k-1} & \text{if } \ a_{i_a(k)} \neq 1 \ \text{ and } \ i_a(k) \ \text{ is even} \\ S_k = S_{k-1} L_{k-1}^{a_{i_a(k)+1}} & \text{if } \ a_{i_a(k)} = 1 \ \text{ and } \ i_a(k) \ \text{ is even} \\ S_k = L_{k-1} S_{k-1}^{-1+a_{i_a(k)}} & \text{if } \ a_{i_a(k)} \neq 1 \ \text{ and } \ i_a(k) \ \text{ is odd} \\ L_k = L_{k-1}^{1+a_{i_a(k)+1}} S_{k-1} & \text{if } \ a_{i_a(k)} = 1 \ \text{ and } \ i_a(k) \ \text{ is odd,} \end{cases} \tag{5}$$

*where the function i_a is defined in Def. 1. The meaning of the symbols is the following: for $k \geq 1$, P_k - **P**refix number k, S_k - **S**hort* run_k *and L_k - **L**ong* run_k. *To make the recursive formula (5) complete, we add that for each $k \geq 2$, if $P_k = S_k$, then L_k is defined in the same way as S_k, with the only difference that the exponent defined by $a_{i_a(k)}$ (or by $a_{i_a(k)+1}$) is increased by 1. If $P_k = L_k$, then S_k is defined in the same way as L_k, with the only difference that the exponent defined by $a_{i_a(k)}$ (or by $a_{i_a(k)+1}$) is decreased by 1.*

Proof. We use Theorem 2 and the equivalence between the digital half lines $y = ax$ (where $a \in]0,1[\setminus \mathbf{Q}$ and $x > 0$) and the words $s'(a)$. We introduced P_k which corresponds to $\mathrm{run}_k(1)$ for each $k \in \mathbf{N}^+$. According to Theorem 2, $\mathrm{run}_{k-1}(1)$ for $k \geq 2$ is short if $i_a(k)$ is even (this result is represented by the first two rows of (2)) and long if $i_a(k)$ is odd (rows 3 and 4 in (2)). This means that $P_k = \mathrm{run}_k(1)$ is short (S_k) if $i_a(k+1)$ is even and long (L_k) if $i_a(k+1)$ is odd. Because we have

$i_a(k+1) = i_a(k) + 1 + \delta_1(a_{i_a(k)})$, the parity of $i_a(k+1)$ is determined by the parity of $i_a(k)$ and $\delta_1(a_{i_a(k)})$, thus, in the cases described by the first and the fourth rows of (2), $P_k = L_k$, and, in the cases described by the second and the third rows, $P_k = S_k$. The exponents in (5) are computed according to the formula for b_k presented in Theorem 2. $\square$

We have described $s'(a)$ by an increasing sequence of prefixes $(P_k)_{k \in \mathbf{N}^+}$. Prefix P_k for each $k \in \mathbf{N}^+$ corresponds to the first run of level k ($\mathrm{run}_k(1)$) in the digitization of $y = ax$, so this description reflects the hierarchy of runs.

Figure 2 shows a digital straight line segment (a prefix of upper mechanical word) and its hierarchy of runs. The picture shows the first digitization run on level 5, $\mathrm{run}_5(1) = S_5$ (the 5^{th} prefix P_5) for the lines $y = ax$ (the words $s'(a)$) with slopes $a = [0; 1, 2, 1, 1, 3, 1, 1, a_8, a_9, \ldots]$, where $a_8, a_9, \ldots \in \mathbf{N}^+$. The dark squares on Fig. 2 represent the short runs_1. They can occur in sequences, while the long runs_1 (white) can only appear alone. We will revisit this example in Sect. 5 (Example 1). More about the hierarchy of runs can be found in Sect. 3.

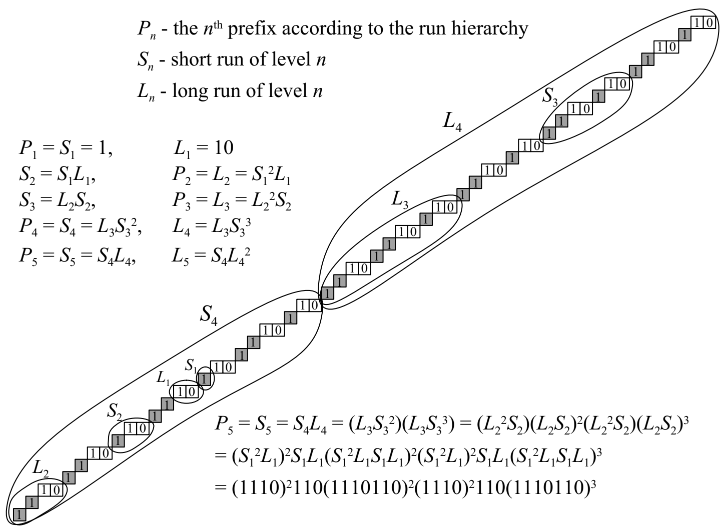

Fig. 2. Hierarchy of runs.

5 Comparison Between our Description by CFs and the Methods Described by Venkov and Shallit

In this section we consider only binary words over the two letter alphabet $\{0, 1\}$. For each such a word A, if it is finite, we denote by $|A|$ the length of A, being the total number of 0's and 1's forming A. In this section we no longer use the cardinality-wise run length as introduced in Section 3; we only use the (binary word)-length as defined above.

Let us first recall the well-known result formulated by the astronomer J. Bernoulli in 1772, proven by A. Markov in 1882 and described by Venkov (1970:67) [11].

Theorem 4 (Markov, Venkov). *For each irrational* $a = [0; a_1, a_2, a_3, \ldots]$*, the characteristic word is* $c(a) = C_1C_2C_3\ldots$*, where*

$$\begin{cases} C_1 = 0^{a_1-1}1 \\ D_1 = 0^{a_1}1 \end{cases}, \begin{cases} C_2 = C_1^{a_2-1}D_1 \\ D_2 = C_1^{a_2}D_1 \end{cases}, \ldots, \begin{cases} C_n = C_{n-1}^{a_n-1}D_{n-1} \\ D_n = C_{n-1}^{a_n}D_{n-1}. \end{cases}$$

Proposition 1 describes the length of C_n and D_n (meaning the number of 0's and 1's occurring in them), which leads immediately to Proposition 2.

Proposition 1. *With all the assumptions and the notation as in Theorem 4, we have* $|C_k| = q_k$ *and* $|D_k| = q_k + q_{k-1}$ *for all* $k \in \mathbf{N}^+$*, where* q_k *is the denominator of the* k^{th} *convergent of the CF expansion of* a*.*

Proof. By induction. For $k = 1$ we have $C_1 = 0^{a_1-1}1$, so $|C_1| = a_1 = q_1$ and $D_1 = 0^{a_1}1$, so $|D_1| = a_1 + 1 = q_1 + q_0$. Let's assume that $|C_k| = q_k$ and $|D_k| = q_k + q_{k-1}$ for some $k \geq 1$. By this assumption, combined with the definition of C_{k+1} and D_{k+1} and Theorem 1, we get $|C_{k+1}| = (a_{k+1} - 1)q_k + q_k + q_{k-1} = q_{k+1}$ and $|D_{k+1}| = a_{k+1}q_k + q_k + q_{k-1} = q_{k+1} + q_k$. □

Proposition 2. *Let* $a \in]0, 1[\setminus \mathbf{Q}$*. For each* $n \in \mathbf{N}^+$*, the length of the* n^{th} *prefix* $C_1 \cdots C_n$ *of* $c(a)$ *as defined in Theorem 4 is* $|C_1 \cdots C_n| = q_1 + \cdots + q_n$*, where* q_k *for* $k \in \mathbf{N}^+$ *is the denominator of the* k^{th} *convergent of the CF expansion of* a*.*

The second CF description of $c(a)$ we consider is that by Shallit (1991) [7], where $c(a)$ is formed as a limit of an increasing sequence of prefixes $(X_n)_{n\in\mathbf{N}^+}$; cf. the method by the *standard sequences* from Lothaire (2002:75, 76, 104, 105) [4].

Theorem 5 (Shallit 1991). *Let* $a = [0; a_1, a_2, a_3, \ldots]$ *be irrational and* $c(a) = (\lfloor (n+1)a \rfloor - \lfloor na \rfloor)_{n\in\mathbf{N}^+}$ *be its characteristic word. Let* $X_0 = 0$*. For the sequence of finite words* $(X_n)_{n\in\mathbf{N}^+}$ *being prefixes* $X_n = c_1c_2 \cdots c_{q_n}$ *of* $c(a)$ *of length* q_n*, where* q_n *are the denominators of the convergents of the CF expansion of* a*, we have*
$X_1 = 0^{a_1-1}1$ *and, for* $n \geq 2$*,* $X_n = X_{n-1}^{a_n}X_{n-2}$*.*

As we have seen (Proposition 2 and Theorem 5), the length of the prefixes of $c(a)$ obtained in both methods (Venkov's, Shallit's) can be expressed by the denominators of the convergents of the CF expansion of a. To be able to compare our result with their methods, we will now express the length of the prefixes P_k (from Theorem 3) of the upper mechanical word $s'(a) = 1c(a)$ in the same terms. The result is contained in Corollary 1. To get the corollary, we need the following theorem, which forms one of the main results in the present paper.

Theorem 6 (Main Result 2; a quantitative description of runs). *Let $a \in \,]0,1[\, \setminus \mathbf{Q}$ and $a = [0; a_1, a_2, \ldots]$. For the word $s'(a)$ we have for all $k \in \mathbf{N}^+$:*

$$|S_k| = q_{i_a(k+1)-1} \quad \textit{and} \quad |L_k| = q_{i_a(k+1)-1} + q_{i_a(k+1)-2},$$

where i_a is the index jump function (Def. 1), $|S_k|$ and $|L_k|$ for $k \in \mathbf{N}^+$ denote the (binary word)-length of short, respectively long runs of level k as in Theorem 3, and q_k are the denominators of the convergents of the CF expansion of a.

Proof. By induction. We also use Def. 1 and Theorem 1. For $k = 1$ the statement is true, because $i_a(2) = 2$ and, due to Theorem 3, $|S_1| = a_1 = q_1$ and $|L_1| = a_1 + 1 = q_1 + q_0$. Let us now assume that the statement is true for some $n - 1 \geq 1$. We will show that it is also true for n. We consider four cases, as in Theorem 3:

- $a_{i_a(n)} \neq 1$ and $i_a(n)$ is even.
  We have $i_a(n+1) = i_a(n) + 1$ and $q_{i_a(n)} = a_{i_a(n)} q_{i_a(n)-1} + q_{i_a(n)-2}$, so:
  $|S_n| = (a_{i_a(n)} - 1) q_{i_a(n)-1} + q_{i_a(n)-1} + q_{i_a(n)-2} = q_{i_a(n)} - q_{i_a(n)-1} + q_{i_a(n)-1} = q_{i_a(n)} = q_{i_a(n+1)-1}$, $|P_n| = |L_n| = a_{i_a(n)} q_{i_a(n)-1} + q_{i_a(n)-1} + q_{i_a(n)-2} = q_{i_a(n)} + q_{i_a(n)-1} = q_{i_a(n+1)-1} + q_{i_a(n+1)-2}$.

- $a_{i_a(n)} = 1$ and $i_a(n)$ is even.
  We have $i_a(n+1) = i_a(n) + 2$ and $q_{i_a(n)} = q_{i_a(n)-1} + q_{i_a(n)-2}$, so:
  $|P_n| = |S_n| = q_{i_a(n)-1} + a_{i_a(n)+1} \cdot q_{i_a(n)} = q_{i_a(n)+1} = q_{i_a(n+1)-1}$,
  $|L_n| = q_{i_a(n)-1} + (1 + a_{i_a(n)+1}) \cdot q_{i_a(n)} = q_{i_a(n)+1} + q_{i_a(n)} = q_{i_a(n+1)-1} + q_{i_a(n+1)-2}$.

- $a_{i_a(n)} \neq 1$ and $i_a(n)$ is odd.
  We have $i_a(n+1) = i_a(n) + 1$ and $q_{i_a(n)} = a_{i_a(n)} q_{i_a(n)-1} + q_{i_a(n)-2}$, so:
  $|P_n| = |S_n| = q_{i_a(n)-1} + q_{i_a(n)-2} + (a_{i_a(n)} - 1) q_{i_a(n)-1} = q_{i_a(n)} + q_{i_a(n)-1} - q_{i_a(n)-1} = q_{i_a(n)} = q_{i_a(n+1)-1}$, $|L_n| = q_{i_a(n)-1} + q_{i_a(n)-2} + a_{i_a(n)} q_{i_a(n)-1} = q_{i_a(n)-1} + q_{i_a(n)} = q_{i_a(n+1)-2} + q_{i_a(n+1)-1}$.

- $a_{i_a(n)} = 1$ and $i_a(n)$ is odd.
  We have $i_a(n+1) = i_a(n) + 2$ and $q_{i_a(n)} = q_{i_a(n)-1} + q_{i_a(n)-2}$, so:
  $|S_n| = a_{i_a(n)+1} q_{i_a(n)} + q_{i_a(n)-1} = q_{i_a(n)+1} = q_{i_a(n+1)-1}$,
  $|P_n| = |L_n| = (1 + a_{i_a(n)+1}) q_{i_a(n)} + q_{i_a(n)-1} = q_{i_a(n)} + q_{i_a(n)+1} = q_{i_a(n+1)-2} + q_{i_a(n+1)-1}$.

The proof is complete. ◻

Corollary 1 (a quantitative description of prefixes). *Let $a \in \,]0,1[\, \setminus \mathbf{Q}$ and $a = [0; a_1, a_2, \ldots]$. The length of the prefixes P_k of the the upper mechanical word $s'(a)$ as defined in Theorem 3 is: $|P_1| = |S_1| = a_1$ and for all $k \geq 2$:*

$$|P_k| = \begin{cases} |L_k| = q_{i_a(k)} + q_{i_a(k)-1} & \textit{if} \quad a_{i_a(k)} \neq 1 \quad \textit{and} \quad i_a(k) \quad \textit{is even} \\ |S_k| = q_{i_a(k)+1} & \textit{if} \quad a_{i_a(k)} = 1 \quad \textit{and} \quad i_a(k) \quad \textit{is even} \\ |S_k| = q_{i_a(k)} & \textit{if} \quad a_{i_a(k)} \neq 1 \quad \textit{and} \quad i_a(k) \quad \textit{is odd} \\ |L_k| = q_{i_a(k)+1} + q_{i_a(k)} & \textit{if} \quad a_{i_a(k)} = 1 \quad \textit{and} \quad i_a(k) \quad \textit{is odd,} \end{cases} \tag{6}$$

where i_a is the index jump function and q_n for $n \in \mathbf{N}^+$ is the denominator of the n^{th} convergent of the CF expansion of a.

Proof. Follows from Theorems 3 and 6, and the fact that, for $k \geq 2$, $i_a(k+1) = i_a(k) + 1$ if $a_{i_a(k)} \neq 1$ and $i_a(k+1) = i_a(k) + 2$ if $a_{i_a(k)} = 1$. □

Let us remark that Corollary 1 shows that the sequences $(P_k)_{k \in \mathbf{N}^+}$ of prefixes of upper mechanical words $s'(a)$ generated by our method are usually (i.e., for most slopes a) **not** subsequences of $(X_k)_{k \in \mathbf{N}^+}$ generated by Shallit, even if we put the letter 1 in the front of each X_k and remove the last letter of each X_k, getting in this way prefixes of $s'(a) = 1c(a)$ with length equal to the denominator of a convergent of a. We have to impose two conditions on the slope a to make the corresponding $(P_k)_{k \in \mathbf{N}^+}$ be a subsequence of the corresponding $(X_k)_{k \in \mathbf{N}^+}$ (after this extra operation of putting the letter 1 in the front of each X_k and taking away the last letter of each X_k). These conditions imposed on the CF elements of a are:

- for each k for which $|P_k| = q_{i_a(k)} + q_{i_a(k)-1}$ it must be $a_{i_a(k)+1} = 1$, in order to get $q_{i_a(k)} + q_{i_a(k)-1} = q_{i_a(k)+1}$ (Theorem 1) so that P_k has the length equal to the denominator of a convergent of a, like $X_{i_a(k)+1}$ (Theorem 5).
- for each k for which $|P_k| = q_{i_a(k)+1} + q_{i_a(k)}$ it must be $a_{i_a(k)+2} = 1$, in order to get $q_{i_a(k)+1} + q_{i_a(k)} = q_{i_a(k)+2}$ so that P_k has the length equal to the denominator of a convergent of a, like $X_{i_a(k)+2}$.

All the lines as described in Example 2 below have this property (that the sequence of prefixes described by our method is a subsequence of the prefixes generated by Shallit's method - we use every second element of the sequence used by Shallit), but for the most slopes this is not the case. This also shows that the method by Shallit does not reflect the run hierarchical structure of words and that our method is different from his. We can say the same about the method by Venkov, but this is obvious, so we leave out the proof in this paper.

Example 1. The line segments $\mathrm{run}_k(1)$ for $k = 1, 2, 3, 4, 5$ on Fig. 2 correspond to the prefixes P_k of $s'(a)$ for all a such that $a = [0; 1, 2, 1, 1, 3, 1, 1, a_8, a_9, \ldots]$, where $a_8, a_9, \ldots \in \mathbf{N}^+$. For these a, $i_a(1) = 1$, $i_a(2) = 2$, $i_a(3) = 3$, $i_a(4) = 5$, $i_a(5) = 6$, $i_a(6) = 8$, and the denominators of the convergents are $q_1 = 1$, $q_2 = 3$, $q_3 = 4$, $q_4 = q_{i_a(4)-1} = 7$, $q_5 = q_{i_a(4)} = 25$, $q_6 = q_{i_a(5)} = 32$, $q_7 = q_{i_a(6)-1} = 57$. It is easy to check that the length $|P_k|$ of prefixes (runs) on Fig. 2 agrees with Corollary 1, so $|P_1| = 1$, $|P_2| = 4$, $|P_3| = 11$, $|P_4| = 25$, $|P_5| = 57$.

Example 2. Let $a = [0; a_1, 1, a_3, 1, a_5, 1, a_7, 1, a_9, \ldots]$, where $a_{2n+1} \in \mathbf{N}^+$ for all $n \in \mathbf{N}$. For $s'(a)$ we have $|P_k| = |S_k| = q_{2k-1}$ and $|L_k| = q_{2k}$ for all $k \in \mathbf{N}^+$ (the notation as in Theorem 3).

Indeed, the index jump function is $i_a(1) = 1$, $i_a(k) = 2k - 2$ for $k \geq 2$, so it is even for all $k \geq 2$. Moreover, $a_{i_a(k)} = 1$ for $k \geq 2$. From Theorem 6, $|L_k| = q_{i_a(k+1)-1} + q_{i_a(k+1)-2} = a_{i_a(k+1)} q_{i_a(k+1)-1} + q_{i_a(k+1)-2} = q_{i_a(k+1)} = q_{2k}$ and $|S_k| = q_{i_a(k+1)-1} = q_{2k-1}$, and from Corollary 1, $|P_k| = |S_k|$ for $k \in \mathbf{N}^+$.

Example 3. Let $a = [0; a_1, a_2, a_3, a_4, \ldots]$, where $a_1 \in \mathbf{N}^+$ and $a_n \geq 2$ for all $n \geq 2$ (thus $i_a(k) = k$ for all $k \in \mathbf{N}^+$). Due to Corollary 1, the lengths of the prefixes P_k (for $k \in \mathbf{N}^+$) of $s'(a)$ as defined in Theorem 3 are:

$$|P_k| = \begin{cases} |S_k| = q_k & \text{if } k \text{ is odd} \\ |L_k| = q_k + q_{k-1} & \text{if } k \text{ is even.} \end{cases} \tag{7}$$

Formulae (7) and the one from Proposition 1 look similar (we get the length q_k and $q_k + q_{k-1}$ in both cases), but they describe different parts of prefixes of $s'(a)$.

Now we will compare our method to that of Shallit.

Proposition 3. *Let $a \in]0, 1[\setminus \mathbf{Q}$ and $a = [0; a_1, a_2, \ldots]$. We have $|P_k| \geq |X_k|$ for all $k \in \mathbf{N}^+$, where P_k is our k^{th} prefix of $s'(a) = 1c(a)$ and X_k is Shallit's k^{th} prefix of $c(a)$. There exists $k \geq 2$ for which the inequality is strict.*

Proof. For any a we have $|P_1| = q_1 = |X_1|$. The sequence $(q_n)_{n \in \mathbf{N}^+}$ is strictly increasing and for each $k \in \mathbf{N}^+$ we have $i_a(k) \geq k$, thus, from Theorems 6 and 5, we get $|P_k| \geq |S_k| = q_{i_a(k+1)-1} \geq q_k = |X_k|$ for $k \in \mathbf{N}^+$. The last statement follows from Corollary 1 and Example 3. The situation when only S_k with length q_k are prefixes is not possible and, if there is an element $a_s = 1$ $(s \geq 2)$ in the CF expansion of a, we have $i_a(s+1) = i_a(s) + 2$, so $q_{i_a(s+1)-1} = q_{i_a(s)+1} > q_s$. $\square$

We have just shown that, for each $a \in]0, 1[\setminus \mathbf{Q}$ and each $k \in \mathbf{N}^+$, our k^{th} prefix of $s'(a) = 1c(a)$ has the same length or is longer than Shallit's k^{th} prefix X_k of $c(a)$. The words are formed more quickly according to our method. Now we will show that our advantage (expressed by quotient) can be arbitrarily large.

Proposition 4. *For the methods from Theorems 3 and 5 we have the following:*

$$\forall \ (E_n)_{n \geq 2} \quad \exists \ a \in]0, 1[\setminus \mathbf{Q} \quad \forall \ k \geq 2 \qquad |P_k| \geq E_k \cdot |X_k|, \tag{8}$$

where $(E_n)_{n \geq 2}$ is any infinite sequence of positive (large) numbers, P_k is our k^{th} prefix of $s'(a) = 1c(a)$ and X_k is Shallit's k^{th} prefix of $c(a)$.

Proof. Let $(E_n)_{n \geq 2}$ be any sequence of (large) positive numbers. We will show how to construct by induction a slope $a = [0; a_1, a_2, \ldots]$ fulfilling (8). We take any $a_1 \in \mathbf{N}^+$ and $a_2 = 1$. Because $a_2 = 1$, then, for every $k \geq 2$, we have $i_a(k+1) \geq k+2$. In the induction step, when we already have defined $a_1, \ldots, a_k$ for some $k \geq 2$, thus also have $q_1, \ldots, q_k$, we define a_{k+1} in order to get $|P_k|/|X_k| \geq E_k$.
According to Corollary 1 and Theorem 1 (the sequence $(q_n)_{n \in \mathbf{N}^+}$ is increasing), we have $|P_k| \geq |S_k| = q_{i_a(k+1)-1} \geq q_{k+1} = a_{k+1} q_k + q_{k-1}$, and, from Theorem 5, $|X_k| = q_k$, so we have $|P_k|/|X_k| \geq (a_{k+1} q_k + q_{k-1})/q_k \geq a_{k+1}$. This means that $|P_k|/|X_k| \geq E_k$ if $a_{k+1} \geq E_k$, so we can take for example $a_{k+1} = \lceil E_k \rceil$. $\square$

Slopes with only one element equal to 1 in the CF expansion can already give us as large an advantage as we define *a priori*. It should be possible to get much better results for the slopes as in Example 2, where the quotient $|P_k|/|X_k|$ is equal to q_{2k-1}/q_k for all $k \in \mathbf{N}^+$. The following lemma (cf. [2] , p. 13) helps us perform further comparisons between our method and the method of Shallit.

Lemma 1. *Let $a = [0; a_1, a_2, \ldots]$. For all $k \geq 2$ we have $q_{2k-1} \geq 2^{\frac{k-2}{2}} q_k$, where q_n for $n \geq 2$ is the denominator of the n^{th} convergent of the CF expansion of a.*

Proof. For $k = 2$ we get $q_3 \geq q_2$, which is true. From Theorem 1 and because the sequence $(q_n)_{n \in \mathbf{N}^+}$ is increasing, we have $q_{4j+1} = a_{4j+1} q_{4j} + q_{4j-1} \geq q_{4j} + q_{4j-1} \geq 2q_{4j-1}$ for $j \geq 1$. Successive application of this inequality yields

$$q_{4j+1} \geq 2^s q_{4j-(2s-1)} \quad \text{for} \quad s = 1, 2, \ldots, 2j. \tag{9}$$

We put $s = j$ in (9) and we get $q_{2k-1} \geq 2^{\frac{k-1}{2}} q_k$, thus $q_{2k-1} \geq 2^{\frac{k-2}{2}} q_k$, for odd k. From Theorem 1 and (9), $q_{4j+3} = a_{4j+3} q_{4j+2} + q_{4j+1} \geq q_{4j+2} + q_{4j+1} \geq 2q_{4j+1} \geq 2 \cdot 2^{j-1} q_{2j+3} \geq 2^j q_{2j+2}$, which gives the statement for even k. $\square$

Theorem 7. *For the slopes a as in Example 2 we have the following:*

- $\forall \;\; k \geq 2 \quad |P_k| = |X_{2k-1}|$,
- $\forall \;\; k \geq 2 \quad |P_k| \geq 2^{\frac{k-2}{2}} \cdot |X_k|$,

where P_k is our k^{th} prefix of $s'(a) = 1c(a)$ and X_k is Shallit's k^{th} prefix of $c(a)$. Moreover, for the methods from Theorems 3 and 5, we have the following:

$$\forall \;\; (E_n)_{n \geq 2} \quad \exists \;\; a \in \,]0,1[\, \setminus \mathbf{Q} \quad \forall \;\; k \geq 2 \quad |P_k| \geq E_k \cdot |X_{2k-2}|, \tag{10}$$

where $(E_n)_{n \geq 2}$ is any infinite sequence of positive (large) numbers.

Proof. From Theorem 5, $|X_k| = q_k$ for $k \in \mathbf{N}^+$. From Example 2, $|P_k| = q_{2k-1}$ for $k \geq 2$, which proves the first two statements (for the second one we also use Lemma 1). To prove (10), we take any sequence $(E_n)_{n \geq 2}$ of positive (large) numbers and construct a slope $a = [0; a_1, 1, a_3, 1, a_5, 1, \ldots]$ as in Example 2. We will show how to choose a_{2k+1} for $k \in \mathbf{N}$ in order to get (10) for this $(E_n)_{n \geq 2}$. We proceed as follows. We take any $a_1 \in \mathbf{N}^+$. We choose a_{2k+1} for $k = 1, 2, 3, \ldots$ by induction. When we already have $a_1, \ldots, a_{2k-1}$ for some $k \geq 1$, then we also have $a_2 = \cdots = a_{2k} = 1$ and the denominators of the convergents $q_1, \ldots, q_{2k}$, and we define a_{2k+1} in order to get $|P_{k+1}|/|X_{2k}| \geq E_{k+1}$. Because, according to Example 2 and Theorem 1, $|P_{k+1}| = q_{2k+1} = a_{2k+1} q_{2k} + q_{2k-1}$ and, from Theorem 5, $|X_{2k}| = q_{2k}$, we have $|P_{k+1}|/|X_{2k}| = (a_{2k+1} q_{2k} + q_{2k-1})/q_{2k} \geq a_{2k+1}$, thus $|P_{k+1}|/|X_{2k}| \geq E_{k+1}$ if $a_{2k+1} \geq E_{k+1}$, and we can take for example $a_{2k+1} = \lceil E_{k+1} \rceil$. $\square$

Theorem 7 shows that the advantage of using our method rather than Shallit's (when forming prefixes of $c(a)$ or $s'(a) = 1c(a)$) can be huge for the words with many

1's in the CF expansion of the slope. It is not only possible to get the advantage we choose *a priori*, but we also get arbitrarily longer prefixes in step k compared to Shallit's prefixes in step $2k-2$ for each $k \geq 2$.

A comparison between our method and the method of Venkov is contained in Theorem 8 and Propositions 5 and 6. Theorem 8 is a Venkov counterpart of Proposition 4 and Theorem 7. The statement (11) there is weaker than (12), but it is still worth to be formulated. The reason is that we can reach the advantage formulated in (11) already for slopes with only one CF element equal to 1. It is easy to find slopes fulfilling (11).

Theorem 8. *For each $a \in]0,1[\setminus \mathbf{Q}$ we have $|P_1| = |C_1|$ and $|P_2| \geq |C_1C_2|$. Moreover, for the methods from Theorems 3 and 4 we have the following:*

$$\forall \ (E_n)_{n\geq 2} \quad \exists \ a \in]0,1[\setminus \mathbf{Q} \quad \forall \ k \geq 2 \qquad |P_k| \geq E_k \cdot |C_1 \cdots C_k|, \tag{11}$$

$$\forall \ (E_n)_{n\geq 2} \quad \exists \ a \in]0,1[\setminus \mathbf{Q} \quad \forall \ k \geq 2 \qquad |P_k| \geq E_k \cdot |C_1 \cdots C_{2k-2}|, \tag{12}$$

where $(E_n)_{n\geq 2}$ is any infinite sequence of positive (large) numbers, P_k is our k^{th} prefix of $s'(a) = 1c(a)$ and $C_1 \cdots C_k$ is Venkov's k^{th} prefix of $c(a)$.

Proof. For all $a \in]0,1[\setminus \mathbf{Q}$ we have $|P_1| = a_1 = |C_1|$. For $k = 2$ we always have $C_1C_2 = q_1 + q_2$ and $|P_2|$ is equal, due to Corollary 1, to $q_2 + q_1$ if $a_2 \neq 1$ and to q_3 if $a_2 = 1$. In the case when $a_2 = 1$ we get $|C_1C_2| = q_1 + q_2 \leq a_3q_2 + q_1 = q_3 = |P_2|$.

To prove (11), we take any sequence $(E_n)_{n\geq 2}$ of (large) positive numbers. We will show how to construct a slope $a = [0; a_1, a_2, \ldots]$ fulfilling (11). The construction will be by induction. We take any $a_1 \in \mathbf{N}^+$ and $a_2 = 1$. Because $a_2 = 1$, then for every $k \geq 2$ we have $i_a(k+1) \geq k+2$. In the induction step, when we already have defined $a_1, \ldots, a_k$ for some $k \geq 2$, thus also have $q_1, \ldots, q_k$, we define a_{k+1} in order to get $|P_k|/|C_1 \cdots C_k| \geq E_k$. From Corollary 1 and Theorem 1, we have $|P_k| \geq |S_k| = q_{i_a(k+1)-1} \geq q_{k+1} = a_{k+1}q_k + q_{k-1}$ and, from Proposition 2, $|C_1 \cdots C_k| = \sum_{i=1}^{k} q_i$. This means that $|P_k|/|C_1 \cdots C_k| \geq (a_{k+1}q_k + q_{k-1})/\sum_{i=1}^{k} q_i \geq (a_{k+1}q_k)/(kq_k) = a_{k+1}/k$, and we get $|P_k|/|C_1 \cdots C_k| \geq E_k$ for $a_{k+1} \geq kE_k$, so we take for example $a_{k+1} = \lceil kE_k \rceil$.

To prove (12), we take any sequence $(E_n)_{n\geq 2}$ of (large) positive numbers. We construct a slope $a = [0; a_1, 1, a_3, 1, a_5, 1, \ldots]$ as in Example 2. We will show how to choose a_{2k+1} for $k \in \mathbf{N}$ in order to get (12) for this $(E_n)_{n\geq 2}$. We proceed as follows. We take any $a_1 \in \mathbf{N}^+$. We choose a_{2k+1} for $k = 1, 2, 3, \ldots$ by induction. Let us say that we already have $a_1, \ldots, a_{2k-1}$ for some $k \geq 1$. Then we also have $a_2 = \cdots = a_{2k} = 1$ and the denominators $q_1, \ldots, q_{2k}$, and we define a_{2k+1} in order to get $|P_{k+1}|/|C_1 \cdots C_{2k}| \geq E_{k+1}$. Because, according to Example 2 and Theorem 1, $|P_{k+1}| = q_{2k+1} = a_{2k+1}q_{2k} + q_{2k-1}$, and, from Proposition 2, $|C_1 \cdots C_{2k}| = \sum_{i=1}^{2k} q_i$, we get $|P_{k+1}|/|C_1 \cdots C_{2k}| = (a_{2k+1}q_{2k} + q_{2k-1})/\sum_{i=1}^{2k} q_i \geq (a_{2k+1}q_{2k})/(2kq_{2k}) = a_{2k+1}/(2k)$, so $|P_{k+1}|/|C_1 \cdots C_{2k}| \geq E_{k+1}$ if $a_{2k+1} \geq 2kE_{k+1}$. We can take for example $a_{2k+1} = \lceil 2kE_{k+1} \rceil$. $\square$

The quotients $|P_k|/|C_1 \cdots C_k|$ and $|P_k|/|C_1 \cdots C_{2k-2}|$ can thus be arbitrarily large. The strongest result is (12), but (11) is the easiest one to reach.

Proposition 4, Theorem 7 and 8 show that, if there are some 1's in the CF expansion of the slope, our method can generate the longest prefixes of all three methods. The greater the number of 1's in the expansion, the greater advantage we get using our method. Because, from Def. 1, $k \leq i_a(k) \leq 2k-2$ for $k \geq 2$ for each $a \in]0,1[\setminus \mathbf{Q}$, slopes as in Example 2 can probably give us the largest possible advantage, depending on the choice of a_{2n+1} for $n \in \mathbf{N}^+$.

Also slopes $a = [0; a_1, a_2, 1, a_4, 1, a_6, \ldots]$ with $a_2 \geq 2$ give us a similar result. For the lines with such slopes we have $i_a(k) = 2k-3$ and $P_k = q_{2k-1}$ for $k \geq 3$.

Proposition 5. *Let* $a = [0; a_1, a_2, a_3, a_4, \ldots]$*, where* $a_n \geq 2$ *for all* $n \geq 2$*. Then* $|P_1| = |C_1|$*,* $|P_2| = |C_1 C_2|$ *and* $|P_k| < |C_1 \cdots C_k|$ *for each* $k \geq 3$*, where* P_k *is our* k^{th} *prefix of* $s'(a) = 1c(a)$ *and* $C_1 \cdots C_k$ *is Venkov's* k^{th} *prefix of* $c(a)$*.*

Proof. From Proposition 2 and Example 3. For $k = 1$ and $k = 2$ we have clearly the above equality. If $k \geq 3$, then $|C_1 \cdots C_k| = q_1 + \cdots + q_k > q_{k-1} + q_k \geq |P_k|$. □

As we have seen in Proposition 5, it can easily happen that $|C_1 \cdots C_k| > |P_k|$ for some $a \in]0,1[\setminus \mathbf{Q}$ and $k \geq 3$. For the slopes as in Example 3, Venkov's prefixes for $k \geq 3$ are longer than ours. It is not possible, though, to make the quotient $|C_1 \cdots C_k|/|P_k|$ arbitrarily large, as it was in the opposite case (Theorem 8).

Proposition 6. *Let* $a \in]0,1[\setminus \mathbf{Q}$*. Then* $|C_1 \cdots C_k| < k \cdot |P_k|$ *for* $k \geq 3$*, where* P_k *is our* k^{th} *prefix of* $s'(a) = 1c(a)$ *and* $C_1 \cdots C_k$ *is Venkov's* k^{th} *prefix of* $c(a)$*.*

Proof. Let $k \geq 3$. It follows from Proposition 2, Corollary 1, and the fact that $(q_n)_{n \in \mathbf{N}^+}$ is strictly increasing, that $|C_1 \cdots C_k| = \sum_{i=1}^{k} q_i < k q_k \leq k \cdot |P_k|$. □

The quotient $|C_1 \cdots C_k|/|P_k|$ is thus bounded by k for each $k \geq 3$.

6 Conclusions and Some Topics for Future Research

We have presented a run-hierarchical CF based description of upper mechanical words $s'(a)$ with slope $a \in]0,1[\setminus \mathbf{Q}$ and intercept 0. We expressed the length of the prefixes obtained according to our method by the denominators of the convergents of the CF expansion of the slope. This allowed us to compare our result with other CF based methods (Venkov's, Shallit's) of forming such words. Due to the special treatment of the CF elements equal to 1, our method gives often longer prefixes after the same number of steps compared to the two other methods.

Our description uses an auxiliary function, the index jump function defined in Def. 1, while the two other methods do not use any extra functions. However, the index jump function is extremely simply constructed and computationally trivial. Another possible drawback of the method could be that it uses more elements of the CF expansion of the slope than the other methods, so the comparison might conceivably be thought of as being unfair. The run-hierarchical method presented in this paper is not meant to replace the existing methods, it should rather be seen as an additional possible method, which gives better results in some cases, as shown for example in Theorems 7 and 8. Moreover, as the author showed in [9] with numerous examples, in case of quadratic irrationals or even some transcendental numbers (like for example $\sqrt[n]{e}-1$ for $n \geq 2$, $\frac{e^2-1}{e^2+1}$), our method gives a compact description of all the runs with the knowledge of the CF elements which form the period (or, in case of the mentioned transcendental numbers, the knowledge of the periodic form of the CF expansion) and then it does not matter any longer that we use CF elements with a large index.

Corollary 1, together with Proposition 2 and Theorem 5, also shows that our method is the only one of the three presented methods which reflects the hierarchy of runs on all the levels. The run-hierarchical description enables us to analyze abstract properties of lines (words), which has been discussed in another paper of the author [10]. We have shown there how we can partition digital lines (upper mechanical words) with slopes $a \in]0,1[\setminus \mathbf{Q}$ into equivalence classes under two equivalence relations defined by means of CFs, based on the description from [9]. Hopefully these partitions can help us gain a better understanding of digital lines and maybe become a useful tool for combinatorics on words. Further work in this field could involve a fixed point theorem for Sturmian words and how to find a Sturmian word such that its letters are coding its own run hierarchical structure as defined in the presented paper. Words like this could be called *words with self-balanced construction*. It would be interesting to express the fixed points described above in terms of generalized balances introduced by I. Fagnot and L. Vuillon in [1].

Acknowledgments. I am grateful to Christer Kiselman for comments on earlier versions of the manuscript.

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Paper IV

PAPER IV, ERRATA

1. p. 3668, item [2] in References
   is: Acta Mathematica Paedagogicae Nyíregyháziensis
   should be: Acta Mathematica Academiae Paedagogicae Nyíregyháziensis

2. p. 3668, item [3] in References
   is: Acta Mathematica Paedagogicae Nyíregyháziensis
   should be: Acta Mathematica Academiae Paedagogicae Nyíregyháziensis.

Theoretical Computer Science 410 (2009) 3655–3669

Contents lists available at ScienceDirect

Theoretical Computer Science

journal homepage: www.elsevier.com/locate/tcs

Two equivalence relations on digital lines with irrational slopes. A continued fraction approach to upper mechanical words

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ARTICLE INFO

Article history:
Received 20 March 2008
Received in revised form 9 April 2009
Accepted 28 April 2009
Communicated by D. Perrin

Keywords:
Digital line
Upper mechanical word
Characteristic word
Irrational slope
Continued fraction
Hierarchy
Run
Classes of digital lines
Classes of words
Fibonacci numbers

ABSTRACT

We examine the influence of the elements of the continued fraction (CF) expansion of irrational positive a less than 1 on the construction of runs in the digitization of the positive half line $y = ax$ or, equivalently, on the run-hierarchical structure of the upper mechanical word with slope a and intercept 0. Special attention is given to the CF elements equal to 1. We define two complementary equivalence relations on the set of slopes, based on their CF expansions. A new description of digital lines is presented; we show how to define a straight line or upper mechanical word by two sequences of positive integers fulfilling some extra conditions. These equivalence relations and this new description enable us to analyze the construction of digital lines and upper mechanical words. The analysis of suprema of equivalence classes under one of these relations leads to a result which involves Fibonacci numbers.

1. Introduction

Combinatorics on words and digital geometry are two relatively new intensively growing areas of discrete mathematics. The first one is about one hundred years old. According to Karhumäki [13], the 1906 paper by Axel Thue (1863–1922) on repetition-free words is considered as a starting point of mathematical research on words; see also [18]. Digital geometry is about fifty years old. Its origin is in computer graphics. The mathematical study of different properties of digital images started in the early 1960s. Azriel Rosenfeld (1931–2004) made pioneering contributions to nearly every area of the field. He wrote the first textbook on computer vision in 1969; see [9].

One of the problems both domains have in common is a description of characteristic words. It is equivalent to a description of corresponding digital lines (with the same slope). This problem has received a lot of attention and has been treated in many different ways, because it has its applications in numerous domains of science. As examples we can cite mathematics (number theory, combinatorics on words, dynamical systems, digital geometry), computer science (e.g. digital straightness), astronomy, and crystallography. Because of this diversity of applications, one can find the same, or very closely related, problems under many different names, e.g., Beatty sequences, Sturmian words, trajectories of rotations, upper (lower) mechanical words, Christoffel words, chain codes of a line, cutting sequences, billiard words; see [7,22,1]. The interdisciplinary character of results obtained in connection with different descriptions of characteristic words made

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doi:10.1016/j.tcs.2009.04.026

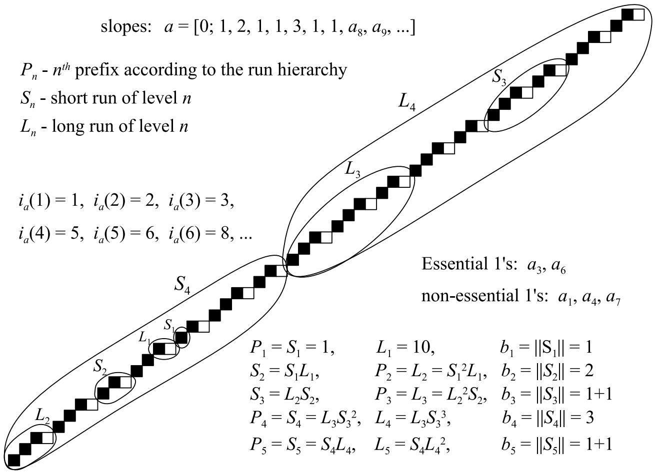


Fig. 1. A CF description according to the hierarchy of runs.

it possible to define independently from digital geometry and combinatorics on words some of the concepts introduced by the author; see [29], a submitted manuscript.

The history of the problem goes back to 1772, when astronomer Johan III Bernoulli applied the continued fraction (CF) expansion of a to describe the sequence $(\lfloor na \rfloor)_{n\in\mathbf{N}^+}$ for an irrational a. Some of many excellent sources of information about different domains of science to which the result was applied, are [8] (section *Concluding remarks* and the list of references, which includes even papers on quasicrystals), [24] (the bibliography which is probably complete at least up to 1972, according to its author), [5,16].

The aim of the present paper is to discuss different aspects of the construction of digital lines and upper mechanical words (Definition 2) with slope $a \in]0, 1[\setminus \mathbf{Q}$ and intercept 0 according to the hierarchy of runs, runs of runs, etc., as described in [21]. Our approach is based on CFs and we perform only simple computations on integer numbers. The formulae we use, (6) for digital lines and (9) for upper mechanical words, have been introduced in two earlier papers by the author.

The run-hierarchical description of lines (6) was first presented in [26]. In this paper and in [27], which is its extended version, we can find references to many other CF-based methods of descriptions of digital lines.

In [28] the author introduced the run-hierarchical formula (9) for upper mechanical words. In the same paper we can find a comparison of this method with two other CF-based descriptions of upper mechanical words. One of them is the one formulated by Bernoulli in 1772, proven by Markov in 1882 and described in [32, p. 67], the second one is the one known as method by *standard sequences*, which can be found in [22]. We have shown in [28] that the three methods are different from each other. In all of them one can express the length of generated prefixes by means of the denominators $(q_1, q_2, \ldots)$ of the convergents of the CF-expansion of the slope a of upper mechanical (characteristic) word we describe, but we get different lengths in each case. For Shallit's method (by standard sequences) the nth generated prefix of $c(a)$ has length $|X_n| = q_n$, for Venkov's method $|C_1 \cdots C_n| = q_1 + \cdots + q_n$, and for our method, the nth prefix P_n of the upper mechanical word $s'(a) = 1c(a)$ has a length $q_{i_a(n)} + q_{i_a(n)-1}$, $q_{i_a(n)+1}$, $q_{i_a(n)}$, or $q_{i_a(n)+1} + q_{i_a(n)}$, depending on the parity of the value $i_a(n)$ of the *index jump function* (Definition 1 in the present paper) at n and on whether $a_{i_a(n)}$ is equal to 1 or not; see Corollary 1 in [28]. This means that the hierarchy of runs in Venkov's method and in the method by standard sequences is different from the hierarchy of runs defined by Rosenfeld.

Fig. 1 illustrates the run-hierarchical construction of both digital lines as in (6) and upper mechanical words as in (9), due to the equivalence which will be discussed in Section 2. The line segment $\text{run}_5(1)$ on Fig. 1 corresponds to the prefix P_5 of upper mechanical words $s'(a)$ for all such a that $a = [0; 1, 2, 1, 1, 3, 1, 1, a_8, a_9, \ldots]$, where $a_8, a_9, \ldots \in \mathbf{N}^+$. Each black square in the figure represents the letter 1 and each white square represents the letter 0 in the binary word $s'(a)$.

An important issue in the present paper is to clarify the role of elements equal to 1 of the CF expansion of the slope $a \in]0, 1[\setminus \mathbf{Q}$ in the run-hierarchical construction of the digitization of the line $y = ax$, equivalently, of the upper mechanical word with slope a and intercept 0. In Section 3, which constitutes the first of the two main topics in this paper, we define *essential* 1*'s* (Definition 6) as those 1's of the CF expansion of the slope which have influence on the *qualitative aspect* of the construction of lines (words). By qualitative aspect we mean that they determine the construction of digitization in terms of long and short runs on all the digitization levels, i.e., *how* the runs are arranged. The essential 1's cause the most frequently appearing run on the level they decide about to be long; see Section 3.2. Other CF elements (meaning non-essential 1's and elements different from 1) take care of the *quantitative aspect* of the digitization only. By this we mean that they determine the run length on each digitization level, i.e., *how many* runs_{k-1} form one run_k for each $k \geq 2$; see Section 3.1. The special treatment the essential 1's get in our description makes our process of forming lines or words for some slopes more efficient than in the case of some other similar recursive formulae, as we have shown in [28].

We define two complementary equivalence relations on the set $]0, 1[\setminus \mathbf{Q}$ of slopes of digital half lines $y = ax$ (or, equivalently, of upper mechanical words). One of them is based on the idea of the *sequence of length specification* (see Definition 3), the second one is defined by the already mentioned index jump function (Definition 1) and can be equivalently defined by the *sequence of the places of essential* 1*'s* (Definitions 6 and 7, Proposition 8).

We consider the problem of existence of least and greatest elements in equivalence classes for both equivalence relations. The solution is presented in Propositions 2 and 3 (for the relation based on the length of runs) and Theorem 5 (for the relation based on essential 1's). Theorem 5 shows the connection between the classes defined by the sequences of the places of essential 1's and the Fibonacci numbers.

Another problem we solve in this paper is how to construct the slope of a straight line which has a digitization fulfilling some *a priori* imposed conditions, in this case, given a short run length on the digitization level k for all $k \in \mathbf{N}^+$ (i.e., the sequence of length specification) and the sequence of the places of essential 1's. The solution of this problem is presented as Theorem 6, which is a converse of Theorem 1. It shows, given the description of a digitization, how to compute the slope of the digitized straight line, while Theorem 1 gives a description of the digitization for a given slope.

As we have stated earlier in this introduction, there exist many different CF-based descriptions for both digital lines and characteristic words. Probably the most commonly used methods are the already mentioned method by Venkov and the standard-sequences method. None of them reflects the run-hierarchical structure of words by analogy to the hierarchy of runs as defined by Rosenfeld for digital lines. There are also other places where we can find similar formulae, see for example [11]. The method presented there is similar to ours, but it does not give any special attention to the CF elements equal to 1, which causes the run-hierarchical structure not to be reflected either.

The method by standard sequences appears in many different places and forms in the literature. Some good sources to consult about the subject are [22] and [17, pp. 75, 76, 104, 105]. On pages 104 and 105 in the latter we can find some exercises which show the connections between both methods (the one by Shallit and the one by Venkov). In some places we can see a very strong connection between the method by standard sequences and the Stern–Brocot algorithm [12, pp. 116–123]; see for example [4,6]. In [4, p. 172] the connection between standard words and *Rauzy's rules* is presented. Compare also [10, pp. 61–67] with [17, pp. 64–77] to see the similarity between the Stern–Brocot tree and the binary tree of standard pairs.

Bates et al. [3] show the link between Stern–Brocot tree and the Gauss map. Our method expressed by (6) and (9) was derived from the method by digitization parameters introduced in [25], which, as shown in [27], are strongly connected to the iterates of the Gauss map. However, in our method we modify the Gauss map each time when the value of the iterate is larger than $\frac{1}{2}$, which corresponds to the special treatment which the essential 1's get in our method.

None of CF-based descriptions of digital lines, even if some of them reflect the hierarchy of runs as defined by Rosenfeld, gives a special attention to those CF elements of the slope which are equal to 1. Some indications concerning influence of CF elements equal to 1 on the construction of digital lines can be found in the Ph.D. thesis of J.-P. Reveillès [20, p. 112], however, his algorithm is written only for rational slopes and the question about the meaning of the CF elements equal to 1 is not treated there. Another place in the literature about digital lines which touches on the problem is the Ph.D. thesis of P. Stephenson [23, chap. 4]. He does not formulate the question either. The author's papers [26–28] and the present paper are probably the only places where the question about the role of the CF elements equal to 1 in the run-hierarchical construction of digital lines is both formulated and thoroughly answered.

2. Digital lines and upper mechanical words

We discuss the digitization of the positive half lines $y = ax$ where $a \in]0, 1[$ is irrational. The standard Rosenfeld digitization (R-digitization) is replaced by the R′-digitization. The modification is very simple and is basically a vertical shift of the grid by $-\frac{1}{2}$ (see Fig. 2; the R′-cross in (k, n), denoted $C_{R'}(k, n)$, is shown there). This results in the following arithmetical description of the digital positive half line l with equation $y = ax$ as a subset of $\mathbf{Z}^2$ (Fig. 3):

$$D_{R'}(l) = \{(k, n) \in (\mathbf{N}^+)^2; l \cap C_{R'}(k, n) \neq \emptyset\} = \{(k, \lceil ak \rceil); k \in \mathbf{N}^+\}. \tag{1}$$

In [26] we presented a description of the construction of digital lines in terms of CFs. In the present paper we will use this description to define two equivalence relations on the set of slopes $a \in]0, 1[\setminus \mathbf{Q}$ of all the digital lines $y = ax$. The results hold also for a kind of binary words, namely for upper mechanical words with slope a and intercept 0 (see Definition 2). This will be shown later in this section.

From now on we assume that the simple CF expansion of $a \in]0, 1[\setminus \mathbf{Q}$ is given, expressed as $a = [0; a_1, a_2, a_3, \ldots]$, and we know the positive integers a_k for all $k \in \mathbf{N}^+$. These are called the *elements* of the CF. We recall that

$$[a_0; a_1, a_2, a_3, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}. \tag{2}$$

In our case, when $a \in]0, 1[\setminus \mathbf{Q}$, we have $a_0 = \lfloor a \rfloor = 0$ and the sequence of the CF elements $(a_1, a_2, \ldots)$ is infinite.

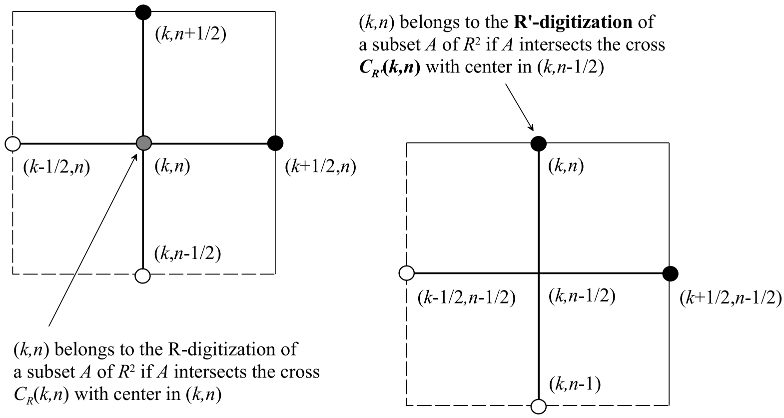


Fig. 2. A comparison between the R-digitization and the R′-digitization.

We call $[a_0; a_1, a_2, \ldots, a_n]$, for each $n \in \mathbf{N}$, the nth *convergent* of the CF $[a_0; a_1, a_2, \ldots]$. If we define

$$p_0 = a_0, \qquad p_1 = a_1 a_0 + 1, \qquad p_n = a_n p_{n-1} + p_{n-2} \quad \text{for } n \geq 2 \tag{3}$$

and

$$q_0 = 1, \qquad q_1 = a_1, \qquad q_n = a_n q_{n-1} + q_{n-2} \quad \text{for } n \geq 2, \tag{4}$$

then

$$[a_0; a_1, a_2, \ldots, a_n] = \frac{p_n}{q_n} \quad \text{for } n \in \mathbf{N}\,. \tag{5}$$

All convergents are irreducible [e.g. 14, p. 12]. For more information about CFs see [14].

Our description from [26] was based on our earlier description by digitization parameters from [25] and the following *index jump function*.

Definition 1. If $a \in\,]0, 1[\, \setminus \mathbf{Q}$ and $a = [0; a_1, a_2, \ldots]$, then the index jump function $i_a : \mathbf{N}^+ \to \mathbf{N}^+$ is defined by $i_a(1) = 1$, $i_a(2) = 2$ and, for $k \geq 2$,

$$i_a(k+1) = i_a(k) + 1 + \delta_1(a_{i_a(k)}),$$

where $\delta_1(x) = \begin{cases} 1, & x = 1 \\ 0, & x \neq 1. \end{cases}$

Our description of the digitization of $y = ax$ for $a \in\,]0, 1[\, \setminus \mathbf{Q}$ reflects the hierarchy of runs on all the digitization levels. The term *run* was already introduced by Azriel Rosenfeld [21, p. 1265]. For a formal definition of runs and for the definition of the R′-digitization see [25].

We call $\text{run}_k(j)$ for $k, j \in \mathbf{N}^+$ a *run of digitization level k*. We use the notation run_k or in plural runs_k, meaning $\text{run}_k(j)$ *for some* $j \in \mathbf{N}^+$, or, respectively, $\{\text{run}_k(i);\quad i \in I\}$ *where* $I \in \mathcal{P}(\mathbf{N}^+)$. We also define the *length of* $\text{run}_k(j)$ (denoted $\|\text{run}_k(j)\|$) as its cardinality, i.e., the number of runs_{k-1} contained in it (for $k = 1$ it is length in the usual meaning).

Each $\text{run}_1(j)$ can be identified with a following horizontal subset of $\mathbf{Z}^2$: $\{(i_0+1, j), (i_0+2, j), \ldots, (i_0+m, j)\}$, where m is the length of the run. For each $a \in\,]0, 1[\, \setminus \mathbf{Q}$ we have only two possible run lengths: $\lfloor \frac{1}{a} \rfloor$ and $\lfloor \frac{1}{a} \rfloor + 1$. All the runs with one of those lengths always occur alone, i.e., do not have any neighbors of the same length in the sequence $(\text{run}_1(j))_{j \in \mathbf{N}^+}$, while the runs of the other length can appear in sequences.

On each level k for $k \geq 2$ we have short runs (S_k) and long runs (L_k), which are composed of the runs of level $k-1$. Only one type of runs_{k-1} (short or long) can appear in sequences, the second type always occurs singly. The first run on level $k-1$ is the run beginning in $(1, 1)$ and will be referred to as $\text{run}_{k-1}(1)$. Each $\text{run}_k(j)$ is composed of a single run_{k-1} (in the beginning or at the end of the $\text{run}_k(j)$) and the maximal number (call it m) of main runs_{k-1} between this single run_{k-1} and the next single one (or the previous one if the $\text{run}_k(j)$ ends with the single run_{k-1}). Then $\|\text{run}_k(j)\| = m + 1$.

We use the notation $S_k^m L_k$, $L_k S_k^m$, $L_k^m S_k$ and $S_k L_k^m$, when describing the form of digitization runs_{k+1}. For example, $S_k^m L_k$ means that the run_{k+1} we consider consists of m short runs_k (S_k) and one long run_k (L_k) in this order, so it is a run_{k+1} with the most frequent element short. The length of such a run_{k+1}, being its cardinality, i.e., the number of runs_k contained in it, is then equal to $m + 1$. We will also use the notation $\|S_{k+1}\|$ and $\|L_{k+1}\|$ for the length of the short resp. long runs_{k+1}. We recall the following theorem describing the digital lines with slope $a \in\,]0, 1[\, \setminus \mathbf{Q}$ in terms of CFs. This theorem formed our main result in [26].

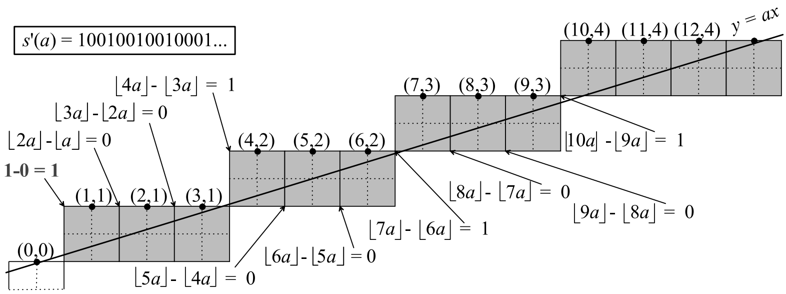


Fig. 3. An R′-digital line $y = ax$ with irrational slope and the corresponding $s'(a)$.

Theorem 1 *([26]; Description by CFs). Let a be irrational and $a = [0; a_1, a_2, \ldots]$. For the digital straight line with equation $y = ax$, we have $\|S_1\| = a_1$, $\|L_1\| = a_1 + 1$, and the forms of* runs_k (form_run_k) *for $k \geq 2$ are as follows:*

$$\text{form\_run}_k = \begin{cases} S_{k-1}^m L_{k-1} & \text{if } a_{i_a(k)} \neq 1 \text{ and } i_a(k) \text{ is even} \\ S_{k-1} L_{k-1}^m & \text{if } a_{i_a(k)} = 1 \text{ and } i_a(k) \text{ is even} \\ L_{k-1} S_{k-1}^m & \text{if } a_{i_a(k)} \neq 1 \text{ and } i_a(k) \text{ is odd} \\ L_{k-1}^m S_{k-1} & \text{if } a_{i_a(k)} = 1 \text{ and } i_a(k) \text{ is odd}, \end{cases} \tag{6}$$

where $m = b_k - 1$ if the run_k *is short and $m = b_k$ if the* run_k *is long. The function i_a is defined in Definition* 1 *and $b_k = a_{i_a(k)} + \delta_1(a_{i_a(k)}) a_{i_a(k)+1}$.*

In [28] we derived a recursive CF description of upper mechanical words from Theorem 1. Let us first recall the definition of those [17, p. 53]:

Definition 2. Given two real numbers α and ρ with $0 \leq \alpha \leq 1$, we define two infinite words $s_{\alpha,\rho} : \mathbf{N} \to \{0, 1\}$ and $s'_{\alpha,\rho} : \mathbf{N} \to \{0, 1\}$ by

$$s_{\alpha,\rho}(n) = \lfloor \alpha(n+1) + \rho \rfloor - \lfloor \alpha n + \rho \rfloor, \qquad s'_{\alpha,\rho}(n) = \lceil \alpha(n+1) + \rho \rceil - \lceil \alpha n + \rho \rceil.$$

The word $s_{\alpha,\rho}$ is the *lower mechanical word* and $s'_{\alpha,\rho}$ is the *upper mechanical word* with *slope* α and *intercept* ρ. A lower or upper mechanical word is *irrational* or *rational* according to whether its slope is irrational or rational.

In the present paper we deal with the special case when $\alpha \in]0, 1[$ is irrational and $\rho = 0$. In this case we will denote the lower and upper mechanical words by $s = s(\alpha)$ and $s' = s'(\alpha)$ respectively. We have $s_0 = s_0(\alpha) = \lfloor \alpha \rfloor = 0$ and $s'_0 = s'_0(\alpha) = \lceil \alpha \rceil = 1$ and, because $\lceil x \rceil - \lfloor x \rfloor = 1$ for non-integer x and $\lceil x \rceil - \lfloor x \rfloor = 0$ only for integers, we have

$$s = s(\alpha) = 0c(\alpha), \qquad s' = s'(\alpha) = 1c(\alpha) \tag{7}$$

(meaning 0, resp. 1 concatenated to $c(\alpha)$). The word $c(\alpha)$ is called the *characteristic word* of α. For each $\alpha \in]0, 1[\setminus \mathbf{Q}$, the characteristic word associated with α is thus the following infinite word $c = c(\alpha) : \mathbf{N}^+ \to \{0, 1\}$:

$$c_n = \lfloor \alpha(n+1) \rfloor - \lfloor \alpha n \rfloor = \lceil \alpha(n+1) \rceil - \lceil \alpha n \rceil, \quad n \in \mathbf{N}^+. \tag{8}$$

The connection between characteristic words and digital lines is a well-known fact. See for example [17, 2.1.2. *Mechanical words, rotations*], [19, chap. 6. *Sturmian Sequences*] or [15]. In [25] we remarked that the R′-digitization of a line with equation $y = ax$, where $a \in]0, 1[\setminus \mathbf{Q}$ and $x > 0$, is the subset of $\mathbf{Z}^2$ defined by (1). Due to (1), (7) and (8), the sequence $s'_0 = 1$, $s'_n = \lfloor (n+1)a \rfloor - \lfloor na \rfloor$ for $n \in \mathbf{N}^+$ describes the R′-digitization of the positive half line $y = ax$.

Fig. 3 illustrates the correspondence between R′-digital lines and upper and lower mechanical and characteristic words for $a \in]0, 1[\setminus \mathbf{Q}$.

For any $a \in]0, 1[\setminus \mathbf{Q}$, the upper mechanical word $s'(a)$, as defined in Definition 2, describes completely the digitization of the positive half line $y = ax$ and we get the following description of upper mechanical words. Because we have (7), our results will give a description of both mechanical and characteristic words.

Theorem 2 *([28]; Upper Mechanical Words by CFs). Let $a \in]0, 1[\setminus \mathbf{Q}$ and $a = [0; a_1, a_2, \ldots]$. If $s'(a)$ is the upper mechanical word with slope a and intercept* 0 *as defined in Definition* 2*, then $s'(a) = \lim_{k\to\infty} P_k$, where $P_1 = S_1 = 10^{a_1-1}$, $L_1 = 10^{a_1}$, and, for $k \geq 2$,*

$$P_k = \begin{cases} L_k = S_{k-1}^{a_{i_a(k)}} L_{k-1} & \text{if } a_{i_a(k)} \neq 1 \text{ and } i_a(k) \text{ is even} \\ S_k = S_{k-1} L_{k-1}^{a_{i_a(k)+1}} & \text{if } a_{i_a(k)} = 1 \text{ and } i_a(k) \text{ is even} \\ S_k = L_{k-1} S_{k-1}^{-1+a_{i_a(k)}} & \text{if } a_{i_a(k)} \neq 1 \text{ and } i_a(k) \text{ is odd} \\ L_k = L_{k-1}^{1+a_{i_a(k)+1}} S_{k-1} & \text{if } a_{i_a(k)} = 1 \text{ and } i_a(k) \text{ is odd}, \end{cases} \tag{9}$$

*where the function i_a is defined in Definition 1. The meaning of the symbols is the following: for $k \geq 1$, P_k - **P**refix number k, S_k - **S**hort run$_k$ and L_k - **L**ong run$_k$. To make the recursive formula (9) complete, we add that for each $k \geq 2$, if $P_k = S_k$, then L_k is defined in the same way as S_k, with the only difference that the exponent defined by $a_{i_a(k)}$ (or by $a_{i_a(k)+1}$) is increased by 1. If $P_k = L_k$, then S_k is defined in the same way as L_k, with the only difference that the exponent defined by $a_{i_a(k)}$ (or by $a_{i_a(k)+1}$) is decreased by 1.*

We have described in [28] the upper mechanical words $s'(a)$ with slope $a \in]0, 1[\setminus \mathbf{Q}$ and intercept 0 by an increasing sequence of prefixes $(P_k)_{k\in\mathbf{N}^+}$. Moreover, the prefix P_k for each $k \in \mathbf{N}^+$ corresponds to the first run of level k in the digitization of $y = ax$, thus we can write $P_k = \text{run}_k(1)$. This description of words according to the hierarchy of runs on all the levels (i.e., runs$_n$ for all $n \in \mathbf{N}^+$) can be helpful for understanding their construction and allows us to define equivalence relations on the set of upper mechanical words with slope $a \in]0, 1[\setminus \mathbf{Q}$ and intercept 0. This will be done in Section 3.

Another important fact about characteristic words, which will be used in the sequel, is the following [17, p. 62, Lemma 2.1.21.]:

Theorem 3. *For any $a, a' \in]0, 1[\setminus \mathbf{Q}$, if $c(a) = c(a')$, then $a = a'$.*

If the R′-digitization of $y = a'x$ is equal to the R′-digitization of $y = ax$ for some $a', a \in]0, 1[\setminus \mathbf{Q}$, then $a' = a$.

3. Main topic I: Two equivalence relations on the set of slopes

The description of digital lines in terms of CFs presented in Theorem 1 gives us possibilities of classifications of digital half lines $y = ax$ (where $a \in]0, 1[\setminus \mathbf{Q}$ and $x > 0$), or, equivalently, of upper mechanical words $s'(a)$ with slope a and intercept 0 (Theorem 2). In this section we will define two equivalence relations on the set of slopes.

3.1. Partition defined by the sequences of length specification

Before we define the equivalence relation based on the sequences of short run length on all the digitization levels, we recall the following theorem, which, in a slightly different form, we already formulated in [25].

Theorem 4. *Let $n \in \mathbf{N}^+$. For each sequence of natural numbers $(b_1, b_2, \ldots, b_n)$ such that $b_1 \geq 1$ and $b_i \geq 2$ for $2 \leq i \leq n$, one can form S_n (the short digitization run on level n) of all the possible lines $y = ax$ (equivalently, upper mechanical word with irrational slope and intercept 0) with $\|S_k\| = b_k$ for $k = 1, \ldots, n$, in exactly m ways, where*

$$m = \begin{cases} 2^{n-1} & \text{if } b_n \neq 2 \\ 2^{n-2} & \text{if } b_n = 2\,. \end{cases}$$

The digitizations of all these lines thus fulfill the following condition:
for $i = 1, \ldots, n$, the short run's length on digitization level i is b_i.

Proof. This follows from Theorem 1. The combinatorial idea behind the proof is the following. All the slopes we consider are of the form $[0; b_1, c_2, \ldots, c_k]$ for some k such that $n \leq k \leq 2n-1$, where for each $i \in [2, k]_{\mathbf{Z}}$ we have $c_i = 1$ (then, if $i \leq k-1$, $c_{i+1} = b_s - 1$ for some $s \in [2, n]_{\mathbf{Z}}$), $c_i = b_r - 1$ for some $r \in [2, n]_{\mathbf{Z}}$ (if $i \geq 3$ and $c_{i-1} = 1$), or $c_i = b_p$ for some $p \in [2, n]_{\mathbf{Z}}$. The problem to solve is thus: in how many ways can we choose which b_q for $q \in [2, n]_{\mathbf{Z}}$ we split into two elements of CF (into 1 and $b_q - 1$)? Any set with $n-1$ elements has 2^{n-1} different subsets, so 2^{n-1} is our answer in general. The exceptions are all the sequences of length specification with $b_n = 2$, because $[0; a_1, \ldots, a_l, 1, 1] = [0; a_1, \ldots, a_l, 2]$ for all sequences $(a_j)_{j\in[1,l]_{\mathbf{Z}}}$. For the sequences $(b_1, \ldots, b_n)$ with $b_n = 2$ we get only 2^{n-2} possible forms of S_4. □

Later in this paper we will formulate a more general theorem (Theorem 6).
Let us consider the following example.

Example 1. We will find all the possible forms of S_4 (short run$_4$) for all the lines with the following first four elements of the sequence of short run length: $(b_1, b_2, b_3, b_4) = (1, 2, 2, 3)$. This means that the short run length on level 1 is 1, on level 2 is 2, on level 3 is 2 and on level 4 is 3. Following the construction given in the proof of Theorem 4, we get $8 = 2^{4-1}$ possible forms of S_4, described by the fourth, fifth, sixth or seventh convergents of the possible slopes, depending on the number of b_i split into 1 and $b_i - 1$ in the part of the CF expansion of the slope describing the first four digitization levels:

(a) $p_4^{(a)}/q_4^{(a)} = [0; b_1, b_2, b_3, b_4] = [0; 1, 2, 2, 3] = \frac{17}{24}$
$S_4 = S_3^2 L_3 = (L_2 S_2)^2 (L_2 S_2^2) = [(S_1^2 L_1)(S_1 L_1)]^2 (S_1^2 L_1)(S_1 L_1)^2$

(b) $p_5^{(b)}/q_5^{(b)} = [0; b_1, 1, b_2 - 1, b_3, b_4] = [0; 1, 1, 1, 2, 3] = \frac{17}{27}$
$S_4 = L_3 S_3^2 = (S_2^2 L_2)(S_2 L_2)^2 = (S_1 L_1)^2 (S_1 L_1^2)[(S_1 L_1)(S_1 L_1^2)]^2$

(c) $p_5^{(c)}/q_5^{(c)} = [0; b_1, b_2, 1, b_3 - 1, b_4] = [0; 1, 2, 1, 1, 3] = \frac{18}{25}$
$S_4 = L_3 S_3^2 = (L_2^2 S_2)(L_2 S_2)^2 = (S_1^2 L_1)^2 (S_1 L_1)[(S_1^2 L_1)(S_1 L_1)]^2$

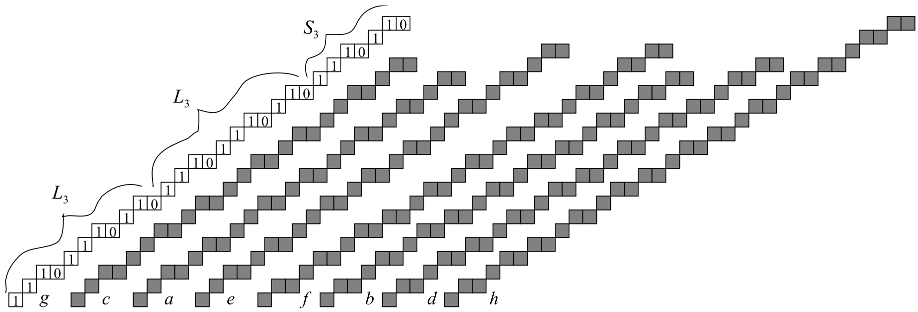


Fig. 4. Example 1. The 8 possible forms of S_4 with the length specification $(1, 2, 2, 3)$.

(d) $p_6^{(d)}/q_6^{(d)} = [0; b_1, 1, b_2 - 1, 1, b_3 - 1, b_4] = [0; 1, 1, 1, 1, 1, 3] = \frac{18}{29}$
$S_4 = S_3^2 L_3 = (S_2L_2)^2(S_2L_2^2) = [(S_1L_1)(S_1L_1^2)]^2(S_1L_1)(S_1L_1^2)^2$

(e) $p_5^{(e)}/q_5^{(e)} = [0; b_1, b_2, b_3, 1, b_4 - 1] = [0; 1, 2, 2, 1, 2] = \frac{19}{27}$
$S_4 = S_3L_3^2 = (L_2S_2)(L_2S_2^2)^2 = (S_1^2L_1)(S_1L_1)[(S_1^2L_1)(S_1L_1)^2]^2$

(f) $p_6^{(f)}/q_6^{(f)} = [0; b_1, 1, b_2 - 1, b_3, 1, b_4 - 1] = [0; 1, 1, 1, 2, 1, 2] = \frac{19}{30}$
$S_4 = L_3^2 S_3 = (S_2^2L_2)^2(S_2L_2) = [(S_1L_1)^2(S_1L_1^2)]^2(S_1L_1)(S_1L_1^2)$

(g) $p_6^{(g)}/q_6^{(g)} = [0; b_1, b_2, 1, b_3 - 1, 1, b_4 - 1] = [0; 1, 2, 1, 1, 1, 2] = \frac{21}{29}$
- gives the max slope (cf. Proposition 3); the 0's and 1's as on the picture:
$S_4 = L_3^2S_3 = (L_2^2S_2)^2(L_2S_2) = [(S_1^2L_1)^2(S_1L_1)]^2(S_1^2L_1)(S_1L_1) =$
$[(1^210)^2(110)]^2(1^210)(110) = 11101110110111011101101110110$

(h) $p_7^{(h)}/q_7^{(h)} = [0; b_1, 1, b_2 - 1, 1, b_3 - 1, 1, b_4 - 1] = [0; 1, 1, 1, 1, 1, 1, 2] = \frac{21}{34}$
- gives the min slope (cf. Proposition 2)
$S_4 = S_3L_3^2 = (S_2L_2)(S_2L_2^2)^2 = (S_1L_1)(S_1L_1^2)[(S_1L_1)(S_1L_1^2)^2]^2$.

All the possible forms of S_4 for all the lines with the four first elements of the corresponding sequence of short run length being 1, 2, 2 and 3 are shown on Fig. 4. To put stress on the equivalence between the descriptions of digital lines and upper mechanical words, we placed in the picture 0's and 1's forming the word corresponding to the S_4 of one of the lines (g). The lines are arranged in decreasing order of slopes, the line g has the largest slope, h the least one.

We are going to define an equivalence relation based on quantitative features of digital lines. We will identify all the lines with the same run length (cardinalities of runs) on all the digitization levels. To get there, we formulate the following definition.

Definition 3. For any irrational $a = [0; a_1, a_2, \ldots]$, the sequence $(b_n)_{n\in\mathbf{N}^+}$, where $b_1 = a_1$ and $b_k = a_{i_a(k)} + \delta_1(a_{i_a(k)})a_{i_a(k)+1}$ for $k \geq 2$ and the function $i_a : \mathbf{N}^+ \to \mathbf{N}^+$ is defined in Definition 1, will be called the sequence of length specification of the digital straight line $y = ax$ (equivalently, of the upper mechanical word with slope a and intercept 0).

These sequences will be used to partition the set of all the digital lines $y = ax$ (upper mechanical words) with irrational slopes $a \in\,]0, 1[$ into classes. We can unify all the lines with the same sequence of length specification in an equivalence class under the following relation.

Definition 4 (*Quantitative Identification; by Run-length*). We define the following relation $\sim_{\text{len}} \subset (]0, 1[\setminus \mathbf{Q})^2$: if $a, a' \in\,]0, 1[\setminus \mathbf{Q}$, then

$$a \sim_{\text{len}} a' \Leftrightarrow \forall n \in \mathbf{N}^+ \quad b_n^{(a)} = b_n^{(a')},$$

where $\left(b_n^{(a)};\quad n \in \mathbf{N}^+\right)$ and $\left(b_n^{(a')};\quad n \in \mathbf{N}^+\right)$ are the corresponding sequences of length specification in the digitization of the lines $y = ax$ and $y = a'x$ respectively (equivalently, in the run-hierarchical construction of $s'(a)$ and $s'(a')$).

The classification above is based on the short run length on each digitization level. Each digital line $y = ax$ generates its sequence of length specification. The motivation for the name of the sequence we can find in Theorem 1.

It is clear from Definition 3 that, for each $a \in\,]0, 1[\setminus \mathbf{Q}$, the sequence of length specification corresponding to a has the following properties: $b_1 \in \mathbf{N}^+$ and, for $n \geq 2$, $b_n \geq 2$. The opposite is also true: all infinite sequences of natural numbers greater than or equal to 2 (and the first element possibly equal to 1) generate the digitization of some lines with short run lengths on each level defined by the elements of the sequence. This means that each sequence $(b_n)_{n\in\mathbf{N}^+}$ where $b_1 \geq 1$ and $b_i \geq 2$ for all $i \geq 2$ is the sequence of length specification for some $a \in]0, 1[\setminus \mathbf{Q}$, so it defines one equivalence class in the set of all digital lines $y = ax$. How to find the elements of the class, is shown in Theorems 4 and 6 and illustrated with Example 1.

It follows from Theorem 4 when we let n go to infinity that, for each infinite sequence $(b_n)_{n\in\mathbf{N}^+}$ such that $b_1 \in \mathbf{N}^+$ and $b_i \geq 2$ for $i \geq 2$, the class under $\sim_{\text{len}}$ generated by this sequence has a cardinality of $2^{\aleph_0}$, thus of the continuum.

This classification is very easy to do when working with CF expansions of the slopes instead of the slopes as real numbers. It involves only the CF expansion of the slope, forming the sequence of length specification (Definition 3) and a comparison of integers.

Remark 1. All the lines contained in the same equivalence class under $\sim_{\text{len}}$ have different index jump functions (see Theorem 6) — if there were two lines with the same index jump function, they would have identical digitization, thus, according to Theorem 3, they would be the same line.

To be able to formulate some propositions about equivalence classes under $\sim_{\text{len}}$, we have to know how to compare two CFs with each other.

Proposition 1.

$$[a_0; a_1, a_2, \ldots] < [b_0; b_1, b_2, \ldots] \quad \Leftrightarrow \quad (a_0, -a_1, a_2, -a_3, a_4, -a_5, \ldots) \overset{\textit{lexic.}}{<} (b_0, -b_1, b_2, -b_3, b_4, -b_5, \ldots).$$

The second inequality is according to the lexicographical order on sequences.

Proposition 2. *For any sequence $(a_{2i+1})_{i\in\mathbf{N}}$ of positive integers, the line with the slope $[0; a_1, 1, a_3, 1, a_5, 1, a_7, 1, \ldots]$ has the least slope compared to all the other lines in its equivalence class under $\sim_{\text{len}}$.*

Proof. According to Definition 3, the line with such a slope belongs to the class defined by $(b_n)_{n\in\mathbf{N}^+} = (a_1, 1 + a_3, 1 + a_5, 1 + a_7, \ldots)$. It follows from Proposition 1, that $[0; a_1, 1, a_3, 1, a_5, 1, a_7, 1, \ldots] < [0; a_1, 1 + a_3, 1 + a_5, 1 + a_7, \ldots]$, because the first difference between the elements occurs on a place with an even number and $a_2 = 1 < 1 + a_3$, because $a_3 \geq 1$. Generally, for any line belonging to the equivalence class defined by $(b_n)_{n\in\mathbf{N}^+}$, the first difference between the elements of $[0; a_1, 1, a_3, 1, a_5, 1, a_7, 1, \ldots]$ and the elements of the CF expansion of the slope of this line occurs on a place with an even number, because it is preceded by a_1 which always remains unchanged for all the lines from the class and a number of pairs $(1, a_i)$ of the elements which are the same as for $[0; a_1, 1, a_3, 1, a_5, 1, a_7, 1, \ldots]$. This means that the first difference between the elements of the CF expansions of the slopes will occur on an even place and it will give the inequality $a_{2i} = 1 < 1 + a_{2i+1}$ for some $i \in \mathbf{N}^+$. The first difference between the CF elements on the place with an even index has the same inequality as the inequality between the CFs, so $a_{2i} = 1 < 1 + a_{2i+1}$ implies

$$[0; a_1, 1, a_3, \ldots, 1, a_{2i-1}, 1, a_{2i+1}, 1, a_{2i+3}, \ldots] < [0; a_1, 1, a_3, \ldots, 1, a_{2i-1}, 1 + a_{2i+1}, \ldots].$$

The proof is thus complete. □

Proposition 3. *For any $a_1 \in \mathbf{N}^+$ and any sequence $(a_{2i})_{i\in\mathbf{N}^+}$ of positive integers such that $a_2 \geq 2$, the line with the slope $[0; a_1, a_2, 1, a_4, 1, a_6, 1, \ldots]$ has the largest slope in its equivalence class under $\sim_{\text{len}}$.*

Proof. The equivalence class of the line with such a slope is defined by the sequence $(b_n)_{n\in\mathbf{N}^+} = (a_1, a_2, 1+a_4, 1+a_6, \ldots)$. Let us take any other line belonging to the same class. If this line has slope $[0; a_1, 1, a_2 - 1, \ldots]$, it is clearly a slope which is less than the slope defined in the statement of the proposition. If the slope is $[0; a_1, a_2, \ldots]$, then we use analogous reasoning as in the proof of Proposition 2 and say that the first difference between the elements of the CF expansion of this slope and the slope which is supposed to be maximal occurs on a place with an odd number. According to Proposition 1, the first difference on an odd place gives the opposite direction of the inequality between the CFs compared to the inequality on the elements on this place. For the slopes defined in the text of the proposition, all a_{2i+1} for $i \in \mathbf{N}^+$ are equal to 1, they are thus minimal, which makes the slopes maximal. □

To summarize, each equivalence class defined by a sequence $(b_n)_{n\in\mathbf{N}^+}$, where $b_1 \in \mathbf{N}^+$ and $b_i \geq 2$ for $i \geq 2$, has both least and largest elements, being $[0; b_1, \overline{1, b_n - 1}]_{n=2}^{\infty}$ and $[0; b_1, b_2, \overline{1, b_n - 1}]_{n=3}^{\infty}$ respectively.

3.2. Partition defined by the index jump function

In this section we will present another partition of digital lines into classes. This partition will have a qualitative character. We will identify lines with the same construction (in terms of long and short runs) on all the levels.

As we can see in Theorems 1 and 2, some elements equal to 1 of the CF expansion of the slope play an important role in the construction of the digitization runs. Analyzing (6) in Theorem 1 we observe that for each level $k - 1$, where $k \geq 2$, the short run S_{k-1} is the most frequent one if $a_{i_a(k)} \neq 1$. In the opposite case, when $a_{i_a(k)} = 1$, the most frequent one is the long run L_{k-1}.

The parity of the index jump function decides only about the starting point in (1, 1), not about the essential form of the digitization as a sequence of runs on all the digitization levels. The most important factor for the construction is this if $a_{i_a(k)}$ is equal to 1 or different from 1. This gives the qualitative description of runs. From Definition 1 we can see that the places of CF elements equal to 1 determine the index jump function. This is the reason why we define the second equivalence relation on the set of slopes in the following way.

Definition 5 (*Qualitative Identification; by Run-Construction*). We define the following relation $\sim_{\text{con}} \subset (]0, 1[\setminus \mathbf{Q})^2$. If $a, a' \in]0, 1[\setminus \mathbf{Q}$, then

$$a \sim_{\text{con}} a' \Leftrightarrow i_a = i_{a'},$$

where i_a and $i_{a'}$ are the corresponding index jump functions in digitization of the lines $y = ax$ and $y = a'x$ respectively.

The relation $\sim_{\text{con}}$ is defined on the set of slopes $]0, 1[\setminus \mathbf{Q}$, i.e., it also partitions the set of all upper mechanical words with slope $a \in]0, 1[\setminus \mathbf{Q}$ and intercept 0. It is clearly an equivalence relation. The following proposition justifies the name of the relation $\sim_{\text{con}}$. It follows immediately from Theorem 1.

Proposition 4. *If $a \sim_{\text{con}} a'$, then the digitization runs of the lines $y = ax$ and $y = a'x$ have exactly the same construction (with respect to short and long runs) for all the digitization levels, only run lengths can be different.*

In (6) we can see that this construction depends on the places of some elements equal to 1 in the CF expansion of the slope. We will call those elements *essential* 1's. Then we will describe which of the CF elements equal to 1 decide about the construction of runs on the level they correspond to, i.e., which 1's are essential (Propositions 5 and 6).

Definition 6. Let $a \in]0, 1[\setminus \mathbf{Q}$ and $a = [0; a_1, a_2, a_3, \ldots]$. Then

$$a_k \text{ is an } essential\ 1 \Leftrightarrow [\ a_k = 1 \wedge \exists m \geq 2, \quad k = i_a(m)\ ].$$

In Proposition 5 we will show that, if a is irrational and $a = [0; a_1, a_2, \ldots]$, then only the $a_k = 1$ $(k > 1)$ which are directly preceded by an even number $(0, 2, 4, \ldots)$ of consecutive 1's (also with an index greater than 1) are essential.

We will now formulate a definition which we need for the proof of Theorem 5 and for the description of digital lines presented in Section 4.

Definition 7. Let $a = [0; a_1, a_2, \ldots]$ be irrational. Let $J = \emptyset$ if there are no 1's in the CF expansion of a (except maybe for a_1), $J = \mathbf{N}^+$ if there are infinitely many 1's in the CF expansion of a and $J = [1, M]_{\mathbf{Z}}$ for some $M \in \mathbf{N}^+$ if there are M essential 1's in the CF expansion of a. The following sequence $(s_j)_{j \in J}$: $s_1 = \min\{k \in \mathbf{N}^+;\ a_k \text{ is essential}\}$, and, for $n \in J \setminus \{1\}$, $s_n = \min\{k > s_{n-1};\ a_k \text{ is essential}\}$ (and $(s_i)_{i \in \emptyset} = \emptyset$ in case $J = \emptyset$) we will call the *sequence of the places of essential* 1*'s* in the CF expansion of a.

Proposition 5. *Let $a = [0; a_1, a_2, \ldots]$ be irrational and let the interval J be as in Definition 7. The sequence of the places of essential 1's in the CF expansion of a is $(s_j)_{j \in J}$, where $s_1 = \min\{k \geq 2;\ a_k = 1\}$ and, for $n \in J \setminus \{1\}$,*

$$s_n = \min\{k \geq s_{n-1} + 2;\ a_k = 1\}.$$

Proof. This follows from Definitions 1, 6 and 7. From Definition 1 we get the following equivalence. For $m \geq 2$

$$a_{i_a(m)} = 1 \Leftrightarrow i_a(m+1) = i_a(m) + 2 \Leftrightarrow i_a(m) + 1 \notin \{i_a(n)\}_{n \in \mathbf{N}^+}. \tag{10}$$

Sequence $(i_a(n))_{n \in \mathbf{N}^+}$ is thus strictly increasing and the difference between each of its two consecutive elements is equal to 1 or to 2.

From (10) one can see that a_k is an essential 1 if and only if $a_k = 1$ $(k \geq 2)$ and the number $k + 1 = i_a(m) + 1$ does not belong to the sequence $(i_a(n))_{n \in \mathbf{N}^+}$, so $(s_j)_{j \in J}$ as defined in Definition 7 is strictly increasing and the difference between each of its two consecutive elements is at least 2. Moreover, according to Definitions 1 and 6, the index of each essential 1 is greater than or equal to 2. □

Proposition 6. *Let $a = [0; a_1, a_2, \ldots]$ be irrational. Then none of $a_{s_m+1} = 1$ for any $m \in J$ is essential.*

Proof. It follows from (10) that $(i_a(k))_{k \in \mathbf{N}^+} = \mathbf{N}^+ \setminus (s_m + 1)_{m \in J}$, thus, according to Definition 6, all the $a_{s_m+1} = 1$ for $m \in J$ are non-essential 1's. □

Each $a \in]0, 1[\setminus \mathbf{Q}$ generates the index jump function i_a, which generates the sequence of the places of important 1's. Each sequence of important 1's has some properties, which will be formulated in the following proposition. We will show that the opposite is also true, i.e., that each sequence with these properties is a sequence of the essential 1's for some $a \in]0, 1[\setminus \mathbf{Q}$.

Proposition 7. *Let $a = [0; a_1, a_2, \ldots]$ be irrational. The corresponding sequence $(s_j)_{j \in J}$ of the places of essential 1's in the CF expansion of a has the following properties:*

- *the set J is as follows: $J = \emptyset$, $J = \mathbf{N}^+$ or $J = [1, M]_{\mathbf{Z}}$ for some $M \in \mathbf{N}^+$, depending on the number of essential 1's in the CF expansion of a.*
- *the sequence $(s_j)_{j \in J}$ is a sequence of positive integers such that $s_1 \geq 2$ and, for $k \in J \setminus \{1\}$, $s_k - s_{k-1} \geq 2$.*

The opposite is also true, i.e., each sequence $(s_j)_{j\in J}$ of positive integers, where J is any index set such that $J = \emptyset$, $J = \mathbf{N}^+$ or $J = [1, M]$ for some $M \in \mathbf{N}^+$ and $s_1 \geq 2$ and, for $k \in J \setminus \{1\}$, $s_k - s_{k-1} \geq 2$, is a sequence of the places of essential 1's for some $a \in]0, 1[\setminus \mathbf{Q}$.

Proof. The first part of the statement follows from Proposition 5. The second part we will prove in Theorem 6. □

Because it is easier to use the sequences of the places of essential 1's than the index jump function, we will formulate the following proposition.

Proposition 8. *If $a, a' \in]0, 1[\setminus \mathbf{Q}$, then $a \sim_{\text{con}} a'$ iff their corresponding sequences of the places of important 1's are equal to each other, i.e.,*

$$a \sim_{\text{con}} a' \Leftrightarrow \left(s_j^{(a)}\right)_{j\in J} = \left(s_k^{(a')}\right)_{k\in J'}.$$

Proof. From Definitions 6 and 7 we can see that the sequences of the places of essential 1's in the CF expansion of a are defined by the index jump function corresponding to a, so, if $i_a = i_{a'}$, then $\left(s_j^{(a)}\right)_{j\in J} = \left(s_k^{(a')}\right)_{k\in J'}$, which proves the implication from the left to the right.

Let us now assume that a and a' have the same sequences of the places of essential 1's. We will show that their index jump functions are also equal to each other. We will show that, for each a, the index jump function can be expressed in terms of the sequence of the places of important 1's. Let us consider the following function $f_a : \mathbf{N}^+ \to \mathbf{N}^+$, where $a \in]0, 1[\setminus \mathbf{Q}$ and $(s_j)_{j\in J}$ is the corresponding sequence of essential 1's:

$$f_a(k) = \begin{cases} k, & 1 \leq k \leq s_1, \\ k+1, & s_1 + 1 \leq k \leq s_2 - 1, \\ k+2, & s_2 \leq k \leq s_3 - 2, \\ k+3, & s_3 - 1 \leq k \leq s_4 - 3, \\ \vdots & \vdots \\ k+m-1, & s_{m-1} - (m-3) \leq k \leq s_m - (m-1), \\ k+m, & s_m - (m-2) \leq k \leq s_{m+1} - m. \\ \vdots & \vdots \end{cases} \tag{11}$$

If $J = \emptyset$, the formula will be reduced to the first line, i.e., $f_a(k) = k$ for all $k \geq 1$. If $J = [1, M]_{\mathbf{Z}}$, the formula will consist of the first $M + 1$ lines, the last one will describe the values of $f_a(k)$ for $k \geq s_M - M + 2$. It follows from Proposition 7 that we have for the function defined by (11):

- $f_a(1) = 1,\ f_a(2) = 2$ (because $s_1 \geq 2$),
- for all the intervals $I_m = [s_m - (m-2), s_{m+1} - m]_{\mathbf{Z}}$ we have $s_{m+1} - m - [s_m - (m-2)] = s_{m+1} - s_m - 2 \geq 0$, and, moreover, $p, p+1 \in I_m \Rightarrow f_a(p+1) - f_a(p) = 1$,
- for each m for which the expressions make sense, we have $f_a(s_m - (m-1)) = s_m - (m-1) + m - 1 = s_m$ and, for the next value of the argument, we have $f_a(s_m - (m-2)) = s_m - (m-2) + m = s_m + 2$. This means that f_a jumps over the value $s_m + 1$ for each $m \in J$.

The function f_a as defined in (11) and the index jump function i_a corresponding to a have the same value for $n = 1$, are both increasing on $\mathbf{N}^+$ and $(f_a(k))_{k\in\mathbf{N}^+} = \mathbf{N}^+ \setminus (s_m + 1)_{m\in J}$, they have thus the same set of values on N^+. This means that $f_a \equiv i_a$. The index jump function can thus be expressed by the sequence of the places of essential 1's, which proves the implication from the right to the left. □

We have just shown that one can identify index jump functions with the corresponding sequences of essential 1's. This means that the equivalence classes under $\sim_{\text{con}}$ can also be defined by sequences as described in Proposition 7.

Let us consider the following examples.

Example 2. Let us consider the slopes with the first CF elements as on Fig. 1. We present an illustration of the forming of the index jump function for those slopes. If $a = [0; a_1, a_2, \ldots]$, where $a_k \geq 2$ for $k = 2, 5, 8, 9, 11, 12, 16, 17$, then the index jump function i_a is formed as follows:

$$\begin{array}{rl} a = & [0; \overset{b_1}{1}, \overset{b_2}{a_2}, \overbrace{\underline{1}, 1}^{b_3}, \overset{b_4}{a_5}, \overbrace{\underline{1}, 1}^{b_5}, \overset{b_6}{a_8}, \overset{b_7}{a_9}, \overbrace{\underline{1}, a_{11}}^{b_8}, \overset{b_9}{a_{12}}, \overbrace{\underline{1}, 1}^{b_{10}}, \overbrace{\underline{1}, a_{16}}^{b_{11}}, \overset{b_{12}}{a_{17}}, \ldots] \\ & \quad\ \downarrow \ \downarrow \quad \downarrow \qquad \downarrow \quad \downarrow \qquad \downarrow \quad \downarrow \quad \downarrow \qquad \downarrow \quad \downarrow \qquad \downarrow \qquad \downarrow \quad \ldots \\ (i_a(k))_{k\in\mathbf{N}^+} = & (\quad 1, 2, \ 3, \quad 5, \ 6, \quad 8, \ 9, \ 10, \quad 12, \ 13, \ 15, \quad 17, \ \ldots) \end{array}$$

In the last row we presented the first twelve elements of the sequence of the values of the index jump function for these a, so $(i_a(k))_{1\leq k\leq 12}$. The essential 1's are underlined in the first row. The sequence of the places of essential 1's is $(s_j)_{j\in J} = (3, 6, 10, 13, 15, \ldots)$. The non-essential 1's are a_1, a_4, a_7, a_{14}. We have also illustrated the sequence of length specification for these slopes. It is $(b_n)_{n\in\mathbf{N}^+} = (1, a_2, 1+1, a_5, 1+1, a_8, a_9, 1+a_{11}, a_{12}, 1+1, 1+a_{16}, a_{17}, \ldots)$.

Example 3. Let $a = [0; a_1, a_2, \ldots]$, where $a_1 \in \mathbf{N}^+$ and $a_n \geq 2$ for all $n \geq 2$. Then we have $i_a(k) = k$ for each $k \in \mathbf{N}^+$ and $b_k = a_k$ for all $k \in \mathbf{N}^+$. This means, according to Theorem 1, that for all such lines $y = ax$ we get the following description of the digitization: $\|S_1\| = a_1$, $\|L_1\| = a_1 + 1$; for $k \in \mathbf{N}^+$: $S_{2k} = S_{2k-1}^{a_{2k}-1} L_{2k-1}$, $L_{2k} = S_{2k-1}^{a_{2k}} L_{2k-1}$, $S_{2k+1} = L_{2k} S_{2k}^{a_{2k+1}-1}$, $L_{2k+1} = L_{2k} S_{2k}^{a_{2k+1}}$. This pattern is valid for all the slopes without any 1's (except possibly for a_1) in the CF expansion. The sequence of the places of essential 1's is $(s_n)_{n\in\emptyset} = \emptyset$.

Example 4. Let $a = [0; a_1, 1, a_3, 1, a_5, 1, \ldots]$, where $a_{2n+1} \in \mathbf{N}^+$ for all $n \in \mathbf{N}$ (cf. Proposition 2). Then we have $i_a(1) = 1$ and $b_1 = a_1$. For $k \geq 2$ we have $i_a(k) = 2k - 2$ and $b_k = a_{2k-2} + a_{2k-1} = a_{2k-1} + 1$. The digitization is thus: $\|S_1\| = a_1$, $\|L_1\| = a_1 + 1$; for $k \geq 2$: $S_k = S_{k-1} L_{k-1}^{a_{2k-1}}$, $L_k = S_{k-1} L_{k-1}^{a_{2k-1}+1}$. This pattern is valid for all the slopes with 1's on all the even places in the CF expansion. The sequence of the places of essential 1's is $(s_n)_{n\in\mathbf{N}^+} = (2n)_{n\in\mathbf{N}^+}$. The class under $\sim_{\text{con}}$ generated by this sequence joins the least elements of all the classes under $\sim_{\text{len}}$ (see Proposition 2).

Example 5. Let $a = [0; a_1, a_2, 1, a_4, 1, a_6, 1, a_8, \ldots]$, where $a_1 \in \mathbf{N}^+$, $a_2 \geq 2$ and $a_{2n} \in \mathbf{N}^+$ for all $n \geq 2$ (cf. Proposition 3). Then we have $i_a(1) = 1$, $i_a(2) = 2$ and $i_a(k) = 2k - 3$ for $k \geq 3$, which means that $i_a(k)$ is odd for all $k \neq 2$. Moreover, $b_1 = a_1$, $b_2 = a_2$ and $b_k = a_{2k-3} + a_{2k-2} = 1 + a_{2k-2}$ for $k \geq 3$. The digitization is thus as follows: $\|S_1\| = a_1$, $\|L_1\| = a_1 + 1$, $S_2 = S_1^{a_2-1} L_1$, $L_2 = S_1^{a_2} L_1$, and for $k \geq 3$ we have $S_k = L_{k-1}^{a_{2k-2}} S_{k-1}$ and $L_k = L_{k-1}^{a_{2k-2}+1} S_{k-1}$. The sequence of the places of essential 1's is $(s_n)_{n\in\mathbf{N}^+} = (2n+1)_{n\in\mathbf{N}^+}$. The class under $\sim_{\text{con}}$ generated by this sequence joins largest elements of all the classes under $\sim_{\text{len}}$ (see Proposition 3).

To complete this section, we would like to formulate a qualitative counterpart of Propositions 2 and 3 which were formulated in Section 3.1 for the quantitative relation $\sim_{\text{len}}$. We describe the solution in Theorem 5, which is one of the main results in this paper. First we will formulate the following lemma [31, pp. 101–102].

Lemma 1. *Let $(F_n)_{n\in\mathbf{N}^+}$ denote the Fibonacci sequence, i.e.,*

$$F_1 = 1, \qquad F_2 = 1 \quad \textit{and}, \quad \textit{for } n \geq 3, \quad F_n = F_{n-2} + F_{n-1}\,. \tag{12}$$

If p_n/q_n for $n \in \mathbf{N}^+$ denotes the nth *convergent of* $[0; \overline{1}\,] = (\sqrt{5} - 1)/2$*, then*

$$p_n = F_n, \qquad q_n = F_{n+1}. \tag{13}$$

Proof. We remark first that $x = [0; \overline{1}\,]$ is the positive root of the equation $x = 1/(1 + x)$, thus $x^2 + x - 1 = 0$, so it is indeed $(\sqrt{5} - 1)/2$.

For the CF $[0; \overline{1}\,]$ we have $a_n = 1$ for $n \in \mathbf{N}^+$, so the formulae (3) and (4) together with (12) give us (13). □

Theorem 5. *There exists no equivalence class under $\sim_{\text{con}}$ with a least element. The infimum is equal to 0 for all the classes.*

There exists exactly one class under $\sim_{\text{con}}$ which has a greatest element. This class is defined by the sequence $(s_j)_{j\in\mathbf{N}^+} = (2j)_{j\in\mathbf{N}^+}$ and the maximum is the Golden Section $(\sqrt{5} - 1)/2$. The following statement describes the suprema of the classes generated by all the possible sequences of the places of essential 1's:

$$\forall n \in \mathbf{N}^+ \, [(\forall k \in [1, n-1]_{\mathbf{Z}} \quad s_k = 2k) \wedge (s_n > 2n \vee |J| = n - 1)] \Rightarrow$$
$$\sup\{a \in \,]0, 1[\, \setminus \mathbf{Q}; \quad a \in [(s_j)_{j\in J}]_{\sim_{\text{con}}}\} = \frac{F_{2n-1}}{F_{2n}},$$

where $(F_n)_{n\in\mathbf{N}^+}$ is the Fibonacci sequence as defined by (12) *and $|J|$ is the cardinality of J.*

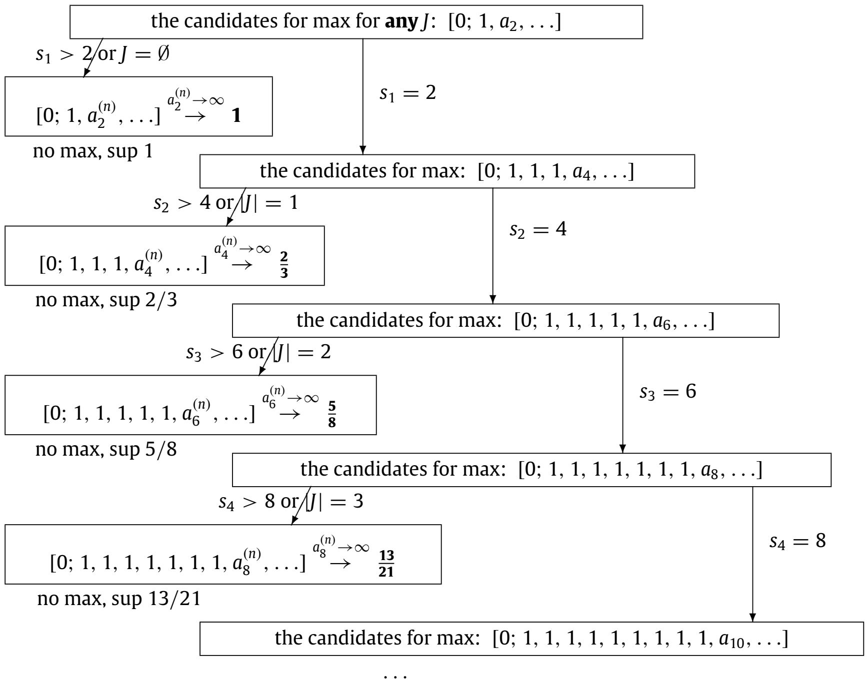
Proof. To prove the statement about the infimum we remark that in each equivalence class under $\sim_{\text{con}}$ there exist slopes $a = [0; a_1, a_2, \ldots]$ with $a_1 = 1$, slopes with $a_1 = 2$ etc. When a_1 tends to infinity, then $a = [0; a_1, a_2, \ldots]$ tends to 0, so we have no least element in $]0, 1[\, \setminus \mathbf{Q}$ and the infimum is 0 for all classes.

To prove the statement about the supremum, we take $J = \emptyset$, $J = \mathbf{N}^+$ or $J = [1, M]_{\mathbf{Z}}$ for some $M \in \mathbf{N}^+$ and any sequence $(s_j)_{j\in J}$ of integers such that $s_1 \geq 2$ and $s_i - s_{i-1} \geq 2$ for all $i \in J \setminus \{1\}$, and we consider the class generated by this sequence. We know, from Propositions 7 and 8, that all the classes under $\sim_{\text{con}}$ are exactly the classes generated by sequences like this.

If $J = \emptyset$, then there is clearly no greatest element in the equivalence class $[\emptyset]_{\sim_{\text{con}}}$, because all the slopes $[0; 1, a_2, \ldots]$ with $a_n \geq 2$ for $n \geq 2$ (which are the only candidates for the position of the maximal element) belong to it and, when a_2 tends to infinity, then $[0; 1, a_2, \ldots]$ tends to 1 (the supremum), which does not belong to $]0, 1[\, \setminus \mathbf{Q}$. The same reasoning holds for $J \neq \emptyset$ in case when $s_1 > 2$ (i.e., $a_2 \geq 2$).

If $J \neq \emptyset$ and $s_1 = 2$ (i.e., $a_2 = 1$), we consider the only (according to Proposition 1) candidate for a greatest element and it is $[0; 1, 1, 1, a_4, \ldots]$. If $s_2 > 4$ (i.e., $a_4 \geq 2$), we repeat the same reasoning again. We go on like this, using Proposition 1 about a comparison of CFs.

The flowchart below analyzes (with respect to largest elements or suprema) all the possible classes generated by all the possible sets J and all the possible sequences $(s_j)_{j\in J}$ as described in Proposition 7:



The rightmost way of the flowchart leads to $[0;\ \overline{1}\ ] = (\sqrt{5}-1)/2$, which is an irrational number. This means that the only class which has largest element is the class as described in Example 4, generated by the sequence of the places of essential 1's $(s_n)_{n\in\mathbf{N}^+} = (2n)_{n\in\mathbf{N}^+}$. The statement about the suprema of all the classes can be derived from the flowchart. It follows from Lemma 1 that the odd-numbered convergents of the Golden Section $[0;\ \overline{1}\ ]$ are $1 = \frac{F_1}{F_2}, \frac{2}{3} = \frac{F_3}{F_4}, \frac{5}{8} = \frac{F_5}{F_6}, \frac{13}{21} = \frac{F_7}{F_8}, \ldots$. Moreover, when analyzing the flowchart one can ensure oneself that it covers all the possible classes under $\sim_{\text{con}}$. $\square$

Let us summarize the results of this section. We defined two equivalence relations on the set $]0, 1[\setminus \mathbf{Q}$.

Each equivalence class under $\sim_{\text{len}}$ has both least (Proposition 2) and greatest (Proposition 3) elements. The least elements of all the classes under $\sim_{\text{len}}$ belong to the equivalence class under $\sim_{\text{con}}$ generated by the following sequence of the places of essential 1's: $(s_n)_{n\in\mathbf{N}^+} = (2n)_{n\in\mathbf{N}^+}$ (Example 4), so

$$\min\{a \in]0, 1[\setminus \mathbf{Q};\ a \in [(b_n)_{n\in\mathbf{N}^+}]_{\sim_{\text{len}}}\} = [0;\ b_1, \overline{1, b_n - 1}]_{n=2}^{\infty}.$$

The largest elements of all the classes under $\sim_{\text{len}}$ belong to the equivalence class under $\sim_{\text{con}}$ generated by the following sequence of the places of essential 1's: $(s_n)_{n\in\mathbf{N}^+} = (2n+1)_{n\in\mathbf{N}^+}$ (Example 5), so

$$\max\{a \in]0, 1[\setminus \mathbf{Q};\ a \in [(b_n)_{n\in\mathbf{N}^+}]_{\sim_{\text{len}}}\} = [0;\ b_1, b_2, \overline{1, b_n - 1}]_{n=3}^{\infty}.$$

In Theorem 5 we answered analogous questions about the relation $\sim_{\text{con}}$. No equivalence class under $\sim_{\text{con}}$ has a least element. The infimum in each class is equal to zero. The answer connected with greatest elements is much more interesting. The partition of all the irrational numbers from the interval $]0, 1[$ into equivalence classes under $\sim_{\text{con}}$ gives the sets with suprema equal to the odd-numbered convergents of the Golden Section, thus with no largest element belonging to the class (which is a set of irrational numbers). The only exception is the class generated by $(s_n)_{n\in\mathbf{N}^+} = (2n)_{n\in\mathbf{N}^+}$, which has a greatest element and it is equal to the Golden Section.

The only class under $\sim_{\text{con}}$ which has a greatest element is the class of least elements of all the classes under $\sim_{\text{len}}$ (cf. Proposition 2).

4. Main topic II: A simple description of digital lines (upper mechanical words) by two sequences of positive integers

In this section we will formulate and prove the converse to Theorem 1, where we have described the construction of digital positive half lines $y = ax$, where $a \in]0, 1[\setminus \mathbf{Q}$, using the index jump function. It was a **necessary condition** for being digital line $y = ax$ with slope $a \in]0, 1[\setminus \mathbf{Q}$. Having the slope, we showed the construction of the digital line. Theorem 6 will be formulated in terms of sequences of the places of essential 1's and proven in terms of CFs. It will show that the condition formulated in Theorem 1 was also **sufficient** (for each such digitization we can find a slope of the real line which is digitized according to the described construction). In Theorem 1 it was: given the slope — describe the digitization. Here (Theorem 6): given the description of the digitization — calculate the slope.

Theorem 6 (*Description by Two Sequences of Positive Integers*). *Each pair $((b_n)_{n\in\mathbf{N}^+}, (s_j)_{j\in J})$ of sequences of positive integers such that*

(1) *$b_i \geq 2$ for all $i \geq 2$,*

(2) *$J = \mathbf{N}^+$, $J = \emptyset$ or $J = [1, M]_\mathbf{Z}$ for some $M \in \mathbf{N}^+$, $s_1 \geq 2$ (if $J \neq \emptyset$) and $s_i - s_{i-1} \geq 2$ for all $i \in J \setminus \{1\}$,*

defines exactly one digital line which has following properties: for all $n \in \mathbf{N}^+$ we have $\|S_n\| = b_n$ and $(s_j)_{j\in J}$ is the sequence of the places of essential 1's in the CF expansion of the slope.

Proof. To construct the slope a of this line, we have to find the element of the equivalence class under $\sim_{\text{len}}$ defined by $(b_n)_{n\in\mathbf{N}^+}$, which fulfills the additional conditions concerning the places of essential 1's.

If $J = \emptyset$, then $a = [0; b_1, b_2, \ldots]$ is the slope we are looking for.

Let $J = \mathbf{N}^+$. To make the following formula, we generalize the method from Theorem 4 and use Definition 6. The slope is $a = [0; a_1, a_2, a_3, \ldots]$, where

$\forall\ \ 1 \leq k \leq s_1 - 1 \qquad a_k = b_k,$

$a_{s_1} = 1,$

$a_{s_1+1} = b_{s_1} - 1,$

$$\forall\ \ 1 \leq m (\leq M - 1) \qquad \begin{array}{l} k \in [s_m - m + 2, s_{m+1} - (m+1)]_\mathbf{Z} \ \ \Rightarrow \ \ a_{k+m} = b_k, \\ a_{s_{m+1}} = 1, \\ a_{s_{m+1}+1} = b_{s_{m+1}-m} - 1. \end{array}$$

We added the restriction $m \leq M - 1$ in the last condition on m (in brackets in the formula above), to cover the case when $J = [1, M]_\mathbf{Z}$ for some $M \in \mathbf{N}^+$. If $M = 1$, the whole second part of the formula disappears. If $J = [1, M]_\mathbf{Z}$ for some $M \in \mathbf{N}^+$, the formula gets an additional row:

$\forall\ \ k \in \mathbf{N}^+ \quad k \geq s_M - M + 2 \quad \Rightarrow \quad a_{k+M} = b_k.$

According to Theorem 1, the line $y = ax$ for a as described above, has the short run lengths determined by the sequence $(b_n)_{n\in\mathbf{N}^+}$ and $(s_j)_{j\in J}$ is clearly the sequence of the places of the essential 1's. Uniqueness follows from Theorem 3 (the same digitization implies the same slope). □

Each digital line is thus fully determined by two sequences of positive integers. One of them (the sequence of length specification $(b_n)_{n\in\mathbf{N}^+}$) fulfills the condition $b_n \geq 2$ for all $n \geq 2$. The second one (the sequence $(s_j)_{j\in J}$ of the places of the essential 1's) fulfills the conditions $s_1 \geq 2$ and $s_i - s_{i-1} \geq 2$ for all $i \in J \setminus \{1\}$. And each digital line gives such two sequences.

We have thus shown equivalence between the set of all digital lines $y = ax$ with $a \in]0, 1[\setminus \mathbf{Q}$ (equivalently, all upper mechanical words with slope a and intercept 0) and the set of all the pairs of sequences of positive integers fulfilling conditions as described in Theorem 6.

In Sections 3.1 and 3.2 we have described two equivalence relations on the set of slopes $]0, 1[\setminus \mathbf{Q}$. A sequence of length specification and a sequence of the places of essential 1's determine together the slope of a line (Theorem 6). This gives us the following corollary.

Corollary 1. *For the two equivalence relations described in Definitions 4 and 5 we have the following. For each $a', a \in]0, 1[\setminus \mathbf{Q}$*

$$a \sim_{\text{len}} a' \wedge a \sim_{\text{con}} a' \Rightarrow a = a'.$$

Each sequence of length specification gives a class of digital lines with b_i as short run length on level i for $i \in \mathbf{N}^+$. Each such class has the cardinality of the continuum. All the lines which belong to the same class generated by the relation $\sim_{\text{len}}$ have different sequences of the places of essential 1's, so they belong to different classes under $\sim_{\text{con}}$. The two relations are complementary.

5. Conclusion

As mentioned in the introduction to this paper, there exist many CF-based descriptions of both digital lines and mechanical words. Our recursive CF description of lines and words with irrational slopes seems to be the only one which reflects the hierarchy of runs on all levels. The hierarchical structure enables us to analyze abstract properties of lines (words).

We have defined two complementary equivalence relations on the set of slopes. One of them (quantitative $\sim_{\mathrm{len}}$) is based on run lengths on all the digitization levels and joins all the lines with the same sequence of length specification $(b_n)_{n\in\mathbf{N}^+}$ in the same class. The second one (qualitative $\sim_{\mathrm{con}}$) joins all the lines with the same construction of digitization in terms of long and short runs on all the digitization levels. We have shown that these two relations are complementary and we have shown how to construct the slope of the (unique) line with the slope from the intersection $[(b_n)_{n\in\mathbf{N}^+}]_{\sim_{\mathrm{len}}} \cap [(s_j)_{j\in J}]_{\sim_{\mathrm{con}}}$ for any pair of sequences $(b_n)_{n\in\mathbf{N}^+}$ and $(s_j)_{j\in J}$ as described in Theorem 6. We have also found a connection between $\sim_{\mathrm{con}}$ and Fibonacci numbers.

It would be interesting to compare our results to those of [2] and examine the relationship between the symmetry partners described there and our equivalence relations. This is a possible topic for future research.

Moreover, (11) invites us to ask the following questions:

- for which $a \in]0, 1[\setminus \mathbf{Q}$ is $(k, i_a(k) - k + 1)_{k\in\mathbf{N}^+}$ a digital half line?
- for which $a \in]0, 1[\setminus \mathbf{Q}$ is $(k, i_a(k) - k + 1)_{k\in\mathbf{N}^+}$ a digital half line with irrational slope?
- for which $a \in]0, 1[\setminus \mathbf{Q}$ are the curves $(k, \lceil ak\rceil)_{k\in\mathbf{N}^+}$ and $(k, i_a(k) - k + 1)_{k\in\mathbf{N}^+}$ equal to each other, i.e., are the same digital half line $D_{R'}(y = ax, x > 0)$?

The last question leads to a fixed-point theorem for Sturmian words (which are irrational, lower or upper, mechanical words), as presented in the author's submitted manuscript [30]. The main result there involves very strongly the partition of the set of slopes by equivalence relation $\sim_{\mathrm{len}}$.

Acknowledgments

I am grateful to Christer Kiselman for comments on earlier versions of the manuscript. I also wish to thank the anonymous referee for useful comments and constructive criticism that helped me improve the quality of this paper.

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Paper V

CONTINUED FRACTIONS, FIBONACCI NUMBERS, AND SOME CLASSES OF IRRATIONAL NUMBERS

HANNA USCKA-WEHLOU

Abstract. In this paper we define an equivalence relation on the set of positive irrational numbers less than 1. The relation is defined by means of continued fractions. Equivalence classes under this relation are determined by the places of some elements equal to 1 (called *essential* 1*'s*) in the continued fraction expansion of numbers. Analysis of suprema of all equivalence classes leads to a solution which involves Fibonacci numbers and constitutes the main result of this paper. The problem has its origin in the author's research on the construction of digital lines and upper and lower mechanical and characteristic words according to the hierarchy of runs.

1. Introduction

Sequences generated by an irrational rotation have been intensively studied by mathematicians, astronomers, crystallographers, and computer scientists; see Venkov (1970) [18, pp. 65–68] and Bruckstein (1991) [4, section *Some consequences and historical remarks*]. These sequences, or related objects, can be found back in the mathematical literature under many different names: rotation sequences, cutting sequences, Beatty sequences, characteristic words, upper and lower mechanical words, balanced words, Sturmian words, Christoffel words, Freeman codes (chain codes) of digital straight lines, and so on; see Pytheas Fogg (2002) [8, p. 143]. There exist some recursive descriptions by continued fractions (CF) of these sequences. The most well known is probably the one formulated by the astronomer J. Bernoulli in 1772, proven by A. Markov in 1882 and described by Venkov (1970) [18, p. 67]. Also well known is the description by Shallit (1991) [11], which can be found in Lothaire (2002) [7, p. 75, 76, 104, 105] as the method by *standard sequences*.

In H.U-W (2008b) [15] the author presented a new CF based description of such sequences. The new description reflects the hierarchy of runs, by analogy to digital straight lines as defined by Azriel Rosenfeld in 1974

2000 *Mathematics Subject Classification.* 11A55, 11B39, 03E20, 68R15.

Key words and phrases. Irrational number, equivalence relation, continued fraction, Fibonacci numbers, digital line, (upper, lower) mechanical word, characteristic word, Beatty sequences, Sturmian word, hierarchy, run.

[10]. This new description appeared to be a good basis for two partitions of upper mechanical words (digital lines) with irrational slopes into equivalence classes according to the length of runs (one of the relations) and the construction of runs (the second one) on all levels in the hierarchy. This has been presented in H.U-W (2009) [16]. Partitions of upper mechanical words with irrational slopes (which are Sturmian words) can give a better understanding of their construction and, as a consequence of that, can be useful in research in combinatorics on words. In H.U-W (2009) [16] the author studied the equivalence classes obtained by both partitions. While examining suprema of the equivalence classes under the relation based on the construction of runs on all the levels in the hierarchy, the author found a solution involving Fibonacci numbers. Now we formulate the essence of this problem, independently from digital geometry and word theory.

The problem we discuss in this paper concerns least and greatest elements in some sets of irrational numbers from the interval $]0,1[$. We define (by means of CFs) an equivalence relation on $]0,1[\setminus \mathbf{Q}$; see Definition 4. This partitions the set of positive irrational numbers less than 1 into equivalence classes. Numbers with the same *sequences of essential places* (Definition 2) in their CF expansions are gathered in the same class. As we will explain in Section 3 (where we present the circumstances in which the presented problem appeared), the upper mechanical words (digital lines) with slopes belonging to the same equivalence class, have the same construction in terms of long and short runs in the hierarchy of runs, because this is fully determined by *essential places* of the slopes (Definition 1), as shown in Proposition 2. The *essential* 1*'s* make that the most frequently appearing run on the level they decide about is long (instead of short, as in case of CF elements different from 1; *non-essential* 1*'s* do not decide about the construction at all, they only determine run length). Sturmian words with slopes belonging to the same equivalence class thus share some construction-related properties, which can give rise to a new tool to the research in combinatorics on words.

The main theorem of the presented paper (Theorem 1) is a description of infima and suprema of all equivalence classes under the relation. The only class which has a greatest element is the one which contains $(\sqrt{5}-1)/2 = [0;\ \overline{1}\ ]$, the Golden Section, and the greatest element is the Golden Section itself. Suprema of all the other equivalence classes are expressed by the odd-numbered convergents of $[0;\ \overline{1}\ ]$. They are thus fractions with numerators and denominators being consecutive Fibonacci numbers.

2. An equivalence relation on the set of positive irrational numbers less than 1

In this paper we assume that the simple continued fraction (CF) expansion of each $a \in \,]0,1[\setminus \mathbf{Q}$ is given, expressed as $a = [0; a_1, a_2, a_3, \dots]$, and we know the positive integers a_k for all $k \in \mathbf{N}^+$. These are called the *elements* of the CF. By *index* of a CF element a_k we mean the positive integer k which describes the place of the element a_k in the CF expansion of a. We recall that

$$[a_0; a_1, a_2, a_3, \dots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} \,. \tag{1}$$

In our case, when $a \in \,]0,1[\setminus \mathbf{Q}$, we have $a_0 = \lfloor a \rfloor = 0$ and the sequence of the CF elements $(a_1, a_2, \dots)$ is infinite. We call $[a_0; a_1, a_2, \dots, a_n]$, for each $n \in \mathbf{N}$, the n^{th} *convergent* of the CF $[a_0; a_1, a_2, \dots]$. If we define

$$p_0 = a_0, \;\; p_1 = a_1 a_0 + 1, \quad \text{and} \quad p_n = a_n p_{n-1} + p_{n-2} \quad \text{for} \;\; n \geq 2 \tag{2}$$

and

$$q_0 = 1, \;\; q_1 = a_1, \quad \text{and} \quad q_n = a_n q_{n-1} + q_{n-2} \quad \text{for} \;\; n \geq 2, \tag{3}$$

then

$$[a_0; a_1, a_2, \dots, a_n] = \frac{p_n}{q_n} \quad \text{for} \;\; n \in \mathbf{N}, \tag{4}$$

see for example Vajda (2008) [17, pp. 158–159]. For more information about CFs see Khinchin (1997) [5].

Some elements equal to 1 in the CF expansion of $a \in \,]0,1[\setminus \mathbf{Q}$ will receive special attention. The reason for this has its roots in the theory of digital lines or, equivalently, upper mechanical words with slope a and intercept 0. This will be explained in Section 3.

Definition 1. Let $a \in \,]0,1[$ be an irrational number and let $[0; a_1, a_2, a_3, \dots]$ be its CF expansion. Let $k \in \mathbf{N}^+$ be such that $a_k = 1$. The integer k is called an *essential place* for a if the following assertions hold:

- $k \geq 2$
- $\exists \; j \in \mathbf{N}, \;\; [0; a_1, a_2, \dots] = [0; a_1, a_2, \dots, a_{k-2j-1}, \underbrace{1, 1, \dots, 1, 1}_{2j}, a_k, \dots]$

  and, if $k - 2j - 1 \geq 2$, then $a_{k-2j-1} \geq 2$.

In other words, a natural number $k \geq 2$ is an essential place of $a = [0; a_1, a_2, \dots]$ iff $a_k = 1$ and a_k is directly preceded by an even number (i.e., by $0, 2, 4, \dots$) of consecutive 1's (i.e., elements $a_m = 1$) with an index m greater than 1. Such elements a_k (where k is an essential place) we called *essential* 1*'s* in H.U-W (2009) [16, Definition 6]. The CF elements $a_k = 1$ which are not in essential places in the CF expansion of a (i.e., if $k = 1$, or if $k \geq 3$ and a_k is directly preceded by an odd number of consecutive 1's with an index greater than 1), are called *non-essential* 1*'s*.

Definition 2. Let $a = [0; a_1, a_2, \dots]$ be irrational. We denote by A the set of all essential places for a, i.e., $A = \{k \in \mathbf{N}^+;\ k \text{ is an essential place for } a\}$, and by $|A|$ the cardinality of A. Let the set J be as follows:

- $A = \emptyset \quad \Rightarrow \quad J = \emptyset$,
- $|A| = \aleph_0 \quad \Rightarrow \quad J = \mathbf{N}^+$,
- $[\ \exists\ M \in \mathbf{N}^+,\ |A| = M\ ] \quad \Rightarrow \quad J = [1, M]_{\mathbf{Z}}$.

We define $(s_j)_{j \in J}$, the *sequence of essential places* of the CF expansion of a as follows:

- $J = \emptyset \quad \Rightarrow \quad (s_j)_{j \in \emptyset} = \emptyset$,
- $J \neq \emptyset \quad \Rightarrow \quad (s_j)_{j \in J}$ is such that $s_1 = \min\{k \in \mathbf{N}^+;\ k \in A\}$ and, if $n \in J \setminus \{1\}$, then $s_n = \min\{k > s_{n-1};\ k \in A\}$.

In words, $J = \emptyset$ if there are no 1's in the CF expansion of a (except maybe for a_1), $J = \mathbf{N}^+$ is there are infinitely many 1's in the CF expansion of a, and $J = [1, M]_{\mathbf{Z}}$ for some $M \in \mathbf{N}^+$ if there are exactly M essential places (essential 1's) in the CF expansion of a. The sequence of essential places for a is indexed by J and we put the smallest essential place first, the next one on the second place, and so on. The sequence of essential places defined above was called the *sequence of the places of essential* 1*'s* in H.U-W (2009) [16, Definition 7].

The following lemma shows how to find essential places in an easy way.

Lemma 1. *Let $a = [0; a_1, a_2, \dots]$ be irrational and let the set J for this slope be as described in Definition 2. Then the sequence $(s_j)_{j \in J}$ of essential places of the CF expansion of a is $(s_j)_{j \in \emptyset} = \emptyset$ if $J = \emptyset$ and, if $J \neq \emptyset$, then $(s_j)_{j \in J}$ is as follows: $s_1 = \min\{k \geq 2;\ a_k = 1\}$ and*

$$n \in J \setminus \{1\} \quad \Rightarrow \quad s_n = \min\{k \geq s_{n-1} + 2;\ a_k = 1\}. \tag{5}$$

Proof. Let us consider any irrational $a = [0; a_1, a_2, \dots]$ and the corresponding $(s_j)_{j \in J}$ as in Definition 2. If $J = \emptyset$, then $(s_j)_{j \in \emptyset} = \emptyset$. If $J \neq \emptyset$, then we can prove the statement by induction (if $|J| = M$ for some $M \in \mathbf{N}^+$, the proof has only a finite number of steps). It follows from Definition 1 that $s_1 \geq 2$. Let us take any $m \in J \setminus \{1\}$ such that s_{m-1} is an essential place. We will show that the next essential place is $s_m = \min\{k \geq s_{m-1} + 2;\ a_k = 1\}$. First we show that $s_m - s_{m-1} \geq 2$. Suppose not, i.e., $s_m = s_{m-1} + 1$. Then both $a_{s_{m-1}} = 1$ and $a_{s_{m-1}+1} = 1$, which are consecutive CF elements of a, are essential 1's. This is not possible, however, because, as consecutive CF elements equal to 1, they cannot both be directly preceded by an even number of CF elements equal to 1 and with an index greater than 1. We get a contradiction, so it must be $s_m - s_{m-1} \geq 2$.

If the difference between s_m (as defined by (5)) and s_{m-1} is greater than 2, then $a_{s_m} = 1$ is the next essential 1 following after $a_{s_{m-1}}$, because, according to (5), there are no other 1's between $a_{s_{m-1}}$ and a_{s_m} (maybe $a_{s_{m-1}+1} = 1$, but then it is a non-essential 1, as it is directly preceded by an odd number of $a_j = 1$ with $j > 1$, and $s_{m-1}+1 < s_m - 1$), so $a_{s_m} = 1$ is directly preceded by zero (i.e., an even number) 1's.

If $s_m - s_{m-1} = 2$ (where s_m is defined by (5)), then $a_{s_m} = 1$ is the next essential 1 following after $a_{s_{m-1}}$, because it is directly preceded by an even number of CF elements equal to 1 and with index greater than 1. This number is equal to zero if $a_{s_{m-1}+1} \geq 2$ and to the even number of such 1's corresponding to s_{m-1}, increased by 2, in case $a_{s_{m-1}+1} = 1$. $\square$

Example 1. Let $a = [0; a_1, a_2, a_3, a_4, \dots]$, with $\forall\ k < 16,\ \ a_k = 1$ if and only if $k \in \{1, 3, 4, 6, 7, 10, 13, 14, 15\}$. We find the sequences of essential places in the CF expansion of such a in the following way:

$$\begin{array}{rl} a = & [0;\ 1,\ a_2,\ \underline{1},\ 1,\ a_5,\ \underline{1},\ 1,\ a_8,\ a_9,\ \underline{1},\ a_{11},\ a_{12},\ \underline{1},\ 1,\ \underline{1},\ \dots\ ] \\ & \qquad\qquad\ \downarrow \qquad\quad\ \ \downarrow \qquad\qquad\qquad \downarrow \qquad\qquad\quad\ \downarrow \quad\ \ \downarrow\ \ \dots \\ (s_j)_{j\in J} = & (\qquad\quad 3, \qquad\quad 6, \qquad\qquad\ \ 10, \qquad\qquad 13 \quad 15\ \dots). \end{array}$$

All essential 1's with index less than 16 are underlined. We have $(s_j)_{j\in J} = (3, 6, 10, 13, 15, \dots)$. The first four non-essential 1's are a_1, a_4, a_7, a_{14}.

The following proposition describes all the possible sequences of essential places for CF-expansions of positive irrational a less than 1. First, we will introduce the following definition.

Definition 3. A sequence $(t_j)_{j\in J}$ of positive integer numbers will be called an *essential sequence* iff:

- the set J is as follows: $J = \emptyset$, $J = \mathbf{N}^+$ or $J = [1, M]_{\mathbf{Z}}$ for some $M \in \mathbf{N}^+$,
- the sequence $(t_j)_{j\in J}$ (if not empty) is a sequence of positive integers such that $t_1 \geq 2$ and, for $k \in J \setminus \{1\}$, $t_k - t_{k-1} \geq 2$.

Proposition 1. *A sequence of positive integer numbers is an essential sequence iff it is the sequence of essential places for some irrational* $a = [0; a_1, a_2, \dots]$.

Proof. Let $(t_j)_{j\in J}$ be an essential sequence (Definition 3). We define $a = [0; a_1, a_2, \dots]$ in the following way: if $J = \emptyset$, then we take any $a_1 \in \mathbf{N}^+$ and, for each $n \geq 2$, we choose any $a_n \geq 2$. If J is not empty, we define $a_{t_i} = 1$ for all $i \in J$ and a_k for $k \in \mathbf{N}^+ \setminus \{t_j\}_{j\in J}$ can be any interger greater than or equal to 2. It follows trivially from Definitions 1 and 2 that $(t_j)_{j\in J}$ is the sequence of essential places for such a. The second implication in the statement follows from Lemma 1. $\square$

All sequences of essential places have elements greater than or equal to 2, are increasing and the difference between each two consecutive elements is greater than or equal to 2. Each sequence, finite or infinite, with those properties (i.e., an essential sequence), is the sequence of essential places for some $a \in \,]0,1[\, \setminus \mathbf{Q}$.

We can identify with each other all irrational numbers from the interval $]0,1[$ which have the same sequences of essential places.

Definition 4. We define the following relation $\sim_{\text{ess}} \subset (]0,1[\, \setminus \mathbf{Q})^2$. If a and a' are positive irrational numbers less than 1, then

$$a \sim_{\text{ess}} a' \quad \Leftrightarrow \quad \left(s_j^{(a)}\right)_{j\in J} = \left(s_k^{(a')}\right)_{k\in J'},$$

where $\left(s_j^{(a)}\right)_{j\in J}$ and $\left(s_k^{(a')}\right)_{k\in J'}$ are the corresponding sequences of essential places in the CF expansion of a and a' respectively.

The relation $\sim_{\text{ess}}$ partitions the set $]0,1[\, \setminus \mathbf{Q}$ into equivalence classes defined by essential sequences (Definition 3, Proposition 1). Let us consider the following examples.

Example 2. Let $(t_j)_{j\in\emptyset} = \emptyset$. The class under $\sim_{\text{ess}}$ generated by this sequence is the set of all $a \in \,]0,1[\, \setminus \mathbf{Q}$ such that $a = [0; a_1, a_2, \dots]$, where $a_1 \in \mathbf{N}^+$ and $a_n \geq 2$ for all $n \geq 2$.

Example 3. Let $(t_j)_{j\in\mathbf{N}^+} = (2j)_{j\in\mathbf{N}^+}$. The class under $\sim_{\text{ess}}$ generated by this sequence is the set of all positive irrational numbers with the CF expansion $a = [0; a_1, 1, a_3, 1, a_5, 1, \dots]$, where $a_{2n+1} \in \mathbf{N}^+$ for all $n \in \mathbf{N}$. The Golden Section $(\sqrt{5}-1)/2$ belongs to this class.

The problem we want to solve in this paper is the question about supremum and infimum of each class under $\sim_{\text{ess}}$. The following lemma, which shows how to compare two CFs with each other, will help us to find the solution.

Lemma 2. *Let $a_0, b_0 \in \mathbf{Z}$ and $a_i, b_i \in \mathbf{N}^+$ for all $i \in \mathbf{N}^+$. Then*

$$[a_0; a_1, a_2, \dots] < [b_0; b_1, b_2, \dots] \quad \Leftrightarrow$$

$$(a_0, -a_1, a_2, -a_3, a_4, -a_5, \dots) \overset{lexic.}{<} (b_0, -b_1, b_2, -b_3, b_4, -b_5, \dots),$$

where the first inequality is according to the order $<$ on the real numbers, and the second inequality is according to the lexicographical order on sequences.

Theorem 1 (Main Theorem)**.** *There exists no equivalence class under $\sim_{\text{ess}}$ with a least element according to the order $\leq$ on the real numbers. The infimum is equal to 0 for all the classes.*

There exists exactly one class under $\sim_{\text{ess}}$ which has a greatest element according to the order $\leq$ on the real numbers. This class is defined by the sequence $(t_j)_{j\in\mathbf{N}^+} = (2j)_{j\in\mathbf{N}^+}$ and the maximum is the Golden Section $(\sqrt{5}-1)/2$. Moreover, the following statement describes suprema of all the classes under $\sim_{\text{ess}}$ different from $[(2j)_{j\in\mathbf{N}^+}]_{\sim_{\text{ess}}}$. For all $n \in \mathbf{N}^+$

$$[(\forall\ \ k \in [1, n-1]_{\mathbf{Z}},\ \ t_k = 2k)\ \ \wedge\ \ (t_n > 2n\ \ \vee\ \ |J| = n-1)]$$

$$\Rightarrow\ \ \sup\{a \in\ ]0,1[\ \setminus \mathbf{Q};\ \ a \in [(t_j)_{j\in J}]_{\sim_{\text{ess}}}\} = \frac{F_{2n-1}}{F_{2n}},$$

where $(F_n)_{n\in\mathbf{N}^+}$ is the Fibonacci sequence, i.e.,

$$F_1 = 1,\ F_2 = 1\ \text{ and, for }\ k \geq 3,\ \ F_k = F_{k-1} + F_{k-2}, \tag{6}$$

$|J|$ denotes the cardinality of J, and $(t_j)_{j\in J}$ is any essential sequence different from $(2j)_{j\in\mathbf{N}^+}$.

Proof. To prove the statement about infimum we observe that in each equivalence class under $\sim_{\text{ess}}$ there exist numbers $a = [0; a_1, a_2, \dots]$ with $a_1 = 1$, numbers with $a_1 = 2$, etc. When a_1 tends to infinity, then $a = [0; a_1, a_2, \dots]$ tends to zero, so zero is infimum and we have no least element in the class (which is a subset of $]0,1[\ \setminus \mathbf{Q}$).

To prove the statement about supremum, we take $J = \emptyset$, $J = \mathbf{N}^+$ or $J = [1, M]_{\mathbf{Z}}$ for some $M \in \mathbf{N}^+$ and we consider all the classes generated by all possible essential sequences, i.e., by sequences $(t_j)_{j\in J}$ of integers such that $t_1 \geq 2$ and $t_i - t_{i-1} \geq 2$ for all $i \in J \setminus \{1\}$.

The flowchart on p. 8 analyzes all such possible classes with respect to greatest elements and suprema. We use Lemma 2 in each step of the flowchart. To make a CF as large as possible, the even-numbered CF elements must be as large as possible (it is represented by the left-hand side of the flowchart) and the odd-numbered CF elements must be as small as possible, thus equal to 1 (see the right-hand side of the flowchart).

If $J = \emptyset$, then there is clearly no greatest element in the equivalence class $[\emptyset]_{\sim_{\text{ess}}}$, because all the numbers $[0; 1, a_2, \dots]$ with $a_n \geq 2$ for $n \geq 2$ (which are the only candidates for the position of maximum) belong to it and, when a_2 tends to infinity, then $[0; 1, a_2, \dots]$ tends to 1, which does not belong to $]0,1[\setminus \mathbf{Q}$, so there is no greatest element. The supremum is equal to 1. The same reasoning holds for $J \neq \emptyset$ in case when $t_1 > 2$ (thus $a_2 \geq 2$).

If $J \neq \emptyset$ and $t_1 = 2$ (thus $a_2 = 1$), we consider the only (according to Lemma 2) candidate for a greatest element and it is $[0; 1, 1, 1, a_4, \dots]$. If $t_2 > 4$ (thus $a_4 \geq 2$), we repeat the same reasoning again: when a_4 tends to infinity, then $[0; 1, 1, 1, a_4, \dots]$ tends to $\frac{2}{3}$, so it is the supremum. There is no greatest element, because the supremum is rational. We go on like this, using Lemma 2 about comparison of CFs.

$\boxed{\text{the candidates for max for } \textbf{any } J: \ [0;1,a_2,\dots]}$

$t_1 > 2$ or $J = \emptyset$ (i.e., $a_2 \geq 2$)

$\boxed{[0;1,a_2^{(n)},\dots] \overset{a_2^{(n)}\to\infty}{\longrightarrow} \mathbf{1}}$

no max

$t_1 = 2$ (i.e., $a_2 = 1$)

$\boxed{\text{the candidates for max: } [0;1,1,1,a_4,\dots]}$

$t_2 > 4$ or $|J| = 1$ (i.e., $a_4 \geq 2$)

$\boxed{[0;1,1,1,a_4^{(n)},\dots] \overset{a_4^{(n)}\to\infty}{\longrightarrow} \mathbf{\frac{2}{3}}}$

no max

$t_2 = 4$ (i.e., $a_4 = 1$)

$\boxed{\text{the candidates for max: } [0;1,1,1,1,1,a_6,\dots]}$

$t_3 > 6$ or $|J| = 2$ (i.e., $a_6 \geq 2$)

$\boxed{[0;1,1,1,1,1,a_6^{(n)},\dots] \overset{a_6^{(n)}\to\infty}{\longrightarrow} \mathbf{\frac{5}{8}}}$

no max

$t_3 = 6$ (i.e., $a_6 = 1$)

$\boxed{\text{the candidates for max: } [0;1,1,1,1,1,1,1,a_8,\dots]}$

$t_4 > 8$ or $|J| = 3$ (i.e., $a_8 \geq 2$)

$\boxed{[0;1,1,1,1,1,1,1,a_8^{(n)},\dots] \overset{a_8^{(n)}\to\infty}{\longrightarrow} \mathbf{\frac{13}{21}}}$

no max

$t_4 = 8$ (i.e., $a_8 = 1$)

$\boxed{\text{the candidates for max: } [0;1,1,1,1,1,1,1,1,1,a_{10},\dots]}$

$t_5 > 10$ or $|J| = 4$ (i.e., $a_{10} \geq 2$)

$\boxed{[0;1,1,1,1,1,1,1,1,1,a_{10}^{(n)},\dots] \overset{a_{10}^{(n)}\to\infty}{\longrightarrow} \mathbf{\frac{34}{55}}}$

no max

$t_5 = 10$ (i.e., $a_{10} = 1$)

$\boxed{\text{the candidates for max: } [0;1,1,1,1,1,1,1,1,1,1,1,a_{12},\dots]}$

and so on...

The rightmost way of the flowchart from p. 8 leads to $[0;\overline{1}\ ]$, which is irrational, equal to $(\sqrt{5}-1)/2$. This means that the only class which has a greatest element is the class as described in Example 3, generated by the essential sequence $(t_j)_{j\in\mathbf{N}^+} = (2j)_{j\in\mathbf{N}^+}$.

The statement about suprema of all the classes can be derived from the flowchart. It follows from (2), (3), (4) and (6), that the odd-numbered convergents of the Golden Section $[0;\overline{1}\ ]$ are $\frac{p_{2n-1}}{q_{2n-1}} = \frac{F_{2n-1}}{F_{2n}}$ for $n \in \mathbf{N}^+$; see for example Vajda (2008) [17, pp. 101–105] or Benjamin and Quinn (2003) [3, p. 52]. The odd-numbered convergents are thus $\frac{F_1}{F_2} = 1$, $\frac{F_3}{F_4} = \frac{2}{3}$, $\frac{F_5}{F_6} = \frac{5}{8}$, $\frac{F_7}{F_8} = \frac{13}{21}$, $\frac{F_9}{F_{10}} = \frac{34}{55}, \ldots$. Moreover, when analyzing the flowchart one can ensure oneself that it covers all the possible classes under $\sim_{\text{ess}}$. $\square$

In Theorem 1 we answered questions about least and greatest elements in classes generated by the relation $\sim_{\text{ess}}$. No equivalence class under $\sim_{\text{ess}}$ has a least element. The infimum in each class is equal to zero. The answer related to largest elements is much more interesting. The partition of all the irrational numbers from the interval $]0,1[$ into equivalence classes under $\sim_{\text{ess}}$ gives the sets with suprema equal to the odd-numbered convergents of the Golden Section, thus with no largest element belonging to the class (which is a set of irrational numbers). The only exception is the class generated by $(t_j)_{j\in\mathbf{N}^+} = (2j)_{j\in\mathbf{N}^+}$, which has a greatest element and it is equal to the Golden Section.

3. The origin of the problem. Digital lines and Sturmian words

In this section we will give some information about the circumstances in which the presented problem arose. In H.U-W (2009) [16] we analyzed two equivalence relations defined on the set of all slopes $a \in\ ]0,1[\ \setminus \mathbf{Q}$ of digital straight lines $y = ax$ (or, equivalently, of upper mechanical words $u(a)\colon \mathbf{N} \to \{0,1\}$, $u_n(a) = \lceil a(n+1)\rceil - \lceil an \rceil$ for each $n \in \mathbf{N}$). One of those relations is the one discussed in the presented paper. This relation identifies with each other all slopes which have the same sequences of essential places (Definitions 1 and 2) in their CF expansions. We know from H.U-W (2008a) [14] that essential 1's determine the construction of digital lines. How exactly, will be shown in Proposition 2.

General information about digital straightness can be found in the review by R. Klette and A. Rosenfeld from 2004 [6]. Very good sources of information are also Reveillès (1991) [9] and Stephenson (1998) [12]. The digitization $D_{R'}$ of $y = ax$ for some $a \in\ ]0,1[\setminus\mathbf{Q}$ as defined in H.U-W (2007) [13] is the following:

$$D_{R'}(y = ax) = \{(k, \lceil ak \rceil);\ \ k \in \mathbf{Z}\}. \tag{7}$$

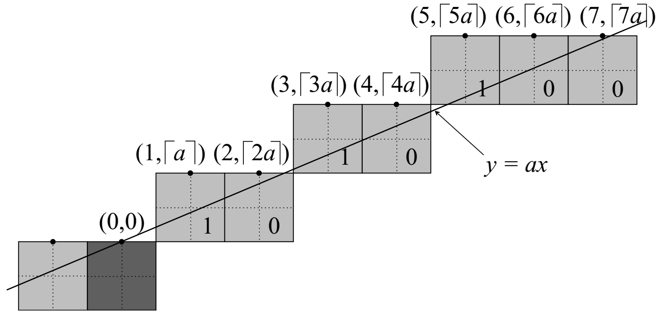

FIGURE 1. Digitization of $y = ax$ for some $a \in \,]0,1[\, \setminus \mathbf{Q}$; $u(a) = 1010100\cdots$.

We illustrate it with an example in Figure 1.

The 0's and 1's on the squares in the picture show the relationship between digital lines and upper and lower mechanical and characteristic words. Let us recall the definitions of those words; see Lothaire (2002) [7, p. 53].

Definition 5. For each $a \in \,]0,1[\, \setminus \mathbf{Q}$ we define two binary words in the following way: $l(a)\colon \mathbf{N} \to \{0,1\}$, $\quad u(a)\colon \mathbf{N} \to \{0,1\}$ are such that for each $n \in \mathbf{N}$

$$l_n(a) = \lfloor a(n+1) \rfloor - \lfloor an \rfloor, \quad u_n(a) = \lceil a(n+1) \rceil - \lceil an \rceil.$$

The word $l(a)$ is the *lower mechanical word* and $u(a)$ is the *upper mechanical word* with *slope* a and *intercept* 0.

We have $l_0(a) = \lfloor a \rfloor = 0$ and $u_0(a) = \lceil a \rceil = 1$ and, because $\lceil x \rceil - \lfloor x \rfloor = 1$ for irrational x, we have

$$l(a) = 0c(a), \quad u(a) = 1c(a) \tag{8}$$

(meaning 0, resp. 1 concatenated to $c(a)$). The word $c(a)$ is called the *characteristic word* of a. For each $a \in \,]0,1[\, \setminus \mathbf{Q}$, the characteristic word associated with a is thus the following infinite word $c(a)\colon \mathbf{N}^+ \to \{0,1\}$:

$$c_n(a) = \lfloor a(n+1) \rfloor - \lfloor an \rfloor = \lceil a(n+1) \rceil - \lceil an \rceil, \quad n \in \mathbf{N}^+. \tag{9}$$

Formulae (7), (8) and (9), together with the 0's and 1's in Figure 1, illustrate and explain the relationship between digital lines and lower and upper mechanical and characteristic words. The developed theory is thus also valid for upper and lower mechanical words and characteristic words; see for example Lothaire (2002) [7, p. 53, 2.1.2 *Mechanical words, rotations*], H.U-W (2008b) [15]. According to Theorem 2.1.13 in Lothaire (2002) [7, p. 57], irrational (lower or upper) mechanical words are Sturmian words.

Our description of digital lines in the author's papers [13, 14, 16] reflected the hierarchy of runs on all digitization levels. The concept of *runs* was already introduced and explored by Azriel Rosenfeld (1974) [10, p. 1265]. We call $\mathrm{run}_k(j)$ for $k, j \in \mathbf{N}^+$ a *run of digitization level* k. Each $\mathrm{run}_1(j)$ can be identified with a subset of $\mathbf{Z}^2$: $\{(i_0+1, j), (i_0+2, j), \ldots, (i_0+m, j)\}$, where m is the length $\|\mathrm{run}_1(j)\|$ of the run. For upper mechanical words, the corresponding run is 10^{m-1}, where $m-1$ is the number of all the letters 0 between the letter 1 in the beginning of the run and the next occurring letter 1 in the word. For each $a \in]0,1[\setminus \mathbf{Q}$ we have only two possible run_1 lengths: $\left\lfloor \frac{1}{a} \right\rfloor$ and $\left\lfloor \frac{1}{a} \right\rfloor + 1$. All runs with one of those lengths always occur alone, i.e., do not have any neighbors of the same length in the sequence $(\mathrm{run}_1(j))_{j \in \mathbf{N}^+}$, while the runs of the other length can appear in sequences. The same holds for the sequences $(\mathrm{run}_k(j))_{j \in \mathbf{N}^+}$ on each level $k \geq 2$. We use the notation $S_k^m L_k$, $L_k S_k^m$, $L_k^m S_k$ and $S_k L_k^m$, when describing the form of digitization runs_{k+1}. For example, $S_k^m L_k$ means that the run_{k+1} consists of m short runs_k (S_k) and one long run_k (L_k) in this order. In Figure 2 we can see an example of the run hierarchical structure for the line $y = ax$ with $a = [0; 1, 2, 1, 1, 3, 1, 1, a_8, a_9, \ldots]$, where $a_8, a_9, \cdots \in \mathbf{N}^+$.

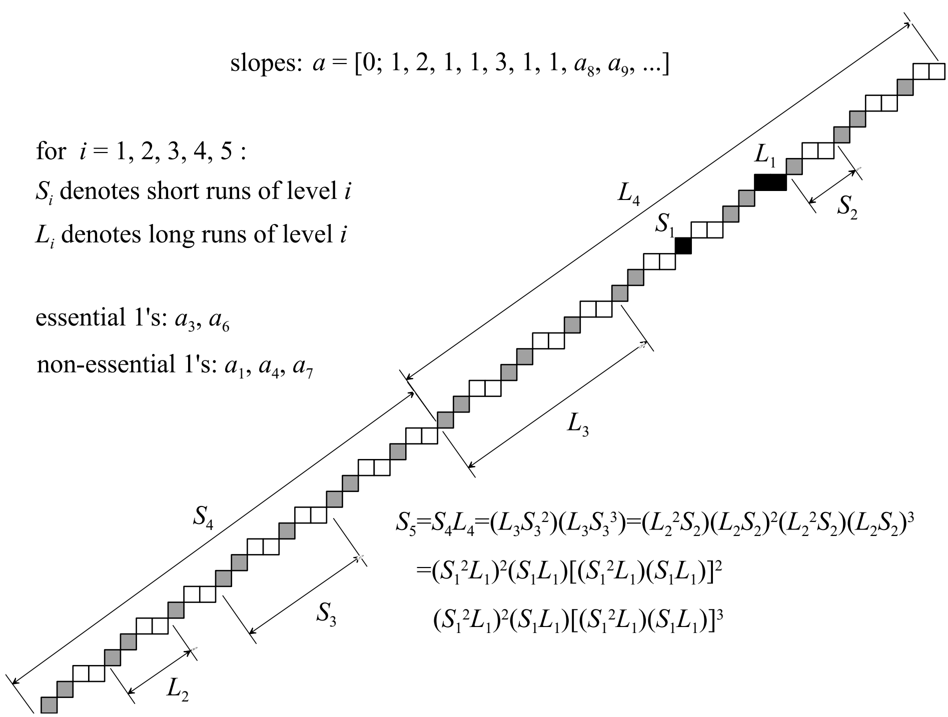

FIGURE 2. Hierarchy of long and short runs on the first four digitization levels; to translate this hierarchy for the case of upper mechanical words, put $S_1 = 1$ and $L_1 = 10$.

The basis for the author's CF description, from H.U-W (2008a) [14], of digital lines $y = ax$ for $a \in]0, 1[\setminus \mathbf{Q}$ according to the definition (7) constitutes the following *index jump function.*

Definition 6. Let $a = [0; a_1, a_2, a_3, \dots]$ be a positive irrational number less than 1. We define the *index jump function* $i_a\colon \mathbf{N}^+ \to \mathbf{N}^+$ for a as follows: $i_a(1) = 1$, $i_a(2) = 2$, and, for $k \geq 2$, $i_a(k+1) = i_a(k) + 1 + \delta_1(a_{i_a(k)})$, where $\delta_1(x) = \begin{cases} 1, & x = 1 \\ 0, & x \neq 1 \end{cases}$ and a_n for $n \in \mathbf{N}^+$ are the CF elements of a.

In H.U-W (2009) [16, Definition 6], essential 1's for $[0; a_1, a_2, \dots]$ were defined as such $a_k = 1$ that $k = i_a(m)$ for some $m \geq 2$ (compare Definition 6 with Lemma 1). The index jump function registers the essential places from the CF expansion of a. We have $(i_a(k))_{k \in \mathbf{N}^+} = \mathbf{N}^+ \setminus (s_j + 1)_{j \in J}$ for all $a \in]0, 1[\setminus \mathbf{Q}$; for more details see H.U-W (2009) [16].

The following proposition, which is an immediate consequence of Theorem 4 from H.U-W (2008a) [14], **explains the role of essential 1's in the construction of digital lines, and, equivalently, in the run hierarchical structure of upper mechanical words.**

Proposition 2. *If a is irrational and $a = [0; a_1, a_2, \dots]$, then for the digitization of $y = ax$ (the run hierarchical structure of $u(a)$) we have the following. The CF elements $a_2, a_3, \dots$ determine the run hierarchical construction of $y = ax$ (of $u(a)$) in the following way. For each $k \in \mathbf{N}^+$*

- $a_{i_a(k+1)} \geq 2 \quad \Rightarrow \quad$ *S_k is the most frequent run on level k,*
- $a_{i_a(k+1)} = 1 \quad \Rightarrow \quad$ *L_k is the most frequent run on level k,*

where i_a is the corresponding index jump function as defined in Definition 6.

The only 1's in the CF expansion of a which influence the run-hierarchical construction of digital line $y = ax$ (upper mechanical word $u(a)$) are thus those which are indexed by the values of the index jump function, equivalently, those which are directly preceded by an even number of consecutive 1's with an index greater than 1. Briefly, only essential 1's cause the change of the most frequent run on the level they correspond to from short (S_k) to long (L_k). Which level they correspond to, is determined by the index jump function generated by a, as shown in Proposition 2. We will illustrate this proposition with the following example.

Example 4. We consider the lines as in Figure 2, thus lines $y = ax$ with slopes $a = [0; 1, 2, 1, 1, 3, 1, 1, a_8, a_9, \dots]$, where $a_8, a_9, \dots \in \mathbf{N}^+$. Each CF element of a is responsible for some digitization level. According to Proposition 2, if $a_{i_a(k+1)} \geq 2$, then the most frequent run on level k is the short one, S_k. Otherwise, i.e., if $a_{i_a(k+1)} = 1$, the dominating run_k is L_k. For the lines as in Figure 2, we have thus the following, which can be compared with the picture:

level k	$a_{i_a(k+1)}$	the most frequent run$_k$
1	$a_{i_a(2)} = a_2 = 2 \geq 2$	S_1
2	$a_{i_a(3)} = a_3 = 1$	L_2
3	$a_{i_a(4)} = a_5 = 3 \geq 2$	S_3
4	$a_{i_a(5)} = a_6 = 1$	L_4

Indeed, we have $L_2 = S_1^2 L_1,\ L_3 = L_2^2 S_2,\ L_4 = L_3 S_3^3,\ L_5 = S_4 L_4^2$.

4. Conclusion

We have presented a partition of the set $]0,1[\setminus \mathbf{Q}$ into equivalence classes under a CF-defined equivalence relation. The relation groups together all positive irrational numbers less than 1 which have the same sequences of essential places in their CF expansions. All digital lines (upper mechanical words) with slopes belonging to the same equivalence class have the same construction in terms of long and short runs on all the levels in the hierarchy of runs. We have proven that the only class which has a greatest element is the class represented by the Golden Section. All the other classes have suprema defined by Fibonacci numbers.

The problem comes originally from digital geometry and word theory, but it can be formulated independently from these domains, as a problem concerning irrational numbers.

Because of the strong relationship between our description of digitization and the Gauss map (see the concept of digitization parameters from [13]), it would be interesting to compare our results to those of Bates et al. (2005) [2] and examine the relationship between the symmetry partners described there and our equivalence relation.

Another possible continuation of the research on our equivalence relation could be analysis of properties of the CFs with sequences of essential places determined by well-known sequences such like the Fibonacci numbers or the Pell numbers. One could try, for example, formulate the rules for transcendentality of CFs depending on the sequences of essential places. Examples of analysis of transcendentality of CFs can be found in Adamczewski et al. (2006) [1].

Acknowledgments. I am grateful to Christer Kiselman for comments on earlier versions of the manuscript. I also wish to thank the anonymous referee for useful comments and constructive criticism which led to improvements in this paper.

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Paper VI

Sturmian words with balanced construction

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Abstract. In this paper we define Sturmian words with balanced construction. We formulate a fixed-point theorem for Sturmian words and analyze the set of all fixed points. The inspiration for this work came from the Kolakoski word and the general idea of self-reading sequences by Păun and Salomaa. The basis for this article is the author's earlier research on the influence of the continued fraction elements in the expansion of $a \in]0,1[\setminus \mathbf{Q}$ on the construction of runs for the upper mechanical word with slope a and intercept 0.

Keywords: upper mechanical word; irrational slope; Sturmian word; continued fraction; hierarchy of runs; fixed point; self-reading sequence; Kolakoski word; Freeman chain code.

1 Introduction

Word theory has grown very intensively during the last century. The theory has found numerous applications in computer science, which has stimulated its fast development. Both mathematicians and theoretical computer scientists have been working on problems connected with word theory and related domains. A very good illustration of the results of this work and of the variety of domains and subjects word theory is connected to, is presented in Pytheas Fogg (2002) [16], Lothaire (2002) [13], Allouche and Shallit (2003) [1], Perrin and Pin (2004) [15], Karhumäki (2004) [10], Berthé, Ferenczi and Zamboni (2005) [3], and Berstel et al. (2008) [2].

This paper about binary words is inspired mainly by ideas of three persons: William G. Kolakoski, Herbert Freeman and Azriel Rosenfeld.

Self-reading sequences have been examined by a lot of researchers. Some general definitions of those can be found in [9, 14]. William G. Kolakoski has described probably the most famous self-reading sequence, very well known to the community of theoretical computer scientists; see [12] and [16, p. 93]. The Kolakoski word is defined as one of the two fixed points of the run-length encoding Δ; see [5, 6]. These words are identical with their own run-length encoding sequences. The one beginning with 2 is: $K = 2211212212211211221211212211211212212211212212\cdots$. Brlek et al. have studied some generalizations of the Kolakoski word to an arbitrary alphabet, which got the name of *smooth words*; see [6] and references there.

A simple example of a self-reading sequence is the Morse sequence u which begins with a and is defined as the fixed point of the *Morse substitution* σ defined over the alphabet $\{a, b\}$ by $\sigma(a) = ab$, $\sigma(b) = ba$, thus $u = abbabaabbaababbabaababbaabbabaabbaababbaabbabaababbabaabbaabab\ldots$; see also [16, p. 7]. Another simple self-reading sequence is the Fibonacci word defined as the fixed point w beginning with 1 of the substitution $\varphi(1) = 10,\ \varphi(0) = 1$; see also [16, p. 7]. We show on Figure 1 how to construct w. The arrows pointing downwards show how we use the definition of the substitution φ, the arrows pointing upwards show how to use the fixed-point condition $w = \varphi(w)$. Because $\varphi(w)$ is being formed faster than w, we get in each step enough information to be able to construct w.

Generally, the *characteristic words* of irrational numbers with purely periodic continued fraction (CF) expansion (i.e., some quadratic surds) are also fixed points of corresponding substitutions, as has been shown in the paper by Shallit (1991) [18]. These fixed points, and, in particular, the Fibonacci word, are Sturmian. They are also examples of self-reading sequences.

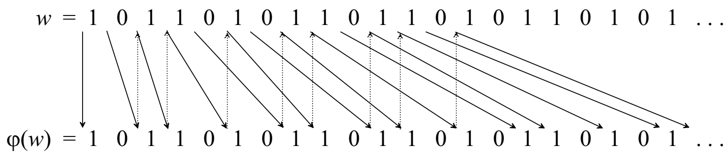

Fig. 1. The Fibonacci sequence as the fixed point w beginning with 1 of the substitution $\varphi(1) = 10,\ \varphi(0) = 1$.

The *cutting sequence* of grid lines by the half-line $y = ax$ for $a \in \,]0, 1[\, \setminus \mathbf{Q}$ and $x > 0$ (i.e., the line passes through no lattice points) is one of binary representations of $y = ax$, where 0 denotes a vertical grid crossing and 1 a horizontal one. Such a sequence for straight lines with irrational slopes is Sturmian [16, p. 143]. There is a close relationship between the cutting sequence of $y = ax$ for $a \in \,]0, 1[\, \setminus \mathbf{Q}$ and the line's *chain code* (which is the same, or the same up to the transformation "replace 10 by 1", as the characteristic word with slope a, depending on whether the line is *naive* or *standard*). Herbert Freeman (1970) [8, p. 260] observed that in the chain code of a digital straight line "successive occurrences of the element occurring singly are as uniformly spaced as possible". This property has been formalized and has got the name of *balance property*; see [25]. The self-similarity properties formulated by Bruckstein (1991) [7] form a quantitative expression of this uniformity principle.

Azriel Rosenfeld described in his paper from 1974 [17] the run-hierarchical structure of digital lines. On each level k (for $k \geq 2$) we have runs$_k$ which are composed of a single occurring run$_{k-1}$ (long L_{k-1} or short S_{k-1}) and a maximal sequence of runs$_{k-1}$ (short S_{k-1} or long L_{k-1}, respectively) following after this single one or preceding it. On some levels the long runs are the most frequent (coming in sequences), while on other levels the short runs are the mainly occurring ones.

In Uscka-Wehlou (2008) [22] we presented a CF-based description of upper mechanical words, which reflects the run-hierarchical structure of words. The present idea is to create a *run-construction encoding* operator, by analogy to the run-length encoding operator. The latter is very well known and was used for coding the Thue–Morse word by Brlek in 1988 [5] and the former is a new concept, defined for the first time in the present paper (Definition 6). We will look for the fixed points of the run-construction encoding operator. For them even the constructional distribution is uniform, in the way as described by Freeman. In the main theorem of this paper (Theorem 4) we show that every infinite sequence of positive natural numbers such that all the elements indexed by numbers greater than 1 are greater than 1 generates exactly one fixed point of the run-construction encoding operator. All of them are self-generating sequences, identical with their own run-construction encoding sequences, by analogy with the Kolakoski word. In the second half of this paper we present a number of examples. We also examine the set of all fixed points (Theorem 5) and formulate a number of questions and combinatorial problems for further research (on p. 8 after Proposition 3, and in Section 6).

2 A continued-fraction-based description of upper mechanical words

In [22] we presented a recursive description by CFs of upper mechanical words. Let us recall the definiton of those; cf. Lothaire (2002) [13, p. 53].

Definition 1. *Given two real numbers a and r with $0 \leq a \leq 1$, we define two infinite words $s(a, r), s'(a, r)\colon \mathbf{N} \to \{0, 1\}$ by $s_n(a, r) = \lfloor a(n+1)+r \rfloor - \lfloor an+r \rfloor$ and $s'_n(a, r) = \lceil a(n+1)+r \rceil - \lceil an+r \rceil$. The word $s(a, r)$ is the* lower mechanical word *and $s'(a, r)$ is the* upper mechanical word *with* slope a *and* intercept r. *A lower or upper mechanical word is* irrational *or* rational *according as its slope is irrational or rational.*

In the present paper we deal with the special case when $a \in]0,1[$ is irrational and $r = 0$. In this case we will denote the lower and upper mechanical words by $s(a)$ and $s'(a)$ respectively. We have $s_0(a) = \lfloor a \rfloor = 0$ and $s'_0(a) = \lceil a \rceil = 1$ and, because $\lceil x \rceil - \lfloor x \rfloor = 1$ for irrational x and $\lceil x \rceil - \lfloor x \rfloor = 0$ only for integers, we have

$$s(a) = 0c(a), \quad s'(a) = 1c(a) \tag{1}$$

(meaning 0, resp. 1 concatenated to $c(a)$). The word $c(a)$ is called the *characteristic word* of a. For each $a \in]0,1[\setminus \mathbf{Q}$, the characteristic word associated with a is thus the following infinite word $c(a)\colon \mathbf{N}^+ \to \{0,1\}$:

$$c_n(a) = \lfloor a(n+1) \rfloor - \lfloor an \rfloor = \lceil a(n+1) \rceil - \lceil an \rceil, \quad n \in \mathbf{N}^+. \tag{2}$$

It is well known that the equality of characteristic words gives the equality of corresponding slopes, i.e., for any $a, a' \in]0,1[\setminus \mathbf{Q}$, if $c(a) = c(a')$, then $a = a'$; cf. Lothaire (2002) [13, p. 62, Lemma 2.1.21].

We assume that, for each $a \in]0,1[\setminus \mathbf{Q}$, its simple CF expansion is given, expressed as $a = [0; a_1, a_2, a_3, \ldots]$, and we know the positive integers a_i for all $i \in \mathbf{N}^+$. These are called the *elements* (or *partial quotients*) of the CF. Let us recall that

$$[a_0; a_1, a_2, a_3, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}\,.$$

In our case, when $a \in]0,1[\setminus \mathbf{Q}$, we have $a_0 = \lfloor a \rfloor = 0$ and the sequence of the CF elements $(a_1, a_2, \ldots)$ is infinite. For more information about CFs see Khinchin (1997) [11].

Our CF description of upper mechanical words from [22] was based on our earlier one by digitization parameters from [19] and the following *index jump function*, introduced by the author in [20].

Definition 2. *For each $a \in]0,1[\setminus \mathbf{Q}$, the* index jump function *$i_a\colon \mathbf{N}^+ \to \mathbf{N}^+$ is defined by $i_a(1) = 1$, $i_a(2) = 2$, and $i_a(k+1) = i_a(k) + 1 + \delta_1(a_{i_a(k)})$ for $k \geq 2$, where $\delta_1(x) = \begin{cases} 1, & x = 1 \\ 0, & x \neq 1 \end{cases}$, and a_j for $j \in \mathbf{N}^+$ are the CF elements of a.*

The index jump function is a renumbering which avoids elements following directly after some 1's in the CF expansion (in particular, it avoids every second element in the sequences of consecutive 1's with index greater than 1); see also [21].

In [22], upper mechanical words were described according to the hierarchy of runs on all levels, as introduced by Azriel Rosenfeld (1974) [17, p. 1265]. A run of the first level is a maximal sequence 10^m, meaning the letter 1 followed by $m \geq 0$ letters 0. For a given slope, there are only two possible run lengths, runs with the smaller length we call *short* runs (S_1) and runs with the largest length we call *long* runs (L_1). The same holds for the other levels: a run of level n is a maximal sequence of runs of level $n-1$, i.e., $S_{n-1}^k L_{n-1}$, $S_{n-1} L_{n-1}^k$, $L_{n-1} S_{n-1}^k$ or $L_{n-1}^k S_{n-1}$ and the cardinality-wise run length of run_n, denoted by $\|\mathrm{run}_n\|$, is the number (here $k+1$) of runs_{n-1} forming it. We denote by $|w|$ the binary-word length of a 0-1 word w, i.e., the total number of its letters. The following theorem shows how exactly the run-hierarchical structure of $s'(a)$ for each $a \in]0,1[\setminus \mathbf{Q}$ depends on the CF elements of a. Because of (1) and (2), this gives also a description of lower mechanical and characteristic words.

Theorem 1 ([22]; a CF description of upper mechanical words). *Let $a \in]0,1[\setminus \mathbf{Q}$ and $a = [0; a_1, a_2, \ldots]$. For $s'(a)$ as in Definition 1 we have $s'(a) = \lim_{k\to\infty} P_k$, where $P_1 = S_1 =$*

10^{a_1-1}, $L_1 = 10^{a_1}$, *and, for* $k \geq 2$,

$$P_k = \begin{cases} L_k = S_{k-1}^{a_{i_a(k)}} L_{k-1} & \text{if} \quad a_{i_a(k)} \neq 1 \quad \text{and} \quad i_a(k) \quad \text{is even} \\ S_k = S_{k-1} L_{k-1}^{a_{i_a(k)+1}} & \text{if} \quad a_{i_a(k)} = 1 \quad \text{and} \quad i_a(k) \quad \text{is even} \\ S_k = L_{k-1} S_{k-1}^{-1+a_{i_a(k)}} & \text{if} \quad a_{i_a(k)} \neq 1 \quad \text{and} \quad i_a(k) \quad \text{is odd} \\ L_k = L_{k-1}^{1+a_{i_a(k)+1}} S_{k-1} & \text{if} \quad a_{i_a(k)} = 1 \quad \text{and} \quad i_a(k) \quad \text{is odd,} \end{cases} \tag{3}$$

where the function i_a *is defined in Definition 2. The meaning of the symbols is the following: for* $k \geq 1$, P_k *is the* ***P****refix number k,* S_k *is the* ***S****hort* run_k *and* L_k *is the* ***L****ong* run_k*. To make the recursive formula (3) complete, we add that for each* $k \geq 2$*, if* $P_k = S_k$*, then* L_k *is defined in the same way as* S_k*, with the only difference that the exponent defined by* $a_{i_a(k)}$ *(or by* $a_{i_a(k)+1}$*) is increased by* 1*. If* $P_k = L_k$*, then* S_k *is defined in the same way as* L_k*, with the only difference that the exponent defined by* $a_{i_a(k)}$ *(or by* $a_{i_a(k)+1}$*) is decreased by* 1*.*

The value of the index jump function for each natural $k \geq 2$ describes the index of the CF element which determines the most frequent run on level $k-1$ (denoted main_{k-1}), which we can formulate as the following corollary. The corollary also describes the cardinality-wise run length on each digitization level and shows how to conclude about the kind of the prefix P_{k-1} as obtained in (3) (long L_{k-1} or short S_{k-1}) from the parity of $i_a(k)$.

Corollary 1. *Let* $a \in \,]0,1[\, \setminus \mathbf{Q}$ *and* $a = [0; a_1, a_2, a_3, \ldots]$*. If* $s'(a)$ *is the upper mechanical word with slope* a *and intercept* 0 *as defined in Definition 1, then, in the run-hierarchic structure of* $s'(a)$ *we have for each* $k \geq 2$

- $a_{i_a(k)} \geq 2 \quad \Rightarrow \quad \mathrm{main}_{k-1} = S_{k-1}, \qquad a_{i_a(k)} = 1 \quad \Rightarrow \quad \mathrm{main}_{k-1} = L_{k-1},$
- $i_a(k)$ *is odd* $\Rightarrow \quad P_{k-1} = L_{k-1}, \qquad i_a(k)$ *is even* $\Rightarrow \quad P_{k-1} = S_{k-1},$

where i_a *is the corresponding index jump function. Moreover, the cardinality-wise run length on each level is the following:* $\|S_n\| = b_n, \|L_n\| = b_n + 1$*, where*

$$b_1 = a_1 \;\; \text{and, for} \;\; n \geq 2, \quad b_n = \begin{cases} a_{i_a(n)}, & a_{i_a(n)} \neq 1 \\ 1 + a_{i_a(n)+1}, & a_{i_a(n)} = 1 \; . \end{cases} \tag{4}$$

Corollary 1 follows immediately from Theorem 1.

Let us recall the concept of the *sequence of length specification* which was first introduced by the author in [23] (Definition 3 there).

Definition 3. *For any irrational* $a = [0; a_1, a_2, \ldots]$*, the sequence* $(b_n)_{n \in \mathbf{N}^+} = (\|S_n\|)_{n \in \mathbf{N}^+}$ *of short run lengths on all levels in the run-hierarchical construction of the upper mechanical word* $s'(a)$ *with slope* a *and intercept* 0*, will be called the* sequence of length specification*.*

It is clear from (4), that for each $a \in \,]0,1[\, \setminus \mathbf{Q}$, the corresponding sequence of length specification $(b_n)_{n \in \mathbf{N}^+}$ fulfills $b_1 \in \mathbf{N}^+$ and, for each $n \geq 2$, $b_n \geq 2$. In [23] we also showed that each sequence fulfilling these condition is the sequence of length specification for some slopes and the cardinality of the set of these slopes is of the continuum. For a fixed index jump function (i.e., a sequence of values $(d_n)_{n \in \mathbf{N}^+}$ such that $d_1 = 1, d_2 = 2$ and, for all $k \geq 2$ $d_k \in \mathbf{N}^+$ and $d_{k+1} - d_k = 1$ or $d_{k+1} - d_k = 2$) there exists exactly one slope with $(b_n)_{n \in \mathbf{N}^+}$ as sequence of length specification [23, 24].

3 The constructional word

In this section we will define (Definition 4) a new binary word associated with the upper mechanical word $s'(a)$ for $a \in \,]0,1[\, \setminus \mathbf{Q}$ and we will call it the *constructional word*. It follows from Definition 2 that, for any $a \in \,]0,1[\, \setminus \mathbf{Q}$ and $n \geq 2$

$$a_{i_a(n)} = 1 \quad \Leftrightarrow \quad i_a(n+1) = i_a(n) + 2 \qquad \text{and} \qquad a_{i_a(n)} \geq 2 \quad \Leftrightarrow \quad i_a(n+1) = i_a(n) + 1. \tag{5}$$

The sequence $(i_a(n))_{n\in\mathbf{N}^+}$ is thus strictly increasing and the difference between each two consecutive elements of this sequence is equal to 1 or to 2. This gives us an idea of defining a new two-letter word associated with a. This word will be called the *constructional word* and it will code the structure of $s'(a)$ in terms of long and short runs on all the levels, according to Corollary 1 and (5).

Definition 4. *Let $a \in \,]0,1[\, \setminus \mathbf{Q}$. The* constructional word *of a is $\gamma = \gamma(a)$, defined by*

$$\gamma_n = i_a(n+2) - i_a(n+1) - 1$$

for $n \in \mathbf{N}^+$, where i_a is the index jump function defined in Definition 2.

It follows from (5) that the constructional word for all $a \in \,]0,1[\, \setminus \mathbf{Q}$ is a 0-1 word, and, for all $n \in \mathbf{N}^+$, $\gamma_n = 1 \;\Leftrightarrow\; a_{i_a(n+1)} = 1$ and $\gamma_n = 0 \;\Leftrightarrow\; a_{i_a(n+1)} \geq 2$. This gives us the following proposition.

Proposition 1. *For each $a \in \,]0,1[\, \setminus \mathbf{Q}$ and for each $n \in \mathbf{N}^+$ we have $\gamma_n = \delta_1(a_{i_a(n+1)})$, where i_a is the index jump function defined in Definition 2.*

Corollary 1 shows clearly why γ got **the name of constructional word**. The elements $a_{i_a(k)}$ for $k \geq 2$ of the CF expansion of the slope $a \in \,]0,1[\, \setminus \mathbf{Q}$ determine the **construction** of runs$_k$ as sets of short and long runs$_{k-1}$. The indices $k \in \mathbf{N}^+$ numbering letters of γ equal to 1 are the same as the indices of the digitization levels with the most frequent long runs$_k$ (L_k).

Example 1. If the slope a is $e - 2 = [0; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, \ldots, 1, 1, 2n, 1, 1, \ldots]$, then the index jump function i_a is formed as follows:

$$\begin{array}{rl}
a = & [0;\overset{b_1}{1},\ \overset{b_2}{2},\ \overbrace{\underline{1},1}^{b_3},\ \overset{b_4}{4},\ \overbrace{\underline{1},1}^{b_5},\ \overset{b_6}{6},\ \overbrace{\underline{1},1}^{b_7},\ \overset{b_8}{8},\ \overbrace{\underline{1},1}^{b_9},\ \overset{b_{10}}{10},\ \overbrace{\underline{1},1}^{b_{11}},\ \ldots] \\
 & \quad\ \downarrow\ \downarrow\ \downarrow\quad \downarrow\ \downarrow\quad \downarrow\ \downarrow\quad \downarrow\ \downarrow\quad \downarrow\ \downarrow\quad \ldots \\
(i_a(k))_{k\in\mathbf{N}^+}= & (\ 1, 2,\ 3,\ \ 5,\ 6,\ \ 8,\ 9,\ \ 11, 12,\ \ 14, 15,\ \ \ldots).
\end{array}$$

In the last row we presented the first eleven elements of the sequence of the values of the index jump function for this a, so $(i_a(k))_{1\leq k\leq 11}$. The sequence of length specification for $a = e - 2$ is $(b_n)_{n\in\mathbf{N}^+} = (1, 2, 2, 4, 2, 6, 2, 8, 2, 10, 2, \ldots 2, 2n, 2, \ldots)$. The constructional word is $\gamma(e - 2) = (01)^\omega$. On odd-numbered levels k short runs (S_k) are the most frequent runs, while on even-numbered levels k long runs (L_k) dominate. The run-hierarchical structure of the digital line $y = (e-2)x$ was thoroughly discussed in the author's paper [21, p. 2252, Example 14].

Definition 4 describes how to form the word γ for $a \in \,]0,1[\, \setminus \mathbf{Q}$, in terms of the corresponding function i_a. The following proposition is a kind of converse to this definition. It says, how to find the function i_a, given the constructional word of a.

Proposition 2. *If $a \in \,]0,1[\, \setminus \mathbf{Q}$ and $\gamma = \gamma(a)$ is the constructional word associated with a, then we have for $n \geq 3$*

$$i_a(n) = n + \sum_{j=1}^{n-2} \gamma_j. \tag{6}$$

Proof. By induction, from Definitions 2 and 4. □

We know, from the author's papers [23, 24], that $(i_a(n))_{n\in\mathbf{N}^+}$ and $(b_n)_{n\in\mathbf{N}^+}$ determine the slope $a \in]0,1[\setminus \mathbf{Q}$. Because of Definition 4 and Proposition 2 we know that also $(\gamma_n)_{n\in\mathbf{N}^+}$ and $(b_n)_{n\in\mathbf{N}^+}$ determine the slope $a \in]0,1[\setminus \mathbf{Q}$.

4 Introduction to Sturmian words

In this section we provide a very brief introduction to Sturmian words, based on Lothaire (2002) [13].

Let $\mathcal{A}$ be a set of symbols usually called the *alphabet*. We denote by $\mathcal{A}^\star$ (in some papers denoted by $\mathcal{A}^{(\mathbf{N})}$) the set of all finite words over $\mathcal{A}$ (i.e., finite sequences of elements from $\mathcal{A}$) and by ε the empty word. We denote by $\mathcal{A}^\omega$ ($\mathcal{A}^{\mathbf{N}}$) the set of (right) infinite words (i.e., sequences of symbols in $\mathcal{A}$ indexed by non-negative integers). In this paper we consider only right infinite words.

A finite word w is a *factor* of a (finite or infinite) word x if there exist words u (finite) and y such that $x = uwy$. **Sturmian words** are defined as infinite words which have exactly $n+1$ different factors of length n for every natural n. In particular, they have 2 factors of length 1, which means that each Sturmian word is constructed of exactly 2 letters, which we can call 0 and 1, thus $\mathcal{A} = \{0, 1\}$.

A word $x \in \mathcal{A}^\omega$ is *periodic* if it is of the form $x = z^\omega$ for some $z \in \mathcal{A}^\star \setminus \{\varepsilon\}$, *eventually periodic* if it is of the form $x = yz^\omega$ for some $y, z \in \mathcal{A}^\star \setminus \{\varepsilon\}$, and *aperiodic* if it is not eventually periodic; cf. Lothaire (2002) [13, p. 9]. We need the following definition to formulate a theorem which shows equivalent characterizations of Sturmian words (Theorem 2).

Definition 5 (Lothaire 2002:48). *For binary words with letters* 0 *and* 1 *we define the following.*

- *The* height *of a finite word x is the number $h(x)$ of letters equal to* 1 *in x.*
- *Given two finite words x and y of the same length, their* balance *is* $\delta(x,y) = |h(x) - h(y)|$.
- *A set of finite words X is* balanced *if* $(\ x, y \in X \ \wedge\ |x| = |y|\ ) \ \Rightarrow\ \delta(x,y) \leq 1$.
- *An infinite word is itself* balanced *if the set of its factors (thus, finite words) is balanced.*

Theorem 2 (Lothaire 2002:57). *Let s be an infinite word. We have the following equivalence: s is Sturmian $\Leftrightarrow$ s is balanced and aperiodic $\Leftrightarrow$ s is irrational (lower or upper) mechanical.*

5 A fixed-point theorem for Sturmian words. Self-generating run construction.

In Sections 2 and 3 we described two words over a two-letter alphabet $\{0, 1\}$ associated with an irrational positive slope $a < 1$. The first of them, the upper mechanical word, is Sturmian (Theorem 2), the second one, the constructional word, can obviously be any 0-1 word. One could try to describe the slopes $a \in]0,1[\setminus \mathbf{Q}$, for which the levels with the most frequent run being long (or, dually, short) are uniformly distributed (for such a we will call $s'(a)$ *words with balanced construction*). And an even more demanding condition would be: find these a for which $\gamma(a) = c(a)$. For these a, $s'(a) = 1c(a)$ will be called *word with self-balanced construction*, because the distribution of the levels with the most frequent run being long (equivalently: the distribution of pairs (a_l, a_{l+1}) of CF elements of a such that $a_l = 1$, $l \geq 2$, and a_l is not immediately preceded by an odd number of consecutive CF elements equal to 1 and with indices greater than 1) is the same as the distribution of the letter 1 in the characteristic word $c(a)$; c.f. Proposition 4 on page 8 and the discussion there.

We consider $\{0, 1\}^\omega$, the set of all right infinite two-letter words composed of 0's and 1's and let $\mathcal{UM}_0 \subset \{0, 1\}^\omega$ be the subset of all upper mechanical words with positive irrational slopes less than 1 and with intercept 0 (which are Sturmian according to Theorem 2).

Definitions 1 and 4 give us two mappings from $]0,1[\setminus\mathbf{Q}$ to $\{0,1\}^\omega$. The first one maps each $a\in\,]0,1[\setminus\mathbf{Q}$ to the associated upper mechanical word $s'(a)=1c(a)$:

$$s'\colon]0,1[\setminus\mathbf{Q}\longrightarrow\mathcal{UM}_0\subset\{0,1\}^\omega,$$

the second one maps each $a\in\,]0,1[\setminus\mathbf{Q}$ to the associated constructional word $\gamma(a)$ concatenated with prefix 1:

$$1\gamma\colon]0,1[\setminus\mathbf{Q}\longrightarrow\{0,1\}^\omega,\quad (1\gamma)(a)=1\gamma(a).$$

Definition 6. *The* run-construction encoding operator $\Delta_c\colon\mathcal{UM}_0\longrightarrow\{0,1\}^\omega$ *is defined as* $\Delta_c=(1\gamma)\circ(s')^{-1}$.

$$\begin{tikzcd} {]0,1[\setminus\mathbf{Q}} \arrow[r, "s'"] \arrow[dr, "1\gamma"'] & \mathcal{UM}_0 \arrow[d, "\Delta_c"] \\ & \{0,1\}^\omega\supset\mathcal{UM}_0 \end{tikzcd}$$

The mapping is well defined (Lemma 2.1.21 from Lothaire 2002:62 mentioned in Section 2). We can also describe this operator by analogy with the run-length encoding operator as in [6]:

$$\Delta_c(s'(a))(0)=1,\quad \Delta_c(s'(a))(n)=\delta_1(a_{i_a(n+1)})\quad\text{for}\quad n\in\mathbf{N}^+$$

which, according to Corollary 1, can be written in the following, more illustrative way:

$$\Delta_c(s'(a))(n)=\begin{cases}0, & S_n \text{ is the most frequent run on level } n\\ 1, & L_n \text{ is the most frequent run on level } n\end{cases}\qquad\text{for}\quad n\in\mathbf{N}^+.$$

Definition 7. *Let $a\in\,]0,1[\setminus\mathbf{Q}$. The upper mechanical word $s'(a)$ has*

- balanced construction *if its constructional word $\gamma(a)$ is a characteristic word $c(\alpha)$ (not necessarily with irrational slope) for some α.*
- Sturmian-balanced construction *if $\gamma(a)$ is a characteristic word $c(\alpha)$ for some $\alpha\in\,]0,1[\setminus\mathbf{Q}$.*
- self-balanced construction *if $1\gamma(a)=\Delta_c(1c(a))=1c(a)$, i.e., its constructional word is equal to its characteristic word, i.e., $s'(a)$ is a fixed point of Δ_c.*

Clearly: self-balanced construction $\Rightarrow$ Sturmian-balanced construction $\Rightarrow$ balanced construction.

Example 2. The words $s'(a)$ with $a=[0;a_1,a_2,a_3,\ldots]$, where $a_k\geq 2$ for all $k\geq 2$, have **balanced construction**. We have $i_a(k)=k$ for all $k\in\mathbf{N}^+$ and $a_{i_a(k)}\geq 2$ for all $k\geq 2$. This means that the constructional word $\gamma=\gamma(a)$ is defined by $\gamma_n=0$ for all $n\in\mathbf{N}^+$, which is the characteristic word with slope 0. This also means that no upper mechanical word with dominating short run on all digitization levels can be a fixed point of Δ_c.

Example 3. The words $s'(a)$ with $a=[0;a_1,1,a_3,1,a_5,1,a_7,\ldots]$, where $a_{2k-1}\in\mathbf{N}^+$ for all $k\in\mathbf{N}^+$, have **balanced construction**. We have $i_a(1)=1$ and $i_a(k)=2k-2$ for $k\geq 2$, and $a_{i_a(k)}=1$ for all $k\geq 2$. This means that the constructional word $\gamma=\gamma(a)$ is defined by $\gamma_n=1$ for all $n\in\mathbf{N}^+$, which is the characteristic word with slope 1. This also means that no upper mechanical word with dominating long run on all digitization levels can be a fixed point of Δ_c.

Let us recall the following theorem, which is a merge of Lagrange's theorem from 1770 with Euler's theorem from 1737; see [4, pp. 66–71].

Theorem 3 (Euler, Lagrange). *Quadratic surds (i.e., algebraic numbers of the second degree), and only they, are represented by periodic or eventually periodic CFs.*

Example 4. A generalization of Example 3: For each $k \in \mathbf{N}^+ \setminus \{1\}$ and for each infinite matrix $A = [a_{ij}]_{i\in\mathbf{N}^+, j\in[1,k]_\mathbf{Z}}$, where $a_{i1} \in \mathbf{N}^+$ and $a_{ij} \in \mathbf{N}^+ \setminus \{1\}$ for $i \in \mathbf{N}^+$ and $j \in [2,k]_\mathbf{Z}$, the upper mechanical words $s'(a)$ with slopes $a = [0; a_{11}, \ldots, a_{1k}, 1, a_{21}, \ldots, a_{2k}, 1, a_{31}, \ldots, a_{3k}, 1, a_{41}, \ldots, a_{4k}, 1, a_{51}, \ldots]$ have **balanced construction**. We have $a_{i_a(nk+1)} = a_{nk+n} = 1$ for all $n \in \mathbf{N}^+$. The constructional words of all these slopes for a fixed k are $0^{k-1}10^{k-1}10^{k-1}1\ldots$. They correspond to the word with slope $\frac{1}{k}$. If all the rows of the matrix A are identical, the CF expansion is periodic and a is quadratic irrational.

Example 5. A generalization of Example 3: For each $k \in \mathbf{N}^+ \setminus \{1\}$ and each pair of infinite matrices $[a_{ij}]_{i\in\mathbf{N}^+, j\in[1,k]_\mathbf{Z}}$ and $[a'_{ij}]_{i\in\mathbf{N}^+, j\in[1,k+1]_\mathbf{Z}}$ such that $a_{i1}, a'_{i1} \in \mathbf{N}^+$ and $a_{ij}, a'_{is} \in \mathbf{N}^+ \setminus \{1\}$ for all indices $i \in \mathbf{N}^+$, $j \in [2,k]_\mathbf{Z}$ and $s \in [2,k+1]_\mathbf{Z}$, the upper mechanical words $s'(a)$ with slopes $[0; a_{11}, \ldots, a_{1k}, 1, a'_{11}, \ldots, a'_{1,k+1}, 1, a_{21}, \ldots, a_{2k}, 1, a'_{21}, \ldots, a'_{2,k+1}, 1, a_{31}, \ldots]$ have **balanced construction**. The constructional words of all these $s'(a)$ for fixed k are $0^{k-1}10^k10^{k-1}10^k10^{k-1}1\ldots$. They correspond to the upper mechanical words $s'(a)$ with slopes $a = \frac{2}{2k+1}$.

Proposition 3. *There exist no quadratic surds which are slopes to upper mechanical words with Sturmian-balanced construction.*

Proof. Let $a \in]0,1[\setminus \mathbf{Q}$ be any quadratic surd. If there are no 1's in the CF expansion of a, then $\gamma(a) = 000\cdots$, which is the characteristic word with slope 0, which is rational. If there is a 1 in the CF expansion of a, then, according to Theorem 3, either this 1 is only in the beginning of the CF (if we have eventual periodicity) or is repeated periodically, which will lead to a characteristic word of a rational number, if any. □

Quadratic surds with purely periodic CF expansion are slopes of fixed points of corresponding substitutions as defined in Shallit (1991) [18]. It follows from Proposition 3, that no quadratic surds can be slopes of fixed points of Δ_c. No quadratic surds can have Sturmian-balanced construction, but some of them have balanced construction. It would be an interesting combinatorial exercise to describe all the quadratic surds with balanced construction (give a necessary and sufficient condition on the CF expansion of slopes) and, generally, to give a necessary and sufficient condition on the elements of the CF expansion of $a = [0; a_1, a_2, \ldots]$ to generate an upper mechanical word $s'(a)$ with balanced construction, Sturmian-balanced construction, self-balanced construction. Proposition 1 and Definition 7 give us the following characterization of the CF expansion of the slopes of fixed points of Δ_c. Let $a = [0; a_1, a_2, \ldots]$. A pair (a_l, a_{l+1}) of CF elements in the expansion of a will be called an *essential pair* if $a_l = 1$, $l \geq 2$, and the element $a_l = 1$ is immediately preceded by an even number, i.e., $0, 2, 4, \ldots$, of consecutive 1's with index greater than 1, i.e., $\exists\ k \in \mathbf{N}$, $[0; a_1, a_2, \ldots] = [0; a_1, a_2, \ldots, a_{l-2k-1}, \underbrace{1, 1, \ldots, 1, 1}_{2k}, a_l, a_{l+1}, \ldots]$ and, if $l - 2k - 1 \geq 2$, then $a_{l-2k-1} \geq 2$; cf. *essential 1's* in [23, 24].

Proposition 4. *If $a = [0; a_1, a_2, \ldots]$, then $s'(a)$ is a fixed point of Δ_c iff $c_n(a) = \delta_1(a_{i_a(n+1)})$ for all $n \in \mathbf{N}^+$, where $c(a)$ is the corresponding characteristic word.*

This means that for a fixed point $s'(a) = 1c(a)$, each essential pair in the CF expansion of a reflects in the letter 1 on the corresponding place in $c(a)$, while the letters 0 of $c(a)$ appear on places corresponding to the places of remaining (i.e., no members of essential pairs) CF elements a_k $(k \geq 2)$ of a.

An upper mechanical word with slope $a \in]0,1[\setminus \mathbf{Q}$ and intercept 0 is a word with self-balanced construction (a fixed point of Δ_c) if its construction according to the hierarchy of runs and the arrangement of 0's and 1's in the word itself are made according to the same rules. The levels with dominating long runs are uniformly distributed like the 1's in the original characteristic word corresponding to the upper mechanical word. The following theorem is the main result of this paper.

Theorem 4 (main result). *Let $(b_n)_{n\in\mathbf{N}^+}$ be any sequence of natural numbers such that $b_1 \in \mathbf{N}^+$ and $b_n \geq 2$ for all $n \geq 2$. There exists exactly one fixed point of Δ_c with $(b_n)_{n\in\mathbf{N}^+}$ as the sequence of its length specification as defined in Definition 3.*

Proof. We will show how to find the fixed point $w = s'(a)$ corresponding to given $(b_n)_{n\in\mathbf{N}^+}$. The uniqueness will follow from the construction. In our reasoning we will use the following rules:

R1. Fixed point condition: for each $k \in \mathbf{N}$, $\mathrm{pref}_{k+1}(w) = 1\gamma_1 \cdots \gamma_k$, where $\mathrm{pref}_{k+1}(w)$ denotes the $k+1$ letters long prefix of the upper mechanical word $w = s'(a)$ we are looking for, and $\gamma = \gamma(a)$.

R2. $(\gamma_1, \gamma_2, \ldots, \gamma_k)$ determines $(i_a(1), i_a(2), \ldots, i_a(k+2))$ according to (6)

R3. $(\gamma_1, \gamma_2, \ldots, \gamma_k)$ and $(b_1, b_2, \ldots, b_{k+1})$ determine $(a_1, a_2, \ldots, a_{i_a(k+1)})$ according to **R2**, Proposition 1 and (4) in the following way. For $j = 1, 2, \ldots, k$

$$\gamma_j = 1 \quad \Rightarrow \quad \left[a_{i_a(j+1)} = 1 \ \wedge \ a_{i_a(j+1)+1} = b_{j+1} - 1\right], \qquad \gamma_j = 0 \quad \Rightarrow \quad a_{i_a(j+1)} = b_{j+1}$$

R4. According to **R2** and **R3**, $(\gamma_1, \ldots, \gamma_k)$ and $(b_1, \ldots, b_{k+1})$ determine (uniquely!) the prefix P_{k+1} (as in the run-hierarchical description (3)) of the upper mechanical word w we are looking for.

One can see that we need to describe a way of finding $(\gamma_1, \gamma_2, \ldots)$ to be able to reconstruct the fixed point (according to the condition **R1**) with the length specification $(b_1, b_2, b_3, \ldots)$. Because we do have whole $(b_1, b_2, b_3, \ldots)$, **R1**–**R4** imply that it is enough to show that for any $k \in \mathbf{N}^+$ we have $|P_{k+1}| > k+1 = |1\gamma_1 \cdots \gamma_k|$, i.e., that the prefixes produced of $(\gamma_1, \gamma_2, \ldots, \gamma_k)$ and $(b_1, b_2, b_3, \ldots)$ are on each step of the construction long enough to supply us with more information about the constructional word, which enables us to continue the construction using **R1**. So we have to prove $|P_{k+1}| > k+1$ for each $k \in \mathbf{N}^+$ (which is actually a severe understatement; see Corollary 1 in [22] and the last but one column in the table in Example 6).

Let us first suppose that $b_1 \geq 2$. We know from Theorem 1, that P_1 is short and $|P_1| = b_1 \geq 2 > 1$, so, because of the recursive construction (3), we get by easy induction $|P_{k+1}| \geq 2^{k+1} > k+1$.

If $b_1 = 1$, we get from Theorem 1 $|P_1| = 1$, which does not look well, because $|P_1| < 2$. To continue our construction, we have to get our information about γ_1 from somewhere else than P_1 and **R1**. Because the first run of level 1 is always short, we know that $s'(a) = 1c(a) = 11\ldots$, thus, **R1** gives us $\gamma_1 = 1$. This implies (rule **R3**) that $a_2 = a_{i_a(1+1)} = 1$ (and $a_3 = b_2 - 1$) and we get the following prefix of $s'(a)$: $P_2 = S_2 = S_1 L_1^{a_3} = 1(10)^{b_2-1}$ from (3) (the second row of the formula, because $i_a(2) = 2$ and $a_2 = 1$). Now we have already $|P_2| \geq 3 > 1+1$ for any b_2 and again, we obtain by induction $|P_{k+1}| \geq 2^{k-1} \cdot 3 > k+1$ for $k \geq 1$, which completes the proof. □

The speed of finding the fixed point grows together with b_1, but we have shown that even in case $b_1 = 1$ we can both get started and go on with our construction. Let us take the length specification $b_1 = 1$ and $b_n = 2$ for $n \geq 2$. This gives the slowest possible process of finding of the slope of the fixed point, but still, even in this worst case, it is possible to construct the unique fixed point:

Example 6. We will find the fixed point of Δ_c with the length specification $(1, 2, 2, 2, \ldots)$. We are thus looking for such $a \in]0,1[\setminus \mathbf{Q}$ that $\mathrm{pref}_{k+1}(s'(a)) = 1\gamma_1 \cdots \gamma_k$ for each $k \in \mathbf{N}$, and $(1, 2, 2, 2, 2, \ldots)$ is the corresponding sequence of length specification. At the starting point we only know that the first letter of $s'(a)$ is, by definition, 1. In each of the following steps we get $P_2, P_3, \ldots$ (step n gives us P_{n+1}).

The facts that $b_1 = 1$ and that the first run of level 1 is short, gives us only the information, that $s'(a) = 1c(a) = 11\ldots$ thus, because $\mathrm{pref}_2(s'(a)) = 1\gamma_1$, we get $\gamma_1 = 1$, which implies (rule **R3**) that $a_2 = a_{i_a(1+1)} = 1$ (and $a_3 = b_2 - 1 = 1$) and we get the following prefix of $s'(a)$: $P_2 = S_2 = S_1L_1 = 110$ from (3) (the second row of the formula, because $i_a(2) = 2$ and $a_2 = 1$). We have moreover $i_a(3) = 3 + \gamma_1 = 4$ (even number).

Further, because $110 = \mathrm{pref}_3(s'(a)) = 1\gamma_1\gamma_2$, we get $\gamma_2 = 0$, which means that $a_{i_a(2+1)} = a_4 = b_3 = 2$ and $i_a(4) = 4 + \gamma_1 + \gamma_2 = 5$. We get $P_3 = L_3 = S_2^2L_2 = 11011011010$ (from the first row of (3), because $a_{i_a(3)} \neq 1$ and $i_a(3)$ is even). This gives us, because of **R1**, $\gamma_3 = 1$, $\gamma_4 = 1$, $\gamma_5 = 0$, $\gamma_6 = 1$, $\gamma_7 = 1$, $\gamma_8 = 0$, $\gamma_9 = 1$, $\gamma_{10} = 0$, which, according to **R4**, allows us to get $P_4, \cdots, P_{11}$. Prefixes P_2, P_3 and P_4 are illustrated on Figure 2. One can see the analogy to Figure 1.

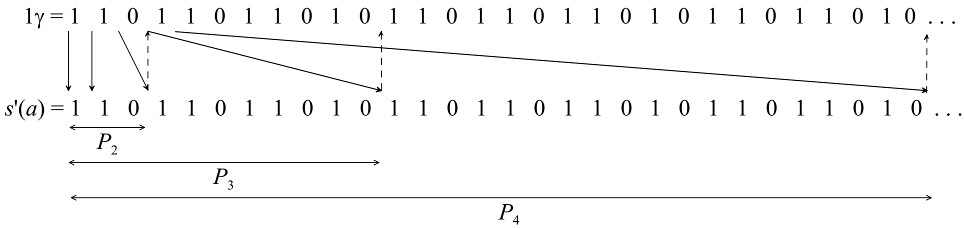

Fig. 2. The prefixes P_2, P_3 and P_4 of the fixed point of Δ_c with the length specification $(b_n)_{n\in\mathbf{N}^+} = (1, 2, 2, 2, \ldots)$.

We can summarise the data we have until now in the following table. In the next to last column, $|P_{k+1}|$ denotes the binary-word length of prefix P_{k+1} (total number of 0's and 1's forming it).

given	k	$i_a(k+1)$	$a_{i_a(k+1)}$	b_{k+1}	S_{k+1}	L_{k+1}	gives P_{k+1}	$\lvert P_{k+1}\rvert$	gives
$\gamma_1 = 1$	1	2	$= 1$	2	S_1L_1	$S_1L_1^2$	S_1L_1	3	γ_2
$\gamma_2 = 0$	2	4	$\neq 1$	2	S_2L_2	$S_2^2L_2$	$S_2^2L_2$	11	$\gamma_3, \ldots, \gamma_{10}$
$\gamma_3 = 1$	3	5	$= 1$	2	L_3S_3	$L_3^2S_3$	$L_3^2S_3$	30	$\gamma_{11}, \ldots, \gamma_{29}$
$\gamma_4 = 1$	4	7	$= 1$	2	L_4S_4	$L_4^2S_4$	$L_4^2S_4$	79	$\gamma_{30}, \ldots, \gamma_{78}$
$\gamma_5 = 0$	5	9	$\neq 1$	2	L_5S_5	$L_5S_5^2$	L_5S_5	128	$\gamma_{79}, \ldots, \gamma_{127}$
$\gamma_6 = 1$	6	10	$= 1$	2	S_6L_6	$S_6L_6^2$	S_6L_6	305	$\gamma_{128}, \ldots, \gamma_{304}$
$\gamma_7 = 1$	7	12	$= 1$	2	S_7L_7	$S_7L_7^2$	S_7L_7	787	$\gamma_{305}, \ldots, \gamma_{786}$
$\gamma_8 = 0$	8	14	$\neq 1$	2	S_8L_8	$S_8^2L_8$	$S_8^2L_8$	2843	$\gamma_{787}, \ldots, \gamma_{2842}$
$\gamma_9 = 1$	9	15	$= 1$	2	L_9S_9	$L_9^2S_9$	$L_9^2S_9$	7742	$\gamma_{2843}, \ldots, \gamma_{7741}$
$\gamma_{10} = 0$	10	17	$\neq 1$	2	$L_{10}S_{10}$	$L_{10}S_{10}^2$	$L_{10}S_{10}$	12641	$\gamma_{7742}, \ldots, \gamma_{12640}$

We proceed in this way. The fixed point $s'(a) = 1c(a)$ is

$$\underbrace{\overbrace{110}^{S_2}\,\overbrace{110}^{S_2}\,\overbrace{11010}^{L_2}\,\overbrace{11011011010}^{L_3}\,\overbrace{11011010}^{S_3}}_{L_4}\,\underbrace{110110110101101101101011011010}_{L_4}\,\underbrace{1101101101011011010}_{S_4}\ldots$$

so the constructional word $\gamma(a) = 10110110101101101101 0\ldots$, which gives the following slope a:

$$[0;1,\underbrace{1,1}_{a_{i_a(2)}=1},2,\underbrace{1,1}_{a_{i_a(4)}=1},\underbrace{1,1}_{a_{i_a(5)}=1},2,\underbrace{1,1}_{a_{i_a(7)}=1},\underbrace{1,1}_{a_{i_a(8)}=1},2,1,1,2,1,1,1,1,2,1,1,1,1,2,1,1,1,1,2,1,1,2,\ldots].$$

Let us analyze the set of all fixed points of Δ_c.

Theorem 5. *Let* $\mathrm{Fix}(\Delta_c) \subset \mathcal{UM}_0$ *denote the set of all fixed points of* Δ_c. *Then:*

1. $\mathrm{Fix}(\Delta_c) \subset s'(]0, \frac{2}{3}[\setminus\mathbf{Q})$*; numbers* 0 *and* $\frac{2}{3}$ *are accumulation points of* $(s')^{-1}(\mathrm{Fix}(\Delta_c))$.
2. $\mathrm{card}(\mathrm{Fix}(\Delta_c))$ *is equal to that of the continuum.*

Proof. It is clear that we can go as near as we want towards 0. If we take $b_1 \to \infty$ then $[0; b_1, b_2, \ldots] \to 0$.

It remains to be shown that we cannot have a fixed point with slope larger than $\frac{2}{3}$. We will look for maximal a such that $1c(a)$ is a fixed point of Δ_c. First, to get as large slope as possible, we have to have $b_1 = 1$ (otherwise the slope is less than $\frac{1}{2}$). So, we proceed as in Example 6, $s'(a) = 11\ldots$, thus $\gamma_1 = 1$, so $a_2 = 1$, which means that $a_{i_a(2)} = 1$, so $i_a(3) = 4$ (a_3 is not of the form $a_{i_a(k)}$ for any $k \in \mathbf{N}^+$). The maximal possible slope of a fixed point begins with $[0; 1, 1, \ldots]$. We are absolutely free in the choice of the next element, because it does not affect the constructional word, as it is not a value of the index jump function. So, to make the slope maximal, we choose 1, because $[a_0; a_1, a_2, \ldots] < [a_0'; a_1', a_2', \ldots]$ iff $(a_0, -a_1, a_2, -a_3, a_4, -a_5, \ldots) \overset{\text{lexic.}}{<} (a_0', -a_1', a_2', -a_3', a_4', -a_5', \ldots)$, where the second inequality is according to the lexicographical order on sequences. Taking $b_3 \to \infty$ (thus, making the slope as large as possible), we get the limit value of $\frac{2}{3}$, because $[0; 1, 1, 1, b_3, \ldots] \to \frac{2}{3}$. We can also illustrate the solution with the following table:

given	k	$i_a(k+1)$	$a_{i_a(k+1)}$	b_{k+1}	S_{k+1}	L_{k+1}	gives P_{k+1}
$\gamma_1 = 1$	1	2	$= 1$	2	S_1L_1	$S_1L_1^2$	S_1L_1
$\gamma_2 = 0$	2	4	$\neq 1$	b_3	$S_2^{b_3-1}L_2$	$S_2^{b_3}L_2$	$S_2^{b_3}L_2$

so the largest slopes of fixed points have the form $[0; 1, 1, 1, b_3, \ldots]$ and tend to $\frac{2}{3}$ when $b_3 \to \infty$.

$$s'(a) = S_2^{b_3}L_2\ldots = (110)^{b_3}(11010)\ldots \overset{b_3\to\infty}{\to} s'\left(\tfrac{2}{3}\right).$$

To prove the second statement of the theorem, we only need to recall that, according to Theorem 4, each sequence of length specification generates exactly one fixed point and each fixed point has its length specification (the same). The set of all fixed points has thus the same cardinality as the set of all sequences of length specification, which is the same as this of $\mathbf{N}^{\mathbf{N}}$. ◻

6 Conclusions and open problems

In this paper we have defined a run-construction encoding operator by analogy to the well-known run-length encoding operator and we formulated and proved a fixed-point theorem for Sturmian words. We also presented some combinatorial problems concerning quadratic surds (on p. 8, after Proposition 3). Some questions and problems arise also in connection with the run-construction encoding operator and the set of its fixed points. Theorem 5 gives us some answers. It states that the cardinality of the set of all fixed points is equal to that of the continuum (which follows from the main theorem of this paper, Theorem 4) and that no slopes of fixed points are larger than $\frac{2}{3}$. No fixed point of substitutions as described in Shallit (1991) [18] can be a fixed point of the run-construction encoding operator. Proposition 3 states that no quadratic surds can be slopes of fixed points of the operator.

There are still some problems to be solved. For example:

- Is the set of slopes of all fixed points of the run-construction encoding operator, i.e. the set $(s')^{-1}(\mathrm{Fix}(\varDelta_c))$, dense in $]0, \frac{2}{3}[\setminus \mathbf{Q}$? Does it have accumulation points different from 0 and $\frac{2}{3}$?
- What kind of irrational numbers are the slopes of fixed points? Are they all transcendental?
- An algorithm finding fixed points related to the equivalence classes defined by sequences of length specification $(b_1, b_2, \ldots)$ could be written.
- How can we use the fixed points in digital geometry?

Acknowledgments. I am grateful to Christer Kiselman for comments on earlier versions of the manuscript.

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ISSN 1401-2049
ISBN 978-91-506-2090-0

Universitetstryckeriet, Uppsala 2009