Precalculus 4: Exponentials and logarithms[1]

Mathematics from high school to university

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A short table of contents

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An extremely detailed table of contents; the videos (titles in green) are numbered

In blue: problems solved on an iPad (the solving process presented for the students; active problem solving)

In red: solved problems demonstrated during a presentation (a walk-through; passive problem solving)

In dark blue: *Read along with this section*: references for further reading and exercises in *Prerequisites* book and in the *Precalculus* book by Carl Stitz and Jeff Zeager.

Books to read along with the course, with more practice problems:

1. *Precalculus Prerequisites, a.k.a. Chapter* 0: Carl Stitz, Ph.D., Lakeland Community College; Jeff Zeager, Ph.D., Lorain County Community College; version from August 16, 2013.
2. *Precalculus*: Carl Stitz, Ph.D., Lakeland Community College; Jeff Zeager, Ph.D., Lorain County Community College; version from July 4, 2013.

These books are added as resources to Video 1, with kind permission of Professor Carl Stitz.

S1 Introduction to the course

You will learn: you will get a very general introduction to exponential and logarithmic functions: how they look and what they describe; you will learn about their simplicity in some aspects, and about some extraordinary complications you don't necessarily have to think about (but you should have a general idea about them); exponential and logarithmic functions are (next to polynomials, rational functions, power functions, trigonometric and inverse trigonometric functions you have learned about in the previous courses in the Precalculus series) examples of *elementary functions*: these functions are the building blocks for all the functions we will work with in the upcoming Calculus series.

1 Introduction to the course. Extra material: this list with all the movies and problems.
Extra material: a formula sheet with the main formulas and graphs discussed in this course.
Extra material: two books named above (*Precalculus Prerequisites* and *Precalculus*).

2 It's all about power, growth, and decay.

3 One graph to rule them all.

4 General questions we ask about functions.

5 Simple answers: domain, range, zeros, y-intercept, monotonicity, invertibility.

6 Some extraordinarily complicated questions.

S2.71828... The noble number e, the Binomial Theorem, and Pascal's Triangle

You will learn: about number e, both in an intuitive way (by images) and in a more formal way. In order to be able to perform some formal proofs, you will need the Binomial Theorem: theorem telling you how to raise a sum of two terms to any positive natural power; you will learn about *factorial*, about binomial coefficients, and Pascal's Triangle (all this will come back in the course in Discrete Mathematics, and then you will get much more practice and combinatorial problems to solve; now we just need the Binomial Theorem as a tool for dealing with e).

Read along with this section: **Precalculus book**: Chapter 9 (pages 651–692): Sections 9.1 (Sequences), 9.2 (Summation notation), 9.3 (Mathematical induction) contain a repetition from *Precalculus* 1; Section 9.4 (The Binomial Theorem) contains new stuff. (For now, ignore the parts containing logarithms, like the second part of Example 9.2.2. on page 668.)

7 Short and sweet about the number e.

8 Summation notation: a repetition and some properties.

Properties: Suppose $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are two sequences of real numbers, and m and p are such natural numbers that $m \leqslant p$. Then:

a) $\displaystyle\sum_{n=m}^{p}(a_n \pm b_n) = \sum_{n=m}^{p} a_n \pm \sum_{n=m}^{p} b_n$

b) $\displaystyle\sum_{n=m}^{p} c\, a_n = c \sum_{n=m}^{p} a_n$ for any real number c.

c) $\displaystyle\sum_{n=m}^{p} a_n = \sum_{n=m}^{j} a_n + \sum_{n=j+1}^{p} a_n$ for any natural number j such that $m \leqslant j < j+1 \leqslant p$.

d) $\displaystyle\sum_{n=m}^{p} a_n = \sum_{n=m+r}^{p+r} a_{n-r}$, for any natural number r.

Extra material: notes with proofs of the properties above.

9 Short and sweet about Pascal's Triangle.

Extra material: notes from the iPad.

10 What about powers of a difference?

Formal proof (from the axioms) that the product of even number of negative factors is positive.

11 Let's practice raising binomials to positive natural powers.

Exercise: Use Pascal's Triangle to compute:

a) $(1+2a)^3$,

b) $(x-2)^5$,

c) $(3x-y)^4$.

Extra material: notes with solved exercises.

12 Factorial and binomial coefficients.

Compute the following:

a) $1!,\ 2!,\ 3!,\ 4!,\ 5!,\quad \dfrac{(k-1)!}{(k+2)!}$ for $k \geqslant 1$

b) $\binom{5}{2}$, c) $\binom{7}{4}$.

Extra material: notes with solved exercises.

13 Short and sweet about the Binomial Theorem.

Apply the Binomial Theorem to compute $(x-2)^4$. Compute $\binom{4}{k}$ for $k = 0, 1, 2, 3, 4$.

Extra material: notes from the iPad.

14 How is it the same?

Show that $\binom{0}{0} = 1$. Let $n, k \in \mathbb{N}^+$ such that $k \leqslant n$. Show the following:

a) $\binom{n}{0} = 1 = \binom{n}{n}$,

b) $\binom{n}{1} = n = \binom{n}{n-1}$,

c) $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$,

d) $\binom{n}{k} = \binom{n}{n-k}$.

Use the results above to motivate that the binomial coefficients are always integer; the last one just shows that Pascal's Triangle has symmetric entries. The combination of $\binom{0}{0} = 1$ with a) and c) shows that both Pascal's Triangles are, indeed, the same.

Extra material: notes with solved exercises.

15 A formal proof of the Binomial Theorem: by induction.

Binomial Theorem: For all non-zero a and b, and for all $n \in \mathbb{N}^+$ holds: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$.

16 **Optional**: A combinatorial proof of the Binomial Theorem.

Let $n, k \in \mathbb{N}^+$ such that $k \leqslant n$. We have n different objects.

a) In how many ways can we arrange these n different objects in a sequence? (permutations)

b) In how many ways can we choose k objects (order important) out of the given n objects? (variations)

c) In how many ways can we choose k objects (order **not** important) out of the given n objects? (combinations)

d) Use the result obtained in c) to prove the Binomial Theorem.

Extra material: notes from the iPad.

17 **Optional**, but really delightful.

Let $n \in \mathbb{N}^+$. Show that

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n.$$

Use the Binomial Theorem (easy), and use a combinatorial reasoning (delightful).

Extra material: notes with the proof of the equality.

18 **Advanced**: Axiom of Completeness in terms of a supremum.

19 **Advanced**: Convergence of a monotone sequence of real numbers.

20 **Advanced**: Convergence of geometric sequences and series.

21 **Advanced**: Motivation of the definition and irrationality of the number e.

Extra material: An article with some definitions, theorems, and proofs:

a) Definitions: upper bound / supremum / maximum; lower bound / infimum / minimum.

b) Theorem 1: A statement equivalent with the Axiom of Completeness.

c) Theorem 2: Convergence of a monotone sequence of real numbers.

d) Theorem 3: Convergence of geometric sequences and series with quotient $0 < q < 1$.

e) Theorem 4: About two increasing sequences with the limit in e.

f) Theorem 5: Number e is irrational.

S3 Powers with various types of exponents

You will learn: about powers with natural, integer, and rational exponents and the computation rules holding for them (the product rules, the quotient rules, the power rule); you will also get an explanation of a more serious stuff: how we can be sure about existence of nth roots of numbers. We will gradually get more and more understanding about the topic of plotting exponential functions. You will also get plenty of exercise to get comfortable with the topic of powers with various types of exponents, and power-related computations.

Read along with this section: **Prerequisites book**: Section 0.2 (pp. 26–37; concentrate on the problems with powers, especially such with fractional exponents, which we didn't work with in Precalculus 2). Section 0.9 (pp. 111–125: Radicals and equations) is recommended for self studies if you don't feel confident working with radicals; you are, of course, welcome with your questions via QA.

22 Our plan.

23 Powers with positive natural exponents.

Exercise 1: Compute: 2^8, $(-3)^4$, $(\frac{1}{2})^8$, $(-5)^3$, 10^9.

Extra material: notes with solved Exercise 1.

24 Two product rules.

Exercise 2: Explain why the product rules are true; use $n = 3$ and $m = 4$.
Compute: $(-2)^9 \cdot 5^9$, $(\frac{3}{7})^{500} \cdot (\frac{21}{9})^{500}$, $\sqrt{2}^{10} \cdot \sqrt{5}^{10}$, $a^2bc^7d^3 \cdot a^7b^2d^4x^7$.

Extra material: notes with solved Exercise 2.

25 Two quotient rules.

Exercise 3: Explain why the quotients rules are true; use $n = 5$ and $m = 3$.
Compute: $\frac{48^3}{12^3}$, $\frac{5^{2005}}{5^{2002}}$.

Extra material: notes with solved Exercise 3.

26 The power rule.

Exercise 4: Explain why the power rule is true; use $n = 3$ and $m = 2$. Compute: $(a^5)^2$, $(x^4)^3$, $(x^4 + 5x^3)^2$.

Extra material: notes with solved Exercise 4.

27 How we expand the definition of powers to other exponents (than just the positive natural ones).

28 Zero powers and indeterminate forms.

Extra material: notes with the explanation why $a^0 = 1$ for all $a \neq 0$.

29 Powers with negative integer exponents.

Extra material: notes with the explanation why $a^{-n} = \frac{1}{a^n}$ for all $a \neq 0$ and $n \in \mathbb{N}^+$.

30 Powers with rational exponents.

Example: Show different ways of computing $25^{-3/2}$.

Extra material: notes with the explanation why $a^{1/n} = \sqrt[n]{a}$ for all $a > 0$ and $n \in \mathbb{N}^+ \setminus \{1\}$. Consequently, $a^{m/n} = \sqrt[n]{a^m}$ for each $m \in \mathbb{Z}$. (You could admit $a < 0$ under some restrictions, but we don't do this in this course; we did it, though, in Precalculus 2.)

31 **Advanced**: Existence of roots.

Extra material: An article with some theorems and proofs:

a) Theorem: Existence of roots. Each equation $x^n = a$ (where $a \geqslant 0$ and $n \in \mathbb{N}^+$) has a solution in $\mathbb{R}$.
b) Corollary 1: The equation $x^n = a$ (with $a > 0$ and $n \in \mathbb{N}^+$, $n \geqslant 2$) has exactly one positive solution. This solution is called the nth root of a.
c) Corollary 2: If the equation $x^n = a$ (with $a > 0$ and $n \in \mathbb{N}^+$, $n \geqslant 2$) has a negative solution, then this negative solution is unique, too.
d) Corollary 3: If n is odd, then the equation $x^n = a$ has exactly one solution.
e) Corollary 4: If n is even, then the equation $x^n = a$ has exactly two solutions: c and $-c$.
f) Corollary 5: If n is odd (and only then), then the Theorem also works if $a < 0$.

32 More about powers with rational exponents.

Discuss the following issues concerning powers with rational exponents:

a) Give an example of a trouble we can get into by allowing raising negative numbers to fractional exponents.
b) Let $a > 1$. Function $f : \mathbb{Q} \to \mathbb{R}$ defined as $f(x) = a^x$ has the property that $f(x) > 0$ for all $x \in \mathbb{Q}$. Show that $f(x) > 1$ for all $x \in \mathbb{Q}^+$; use this fact to motivate that the function is strictly increasing on $\mathbb{Q}$.
c) Let $0 < a < 1$. Use b) to motivate that $h(x) = a^x$ with $D_h = \mathbb{Q}$ is strictly decreasing on $\mathbb{Q}$. (Also $h(x) > 0$.)

Extra material: notes from the iPad.

33 Powers with real exponents; graphs of exponential functions.

34 Computing various powers, Problem 1.

Problem 1: Compute: $(-8)^{2/3} - 9^{-3/2}$, $\left(-\frac{32}{9}\right)^{-\frac{3}{5}}$, $\sqrt{(\sqrt{5} - 2\sqrt{5})^2 + (\sqrt{18} - \sqrt{8})^2}$.

Extra material: notes with solved Problem 1.

35 Computing various powers, Problem 2.

Problem 2: Express the following numbers as powers with integer bases:

$$5 \cdot 3^{7/2} - 6 \cdot 3^{5/2}, \qquad 3^{-3/4} \cdot 27^{2/3}.$$

Extra material: notes with solved Problem 2.

36 Computing various powers, Problem 3.

Problem 3: Express the following numbers as powers with integer bases:

$$\frac{15 \cdot 5^9 + 2 \cdot 5^{10}}{625 \cdot 5^6}, \qquad \frac{3^{-7} + \frac{2}{3} \cdot 3^{-6}}{3^{-5}}.$$

Extra material: notes with solved Problem 3.

37 Computing various powers, Problem 4.

Problem 4: Express number $\sqrt{5\sqrt[3]{5\sqrt{5}}}$ as power with an integer base.

Extra material: notes with solved Problem 4.

38 Computing various powers, Problem 5.

Problem 5: Which of the following numbers are integers:

$$1.\ 16^{0.25}, \qquad 2.\ 125^{-1/3}, \qquad 3.\ 81^{5/4}, \qquad 4.\ 0.01^{-0.5}.$$

Extra material: notes with solved Problem 5.

39 Computing various powers, Problem 6.

Problem 6: Number $\sqrt[3]{16} + \sqrt[3]{54} - \sqrt[3]{250}$ is:

a) rational,

b) irrational,

c) negative,

d) positive.

Extra material: notes with solved Problem 6.

40 Simplifying expressions, Problem 7.

Problem 7: Let $x > 0$. Compute $\dfrac{\sqrt[4]{x^9}}{\sqrt[3]{x^2}}$.

Extra material: notes with solved Problem 7.

41 Simplifying expressions, Problem 8.

Problem 8: Let $a, b, c \geqslant 0$. Simplify as far as you can:

$$\sqrt{ab^3} \cdot \sqrt[4]{a^2b^3} \cdot \sqrt[4]{bc^2}.$$

Extra material: notes with solved Problem 8.

42 Simplifying expressions, Problem 9.

Problem 9: Let $x, y > 0$. Determine p if

$$\sqrt{\frac{x}{y}\sqrt{\frac{y^3}{x^3}\sqrt{\frac{x^5}{y^5}}}} = \left(\frac{x}{y}\right)^p$$

Extra material: notes with solved Problem 9.

43 Simplifying expressions, Problem 10.

Problem 10: Which of the following equalities are true for all real numbers $a > 0$:

$$1.\ \sqrt{a}\cdot\sqrt[3]{a} = \sqrt[6]{a}, \qquad 2.\ \sqrt[3]{\sqrt{a}} = \sqrt[6]{a}, \qquad 3.\ \frac{\sqrt{a}}{\sqrt[3]{a}} = \sqrt[6]{a}, \qquad 4.\ \sqrt[3]{a^2}\cdot\sqrt[6]{a^3} = \sqrt[6]{a^2}.$$

Extra material: notes with solved Problem 10.

44 Simplifying expressions, Problem 11.

Problem 11: Let $a, b, c > 0$ and $x = \dfrac{(\sqrt{ab} - \sqrt{bc})^2}{abc}$. Then:

a) $x = \dfrac{1}{a} + \dfrac{1}{c}$,

b) $x = \dfrac{1}{a} + \dfrac{1}{c} - \dfrac{1}{\sqrt{ac}}$,

c) $x = \dfrac{1}{a} + \dfrac{1}{c} + \dfrac{1}{\sqrt{ac}}$,

d) another answer.

Extra material: notes with solved Problem 11.

45 Simplifying expressions, Problem 12.

Problem 12: Let $a, b, c > 0$ and $x = \dfrac{\sqrt[3]{ab\sqrt{c}} - a\sqrt[4]{b^2c}}{\sqrt[6]{a^3b^2c}}$. Then:

a) $x = \sqrt[6]{\dfrac{1 - a^4b\sqrt{c}}{a}}$,

b) $x = \dfrac{1 - \sqrt[3]{a^2\sqrt[4]{b^2c}}}{\sqrt[6]{a}}$,

c) $x = \dfrac{\sqrt[6]{a}}{\sqrt[6]{1 + a^4b\sqrt{c}}}$,

d) another answer.

Extra material: notes with solved Problem 12.

46 Simplifying expressions, Problem 13.

Problem 13: Compute $\dfrac{4^x}{8^y}$ given that $2x - 3y = 4$.

Extra material: notes with solved Problem 13.

47 Careful with signs, Problem 14.

Problem 14: If $a < 0$ and $x = \sqrt{\sqrt{\sqrt{a^2}}}$, then:

$$1.\ x = \sqrt[3]{a}, \qquad 2.\ x = \sqrt[4]{a}, \qquad 3.\ x = \sqrt[4]{-a}, \qquad 4.\ x \text{ is undefined.}$$

Extra material: notes with solved Problem 14.

48 Careful with signs, Problem 15.

Problem 15: If a is a real number and $f(x) = \sqrt{(x+a)^2} - \sqrt{(x-a)^2}$, then $f(0)$ is equal to:

a) 0, b) $2a$,

c) $-2a$, d) another answer.

Extra material: notes with solved Problem 15.

49 Careful with signs, Problem 16.

Problem 16: If b is a real number and $a = \sqrt{b^2}$, and $x > a$, then $\sqrt{x+a} - \sqrt{x-a}$ is equal to:

a) $\sqrt{x+b}-\sqrt{x-b}$,

b) $2\sqrt{a}$,

c) $|\sqrt{x+b}-\sqrt{x-b}|$,

d) another answer.

Extra material: notes with solved Problem 16.

50 Comparing numbers, Problem 17.

Problem 17: Compare the following numbers (which one is larger) without using any type of calculator:

$$2^{2000} \quad \text{and} \quad 10^{800}.$$

Extra material: notes with solved Problem 17.

51 Comparing numbers, Problem 18.

Problem 18: Compare the following numbers (which one is larger) without using any type of calculator:

$$\left(4^{\sqrt{3}}\right)^{\frac{\sqrt{3}}{2}} \quad \text{and} \quad \left[(3-\sqrt{2})^{\sqrt{3}}\cdot(3+\sqrt{2})^{\sqrt{3}}\right]^{\sqrt{3}}.$$

Extra material: notes with solved Problem 18.

52 Comparing numbers, Problem 19.

Problem 19: Which inequalities are true:

a) $\left(\frac{10}{21}\right)^{11} < \left(\frac{10}{21}\right)^{12}$,

b) $20^{-11} < 20^{-12}$,

c) $(\sqrt{2}-1)^{21} < (\sqrt{2}+1)^{-21}$,

d) $(\sqrt{2}-1)^{21} > (\sqrt{2}+1)^{-21}$.

Extra material: notes with solved Problem 19.

53 The charms of squares and conjugates, Problem 20.

Problem 20: Show that M is integer:

$$M = \frac{1}{\sqrt{1}+\sqrt{4}} + \frac{1}{\sqrt{4}+\sqrt{7}} + \frac{1}{\sqrt{7}+\sqrt{10}} + \ldots + \frac{1}{\sqrt{94}+\sqrt{97}} + \frac{1}{\sqrt{97}+\sqrt{100}}.$$

Extra material: notes with solved Problem 20.

54 The charms of squares and conjugates, Problem 21.

Problem 21: Simplify as far as you can:

$$\frac{\sqrt[4]{3}}{2} - \frac{1}{\sqrt[4]{3}+1}\cdot\frac{1}{\sqrt{3}+1}.$$

Extra material: notes with solved Problem 21.

55 The charms of squares and conjugates, Problem 22.

Problem 22: Show that M is integer:

$$M = \sqrt{3+2\sqrt{2}} + \sqrt{6-4\sqrt{2}}.$$

Extra material: notes with solved Problem 22.

56 The charms of squares and conjugates, Problem 23.

Problem 23: Show that M is integer:

$$M = \left(4 - 2 \cdot 3^{\frac{1}{2}}\right)^{\frac{1}{2}} - \left(4 + 2 \cdot 3^{\frac{1}{2}}\right)^{\frac{1}{2}}.$$

Extra material: notes with solved Problem 23.

57 The charms of squares and conjugates, Problem 24.

Problem 24: Show that M is integer:

$$M = \left(6 - 20^{\frac{1}{2}}\right)^{\frac{1}{2}} - \left(6 + 20^{\frac{1}{2}}\right)^{\frac{1}{2}}.$$

Extra material: notes with solved Problem 24.

58 The charms of squares and conjugates, Problem 25.

Problem 25: So that you don't get the (wrong) impression that numbers like in the previous videos always are integer, you get this one now: If $M = \sqrt{3 + 2\sqrt{2}} + \sqrt{3 - 2\sqrt{2}}$, then M is equal to:

1. $2\sqrt{2}$, 2. $2\sqrt{3}$, 3. 2, 4. another number.

Extra material: notes with solved Problem 25.

59 The charms of squares and conjugates, Problem 26.

Problem 26: Simplify as far as you can:

$$(3 + (2 \cdot 3^{\frac{1}{2}} - 2)(2 + 3^{\frac{1}{2}})^{\frac{1}{2}})^{\frac{1}{2}}.$$

Extra material: notes with solved Problem 26.

60 The cubes, Problem 27.

Problem 27: Show that M is integer:

$$M = \sqrt[3]{2 + \sqrt{5}} + \sqrt[3]{2 - \sqrt{5}}.$$

Extra material: notes with solved Problem 27.

61 The cubes, Problem 28.

Problem 28: Show that M is integer:

$$M = \sqrt[3]{9 + \sqrt{80}} + \sqrt[3]{9 - \sqrt{80}}.$$

Extra material: notes with solved Problem 28.

62 Squares and cubes, Problem 29.

Problem 29: Compare the following numbers (which one is larger) without using any type of calculator:

$$3^{12} - 2^{18} \quad \text{and} \quad 3^6 + 2^9.$$

Extra material: notes with solved Problem 29.

63 The cubes, Problem 30.

Problem 30: The number $\dfrac{1}{\sqrt[3]{5} - \sqrt[3]{4}}$ is equal to:

a) $\sqrt[3]{5} + \sqrt[3]{4}$,

b) $\sqrt[3]{5}+\sqrt[3]{4}+\sqrt[3]{20}$,

c) $\sqrt[3]{25}+2\sqrt[3]{2}+\sqrt[3]{20}$,

d) $\sqrt[3]{25}+\sqrt[3]{16}$.

Extra material: notes with solved Problem 30.

64 Squares and cubes, Problem 31.

Problem 31: Simplify the expression as far as you can:

$$\left(\frac{1}{8}a^{\frac{1}{3}}-b^{\frac{1}{3}}\right)\left(\frac{1}{512}a+b\right)\left(\frac{1}{64}a^{\frac{2}{3}}-\frac{1}{8}a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}\right)^{-1}$$

and compute its numerical value for $a=512$ and $b=\frac{1}{27}$.

Extra material: notes with solved Problem 31.

65 An equation, Problem 32.

Problem 32: Solve the following equation:

$$2^{2^{2^{2^{2}}}}=\left(2^{2^{2^{2}}}\right)^{x}.$$

Extra material: notes with solved Problem 32.

66 The last one, Problem 33.

Problem 33: Determine the domain of the following expression:

$$\left(\frac{4a+20}{\sqrt{a^2+10a+25}}\right)^{\frac{1}{2}}$$

and simplify it as far as you can.

Extra material: notes with solved Problem 33.

S4 Power functions, their properties and graphs

You will learn: about power functions and their properties: monotonicity for positive arguments, monotonicity for negative arguments, and how the curves $y=x^\alpha$ and $y=x^\beta$ are situated in relation to each other for various pairs of α and β (with some cool interactions between power functions and exponential functions); you will also get a glimpse into the world of derivatives, to illustrate the problem of monotonicity of $y=x^\alpha$ for any real non-zero α and positive arguments x, and to explain the cusps and rounding in some graphs.

67 A general introduction to power functions.

An observation: The product of power functions is a power function, but the sum of power functions does not need to be a power function.

Extra material: notes from the iPad.

68 Power functions with positive natural exponents.

An elementary proof of strict monotonicity (being increasing) of $f(x)=x^n$ for $n\in\mathbb{N}^+$ for positive arguments x, and an elementary proof of the way the curves $y=x^n$ for $n\in\mathbb{N}^+$ are situated in relation to each other for positive arguments x.

MANIM: $y=x^3,\ y=x^4,\ y=2x^3,\ y=2x^4,\ y=\frac{1}{2}x^3,\ y=\frac{1}{2}x^4,\ y=-2x^3,\ y=-2x^4,\ y=-\frac{1}{2}x^3,\ y=-\frac{1}{2}x^4$.

69 Power functions with negative integer exponents.

An elementary proof of strict monotonicity (being decreasing) of $f(x)=x^{-n}$ for $n\in\mathbb{N}^+$ for positive arguments x, and an elementary proof of the way the curves $y=x^{-n}$ for $n\in\mathbb{N}^+$ are situated in relation to each other for positive arguments x.

Examples: $y=\dfrac{1}{x},\ \ y=\dfrac{1}{x^2},\ \ y=\dfrac{1}{x^3},\ \ y=\dfrac{1}{x^4}$. Inverses to $f(x)=x^{-2}$ and $f(x)=x^{-5}$.

70 Power functions with positive fractional exponents.

An elementary proof of strict monotonicity (being increasing) of $f(x) = x^{1/n}$ for $n \in \mathbb{N}^+$ for positive arguments x, and an elementary proof of the way the curves $y = x^{1/n}$ for $n \in \mathbb{N}^+$ are situated in relation to each other for positive arguments x.

MANIM: $y = x^{1/2}$,
$y = x^{1/3}$, $y = x^{2/3}$, $y = x^{4/3}$, $y = x^{13/3}$,
$y = x^{1/4}$, $y = x^{3/4}$, $y = x^{5/4}$, $y = x^{9/4}$,
$y = x^{1/5}$, $y = x^{2/5}$, $y = x^{3/5}$, $y = x^{4/5}$, $y = x^{6/5}$, $y = x^{7/5}$, $y = x^{13/5}$.

71 Power functions with negative fractional exponents.

MANIM: $y = x^{1/2}$, $y = x^{-1/2}$
$y = x^{1/3}$, $y = x^{-1/3}$,
$y = x^{2/3}$, $y = x^{-2/3}$,
$y = x^{4/3}$, $y = x^{-4/3}$.

72 Power functions with real exponents.

MANIM: $y = x^2$, $y = x^3$, $y = x^e$, $y = x^\pi$, $y = x^{2.7}$, $y = x^{3.1}$, $y = x^{3.2}$.

73 **Advanced**: What is the derivative telling you about monotonicity and cusps.

Examples:

* The derivative of $f(x) = x^{2/3}$ is $f'(x) = \frac{2}{3}x^{-1/3}$, which is undefined in zero; this explains the cusp.
* The derivative of $f(x) = x^{4/3}$ is $f'(x) = \frac{4}{3}x^{1/3}$, which is defined in zero; this explains the mild rounding.
* Both first and second derivatives of $f(x) = x^{8/3}$ are equal to zero in zero; this explains the broad rounding.

Comparing powers with the same exponent: $90^6 > 80^6$, $90^{1/\pi} > 80^{1/\pi}$, $90^{0.01} > 80^{0.01}$, $90^{-60} < 80^{-60}$.

74 Power functions, Exercise 1.

Exercise 1: The graph of a power function passes through the points $(2, 16)$ and $(3, 54)$. Determine this function.

Extra material: notes with the solution of Exercise 1.

75 Power functions, Exercise 2.

Exercise 2: The graph of a power function passes through the points $(1, -2)$ and $(1/2, -8)$. Determine this function.

Extra material: notes with the solution of Exercise 2.

76 Power functions, Exercise 3.

Exercise 3: The graph of a power function passes through the points $(4, -6)$ and $(9, -9)$. Determine this function.

Extra material: notes with the solution of Exercise 3.

77 Typical graph transformations, Exercise 4.

Exercise 4: Plot step by step the following curves, starting with auxiliary graphs of appropriate power functions:

$$y = \tfrac{1}{2}(x-1)^3 - 2, \qquad y = \frac{2}{(x+2)^3} - 3, \qquad y = \frac{-3}{(x-1)^4} + 1 \qquad y = -\tfrac{1}{2}(x+1)^{\frac{1}{2}} + 3.$$

S5 Exponential functions, their properties and graphs

You will learn: about exponential functions $f(x) = a^x$ with $a > 1$, and $f(x) = a^x$ with $0 < a < 1$, and their various properties; by now you know how to plot these functions, and now you will perform transformations of the well-known graphs for plotting new functions: $g(x) = f(x) \pm c$, $g(x) = f(x \pm c)$, $g(x) = cf(x)$, etc.; you get a long problem-solving session in which various properties of exponential functions will be used; we will also illustrate some interesting interactions between exponential functions and power functions.

Read along with this section: **Precalculus book**: Section 6.1 (pp. 417–436; omit the parts about logarithms for now).

78 Properties of exponential functions.

Example: Determine (in each of the four cases separately) whether the number x is positive or negative if:

$$(1)\ (0.8)^x = \frac{1}{3}, \qquad (2)\ (2.7)^x = \frac{3}{4}, \qquad (3)\ \left(\frac{2}{9}\right)^x = 6, \qquad (4)\ \left(3\frac{1}{3}\right)^x = 15.$$

79 Comparing powers with the same base, Problem 1.

Problem 1: Order the following numbers from the least to the greatest: $5^{\sqrt{3}}$, 5^{π}, $5^{3.1}$, $5^{-\sqrt{2}}$, $5^{\sqrt{6}}$.

Extra material: notes with solved Problem 1.

80 Comparing powers with the same base, Problem 2.

Problem 2: Order the following numbers, sarting with the least: $(1/\sqrt{5})^{3\sqrt{2}}$, $(1/\sqrt{5})^{1/\sqrt{2}}$, $(1/\sqrt{5})^{1+\sqrt{3}}$, $(1/\sqrt{5})^{\pi}$.

Extra material: notes with solved Problem 2.

81 Properties of exponential functions, Problem 3.

Problem 3: If $a > 1$, then the inequality $a^{x^3} > \left(\frac{1}{a}\right)^{x^2-1}$ has the same solution set as:

a) $x^3 - x^2 + 1 > 0$,

b) $x^3 + x^2 - 1 > 0$,

c) $x^3 - x^2 - 1 > 0$,

d) none of the above.

Extra material: notes with solved Problem 3.

82 Properties of exponential functions, Problem 4.

Problem 4: If $a > 0$ and the inequalities $a^{x^3} > \left(\frac{1}{a}\right)^{x+1}$ and $x^3 + x + 1 < 0$ have the same solution sets, then we can draw the conclusion that:

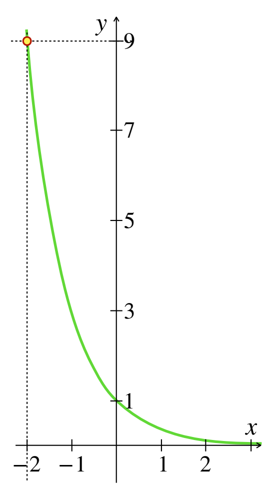
1. $a > 1$, 2. $a = 1$ 3. $a < 1$, 4. none of the above.

Extra material: notes with solved Problem 4.

83 Exponential functions, Problem 5.

Problem 5: Given the graph to an exponential function (see the picture), determine the formula for the function, and compute the value of $g(x) = f(-x)$ for the argument $x = -0.5$.

Extra material: notes with solved Problem 5.



84 Exponential functions, Problem 6.

Problem 6: Determine the domain of

$$f(x) = \frac{\sqrt{8-2^x}}{\sqrt{-x^2+3x}}.$$

Extra material: notes with solved Problem 6.

85 Exponential functions, Problem 7.

Problem 7: Show that the function $f(x) = \dfrac{7^{2x}-1}{7^{x+1}}$ is odd.

Extra material: notes with solved Problem 7.

86 Exponential functions, a trick to remember, Problem 8.

Problem 8: Given that the function $f(x) = 25^x + 25^{-x}$ has value 34 in some argument x_0, determine the value $g(x_0)$ where $g(x) = 5^x + 5^{-x}$.

Extra material: notes with solved Problem 8.

87 Graphs of exponential functions, and their transformations, Problem 9.

Problem 9: Plot the following curves, starting with auxiliary graphs of appropriate exponential functions:

$$y = 2^x - 1, \quad y = \left(\frac{1}{3}\right)^{x-1}, \quad y = 3^{-x} + 2, \quad y = 10^{(x+1)/2} - 20, \quad y = 8 - e^{-x}, \quad y = 10e^{-0.1x}.$$

88 Exponential functions, another trick to remember, Problem 10.

Problem 10: Determine the range of $f(x) = -9^x - 2 \cdot 3^x + 8,\ x \in \mathbb{R}$.

Extra material: notes with solved Problem 10.

89 Exponential functions, Problem 11.

Problem 11: Determine the range of $f(x) = 49^x - 7^x - 6,\ x \in \mathbb{R}$.

Extra material: notes with solved Problem 11.

90 Exponential functions, Problem 12.

Problem 12: Determine the range of $f(x) = \left(\dfrac{1}{2}\right)^{x^2-4x},\ x \in [1,\ 4.5]$.

Extra material: notes with solved Problem 12.

91 Exponential functions, Problem 13.

Problem 13: Determine the range of $f(x) = \left(\dfrac{\sqrt{5}}{5}\right)^{x^2+2x-4},\ x \in [0,\ 2]$.

Extra material: notes with solved Problem 13.

92 Exponential functions, Problem 14.

Problem 14: Plot

$$f(x) = 2^{|x-1|-|x|},\ x \in \mathbb{R}.$$

Extra material: notes with solved Problem 14.

93 Exponential functions, Problem 15.

Problem 15: Plot $f(x) = \left(\dfrac{1}{2}\right)^{|x+2|+|x|},\ x \in \mathbb{R}$.

Extra material: notes with solved Problem 15.

94 Exponential functions, Problem 16.

Problem 16: Plot $f(x) = |-2^{x-5}+1|,\ x \in \mathbb{R}$.

95 Graphical methods for solving equations, Problem 17.

Problem 17: Equation $2^x = \cos x$ has:

a) exactly one solution,

b) infinitely many solutions,

c) no positive solutions,

d) exactly two solutions.

96 How graphs of exponential functions relate to each other.

97 Interactions between power functions and exponential functions.

S6 Important properties of strictly monotone functions

You will learn: about important properties of strictly monotone functions, which will help us understand exponential functions, logarithmic functions as inverses to exponential functions, and solve exponential and logarithmic equations.

98 An adapted definition of strictly monotone functions.

The simplest definition, and a less intuitive (but useful) definition of strictly monotone functions.

99 Invertibility of strictly monotone functions.

Strictly monotone functions are invertible, and their inverses are also strictly monotone (in the same way as the original functions). Functions $x \mapsto x^n$ and $x \mapsto x^{1/n}$ ($n \in \mathbb{N}^+$) are each other's inverses ($x \geqslant 0$ if n is even).

Exercise: Prove in an elementary way (not using the theorem proven in this video) that $f(x) = \sqrt{x}$ is strictly increasing on its entire domain.

Extra material: notes from the iPad.

100 Reciprocals of strictly monotone positive (or negative) functions.

If the function $f(x)$ is monotone on the set $X \subset \mathbb{R}$, and keeps the same sign over X (i.e., $f(x) > 0$ for all $x \in X$, or $f(x) < 0$ for all $x \in X$), then the reciprocal function $g(x) = \dfrac{1}{f(x)}$ has opposite (compared to f) monotonic behaviour on X.

Examples: $y = e^x$ and $y = e^{-x}$; couples of power functions from Video 71: $y = x^{2/3}$ and $y = x^{-2/3}$, $y = x^{1/3}$ and $y = x^{-1/3}$.

Extra material: notes from the iPad.

101 Scalings of strictly monotone functions.

A scaling of a strictly increasing (decreasing) function has the same (monotonic) property (as the original function) if the scalar is positive, opposite (monotonic) property if the scalar is negative.

102 Compositions of strictly monotone functions.

* The composition of two increasing or both decreasing functions is an increasing function. Examples: $h_1(x) = \sqrt{x-4}$, $h_2(x) = (4-x)^{-1/2}$.
* The composition of an increasing and a decreasing function (or: a decreasing and an increasing function) is a decreasing function. Example: $h_3(x) = \sqrt{4-x}$.

Example: Back to the problems from Video 87: $f(x) = e^{0.01x}$ and $f(x) = e^{-0.01x}$.

Example: Back to the problems from Video 88: $f(x) = -9^x - 2 \cdot 3^x + 8$ and from Video 89: $f(x) = 49^x - 7^x - 6$.

Extra material: notes from the iPad.

103 Sums of strictly monotone functions.

The sum of two strictly increasing (decreasing) functions has the same (monotonic) property.

Exercise: Determine the domain, and the type of monotonic behaviour of the function: $f(x) = \frac{1}{x+3} - \sqrt{x+1}$.
Extra material: notes from the iPad.

104 Products of strictly monotone positive functions.

The product of strictly increasing (decreasing) positive functions is also an increasing (decreasing) function.

Extra material: notes from the iPad.

105 **Optional**: Solving equations involving strictly monotone functions.

If the function $y = f(x)$ is strictly increasing or strictly decreasing on the set I, and $d \in \mathbb{R}$, then the equation $f(x) = d$ has **at most one** solution in I.

106 **Optional**: Solving equations involving strictly monotone functions, Problem 1.

Problem 1: Solve the equation $\sqrt{2x-3} + \sqrt{4x+1} = 4$.

Extra material: notes with solved Problem 1.

107 **Optional**: Solving equations involving strictly monotone functions, Problem 2.

Problem 2: Solve the equation $(\sqrt{2-\sqrt{3}})^x + (\sqrt{2+\sqrt{3}})^x = 2^x$.

Extra material: notes with solved Problem 2.

108 **Optional**: Solving equations involving strictly monotone functions, Problem 3.

Problem 3: Solve the equation $x^2 + |x| + \sqrt{x} + 2x = 111$.

Extra material: notes with solved Problem 3.

109 **Optional**: Solving equations involving strictly monotone functions, Problem 4.

Problem 4: Solve the equation $\sqrt[8]{x+1} + \sqrt[8]{x-1} = \sqrt[8]{2}$.

Extra material: notes with solved Problem 4.

S7 Logarithmic functions as inverses to exponential functions

You will learn: the definition and properties of logarithms with various bases; properties of logarithmic functions, their graphs, and graphs of some related functions obtained by transformations of graphs of basic logarithmic functions.

Read along with this section: **Precalculus book**: Section 6.1 (pp. 417–436), Section 6.2 (pp. 437–447).

110 A general introduction to logarithms.

111 Inverse operations to taking powers.

112 Taking logarithms as inverse operation to raising to a power; cancelling exponentials.

Cancelling exponentials: If $a > 0, a \neq 1$, then $\log_a a^x = x$, $\; a^{\log_a x} = x$. The second equality only for $x > 0$.

Seven basic examples from the book: page 424.

113 Logarithms, some exercises.

Plenty of basic exercises from the book: page 429.

$$\log_{36} 216, \quad \log_{\frac{1}{6}} 216, \quad 36^{\log_{36} 216}, \quad \log \sqrt[9]{10^{11}}, \quad \ln \frac{1}{\sqrt{e}}, \quad \log e^{\ln 100}, \quad \log_2 3^{-\log_3 2}, \quad \ln 42^{6 \log 1}.$$

Extra material: notes with the solution to selected exercises from page 429.

114 Logarithmic functions and their properties.

Cancelling exponentials in terms of compositions of $f(x) = a^x$ and $f^{-1}(x) = \log_a x$, where $a > 0, a \neq 1$: $f^{-1} \circ f = \mathrm{Id}_{\mathbb{R}}$, $\; f \circ f^{-1} = \mathrm{Id}_{\mathbb{R}^+}$.

115 Three cool properties.

If $a > 0$, $a \neq 1$ and $x > 0$, then: $\log_{1/a} x = -\log_a x$, $\quad \log_a \frac{1}{x} = -\log_a x$, $\quad \log_{1/a} \frac{1}{x} = \log_a x$.

Extra material: notes from the iPad.

116 Arrangement of the curves.

117 How to determine the signs of logarithms.

Example: Determine whether the following numbers are positive or negative (compare to V78):

$$(1)\ \log_{0.8}\frac{1}{3}, \qquad (2)\ \log_{2.7}\frac{3}{4}, \qquad (3)\ \log_{\frac{2}{9}} 6, \qquad (4)\ \log_{3\frac{1}{3}} 15.$$

118 The product rule for logarithms.

Let $a > 0;\ a \neq 1;\ x, y > 0$. Then $\log_a(xy) = \log_a x + \log_a y$.

Extra material: notes from the iPad.

119 The quotient rule for logarithms.

Let $a > 0;\ a \neq 1;\ x, y > 0$. Then $\log_a(\frac{x}{y}) = \log_a x - \log_a y$.

Extra material: notes from the iPad.

120 The power rule for logarithms, a historical note, and plenty of exercises.

Let $a > 0;\ a \neq 1;\ x > 0,\ \alpha \in \mathbb{R}$. Then $\log_a x^\alpha = \alpha \log_a x$.

Evaluate $\sqrt[3]{84.9 \cdot (1.04)^2}$ by taking the logarithm of this expression and using appropriate computational rules.

Plenty of basic exercises from the book: Section 6.2 (pages 437–447).

Proof 1 of the formula $\log_{n^2} x = \log_n \sqrt{x}$ for $x > 0$ and $n > 0, n \neq 1$.

Extra material: notes from the iPad.

121 The base change rule for logarithms, and for powers.

Let $a, b > 0;\ a, b \neq 1;\ x > 0$. Then $\log_b x = \log_b a \cdot \log_a x$ and $b^x = a^{x \log_a b}$.

Proof 2 of the formula $\log_{n^2} x = \log_n \sqrt{x}$ for $x > 0$ and $n > 0, n \neq 1$.

Extra material: notes from the iPad.

122 The base switch rule for logarithms; more practice problems from the book.

Let $a, b > 0;\ a, b \neq 1$. Then $\log_b a = (\log_a b)^{-1}$.

Extra material: notes from the iPad.

123 Typical graph transformations.

Example: Plot the following curves, starting with auxiliary graphs of appropriate logarithmic functions:

$y = \log_2(x+1), \quad y = \log_{\frac{1}{3}} x + 1, \quad y = -\log_3(x-2), \quad y = 2\log(x+20) - 1, \quad y = -\ln(8-x), \quad y = -10\ln\frac{x}{10}$.

124 Logarithmic functions; exercises from the book.

Exercise: Compare and contrast the graphs of $y = \ln x^2$ and $y = 2\ln x$.

125 Logarithms, Problem 1.

Problem 1: Number $\log_3 2$ is:

1. negative, 2. positive, 3. greater than 1, 4. rational.

Theorem: If $m, n \in \mathbb{N}$ and $m, n > 1$, and $\gcd(m, n) = 1$, then $\log_m n$ is irrational.

126 Logarithms, Problem 2.

Problem 2: Compute:

$$\log_{\frac{1}{2}} \frac{32}{\sqrt[4]{8}}.$$

Extra material: notes with solved Problem 2.

127 Logarithms, Problem 3.

Problem 3: Compute the geometric mean of the numbers $x, y, z > 0$ such that $\log_2 x = 3,\ \log_4 y = \log_4 \sqrt{z} = 2$.

Extra material: notes with solved Problem 3.

128 Logarithms, Problem 4.

Problem 4: Show that the product mn, where:

$$m = \log_3 \log_3 \sqrt[3]{\sqrt[3]{\sqrt[3]{3}}}, \qquad n = \log_2 \log_2 \sqrt[4]{\sqrt{2}}$$

is the square of some natural number.

Extra material: notes with solved Problem 4.

129 Logarithms, Problem 5.

Problem 5: Define the numbers a and b in the following way: $a = \log 3,\ b = \log 2$. Express the numbers $\log 108$, $\log 5$, and $\log_9 20$ in terms of a and b.

Extra material: notes with solved Problem 5.

130 Logarithms, Problem 6.

Problem 6: Show that:

a) if $a = \log_3 2$ and $b = \log_3 7$ then $\log_{81} 84 = \dfrac{2a+b+1}{4}$,

b) if $a = \log_6 5$ and $b = \log_6 2$ then $\log_3 10 = \dfrac{a+b}{1-b}$.

Extra material: notes with solved Problem 6.

131 Logarithms, Problem 7.

Problem 7: Let $\log_6 2 = a$. Then:

1. $\log_3 2 = \frac{a}{1-a}$, 2. $\log_2 3 = \frac{a}{1+a}$, 3. $\log_6 3 = 1-a$, 4. $\log_3 6 = a-1$.

Extra material: notes with solved Problem 7.

132 Logarithms, Problem 8.

Problem 8: If $e^x = \sqrt{2}-1$, then x is equal to:

1. $\ln \frac{1}{\sqrt{2}+1}$, 2. $\frac{1}{2}\ln 2 - 1$, 3. $\frac{1}{2}\ln 2$, 4. another number.

Extra material: notes with solved Problem 8.

133 Logarithms, Problem 9.

Problem 9: If $a = \ln(\sqrt{2}-1) - \ln \frac{1}{\sqrt{2}+1}$, then:

1. $a < 0$, 2. $a = 0$, 3. $a > 0$, 4. such number does not exist.

Extra material: notes with solved Problem 9.

134 Logarithms, Problem 10.

Problem 10: If $x > y > 0$, then:

a) $\ln(x-y) = \ln x - \ln y$,

b) $\ln(x^2 - y^2) = \ln(x+y) + \ln(x-y)$,

c) $\ln \frac{x}{y} = 1 - \ln \frac{y}{x}$,

d) none of the above.

Extra material: notes with solved Problem 10.

135 Logarithms, Problem 11.

Problem 11: If $\ln a + \ln b + \ln c = 0$, then:

a) $abc = 0$,

b) $a+b+c=1$,

c) $a+b+c=0$,

d) none of the above.

Extra material: notes with solved Problem 11.

136 Logarithms, Problem 12.

Problem 12: Let $a=\log_3 10, b=\log_5 10, c=\log_{15} 10$. The sum of the reciprocals of a, b, and c is:

1. integer, 2. greater than 2, 3. not less than 3, 4. not greater than 3.

Generalised product rule: If $x,y,z>0$, and $a>0,\ a\neq 1$, then $\log_a(xyz)=\log_a x+\log_a y+\log_a z$.

Extra material: notes with solved Problem 12 and with a proof of the generalised product rule.

137 Logarithms, Problem 13.

Problem 13: The number $\log_2 3\cdot\log_3 4\cdot\log_4 5\cdot\log_5 6\cdot\log_6 7\cdot\log_7 8$ is:

a) irrational,

b) integer,

c) a square of a natural number,

d) less than 5.

Determine $\log_3 2\cdot\log_4 3\cdot\log_5 4\cdot\log_6 5\cdot\log_7 6\cdot\log_8 7$.

Extra material: notes with solved Problem 13.

138 Logarithms, Problem 14.

Problem 14: Number $2^{\log_3 5}-5^{\log_3 2}$ is:

1. positive, 2. integer, 3. equal to 3, 4. equal to 0.

Theorem: If $x,y,z\in\mathbb{R}^+\setminus\{1\}$, then the numbers $x^{\log_y z}$ and $z^{\log_y x}$ are equal to each other.

Extra material: notes with solved Problem 14 and with a proof of the Theorem.

139 Logarithms, Problem 15.

Problem 15: For all real numbers $x\neq 0$:

a) $\ln|x|+\frac{1}{\ln|x|}=0$,

b) $\ln|\frac{1}{x}|+\ln|x|=0$,

c) $\ln x^2+2\ln(-x)=0$,

d) none of the above.

Extra material: notes with solved Problem 15.

140 Logarithms, Problem 16.

Problem 16: If the function f is defined by

$$f(x)=\frac{1}{\ln(\ln(x^2-1))}$$

then its domain contains all such x that:

1. $|x|>0$, 2. $|x|>1$, 3. $|x|>\sqrt{2}$, 4. none of the above.

Extra material: notes with solved Problem 16.

141 Logarithms, Problem 17.

Problem 17: Determine the largest integer for which the following function is defined:

$$f(x) = \ln\left(\frac{2}{3} - 3^x\right).$$

Extra material: notes with solved Problem 17.

142 Logarithms, Problem 18.

Problem 18: The graphs of $y = \log(x-1)$ and $y = \log(1-x)$ are symmetrical to each other along:

a) the diagonal line $y = x$,

b) the x axis,

c) the line $x = 1$,

d) the y-axis.

Extra material: notes with solved Problem 18.

143 Logarithms, Problem 19.

Problem 19: If $f(x) = \ln(x + \sqrt{x^2+1})$, then:

a) f is defined for all $x \in \mathbb{R}$,

b) f is strictly increasing for $x > 0$,

c) $f(0) = 0$,

d) f is odd, and thus also increasing for $x < 0$.

You will get to see the graph of this function, together with the graph of $g(x) = \ln(2x)$ for comparison; the derivatives of both functions will also be plotted. You will see that f is strictly increasing for all $x \in \mathbb{R}$.

Extra material: notes with solved Problem 19.

144 Logarithms, Problem 20.

Problem 20: Show that the function $f(x) = x \cdot \log \dfrac{x+3}{x-3}$ is even.

Extra material: notes with solved Problem 20.

145 Logarithms, Problem 21.

Problem 21: Let $x > 0, x \neq 1$, and $n \in \mathbb{N}^+ \setminus \{1\}$. Simplify $\dfrac{1}{\log_2 x} + \dfrac{1}{\log_3 x} + \ldots + \dfrac{1}{\log_n x}$.

Extra material: notes with solved Problem 21.

146 Logarithms, Problem 22.

Problem 22: Let $x > 0, x \neq 1$. Simplify: $x^{\frac{1}{\log_2 x}} \cdot x^{\frac{1}{\log_4 x}} \cdot \ldots \cdot x^{\frac{1}{\log_{512} x}}$.

Generalised product rule: If $x, y, z \in \mathbb{R}$, and $a > 0$, then $a^x \cdot a^y \cdot a^z = a^{x+y+z}$.

Extra material: notes with solved Problem 22.

147 Logarithms, Problem 23.

Problem 23: Order the numbers from the least to the greatest: $\log_5 6,\ \log_2 81, \log_3 4,\ \log_{1/2} 3,\ \log_2 3,\ \log_5 \sqrt{6}$.

Theorem: If $n \in \mathbb{N}$ and $n \geqslant 3$, then $\log_{n-1} n > \log_n (n+1)$.

Extra material: notes with solved Problem 23 and a proof of the Theorem.

148 Logarithms, Problem 24.

Problem 24: Compute $\log_{\frac{\sqrt{2}}{2}} \dfrac{y}{x}$ if $x = \dfrac{\log_6^3 4 + \log_6^3 9}{\log_6^2 2 - \log_6 2 \cdot \log_6 3 + \log_6^2 3}$ and $y = \log_{\frac{3}{7}}^2 21 - \log_{\frac{3}{7}} 81 \cdot \log_{\frac{3}{7}} 7$.

Extra material: notes with solved Problem 24.

S8 Exponential equations and inequalities

You will learn: how to solve exponential equations and inequalities, starting with some simple ones, ending with some more complex examples; determining inverse functions by solving exponential equations.

Read along with this section: **Precalculus book**: Section 6.3 (pp. 448–458). **Attention**: Equations containing **radicals** (roots) are **not** discussed (systematically) in this course, as they are typically a topic of College Algebra. If you need them, you can read about them in **Precalculus Prerequisites** book in Section 0.9 (pages 111–125); you can always ask me questions if you need some guidance. Read also section 5.3 (*Other algebraic functions*) on pages 397–415 in the **Precalculus book**.

149 What is an exponential equation or inequality?

150 Strict monotonicity of exponential functions is the key.

151 We have already solved some exponential equations.

152 We have even solved some exponential inequalities.

153 Solving equations by performing inverse operations in reverse.

Three relevant examples:

E1 Inverse operations in solving simple linear equations like $4x + 2 = 14$.

E2 Inverse operations in solving simple exponential equations like $4 \cdot 2^x + 2 = 14$.

E3 Inverse operations in solving simple exponential equations like $2^{4x+2} = 14$.

154 I finally got it: the meaning of the word logarithm.

155 Some basic types of exponential equations, and how to treat them.

156 Exponential equations, Problem 1.

Problem 1: Solve the equations:

$$\sqrt[3]{2^x} = 128, \qquad \left(\frac{1}{3}\right)^{x^2-4x+4.5} = \sqrt{\frac{1}{3}}, \qquad \left(\frac{5}{6}\right)^x = \left(\frac{36}{25}\right)^2.$$

Extra material: notes with solved Problem 1.

157 Exponential equations, Problem 2.

Problem 2: Solve the equations:

$$\left(\frac{1}{2}\right)^{\frac{x-1}{x+2}} = 4, \qquad 4^{x^2} = \left(\frac{1}{2}\right)^{x-1}.$$

Extra material: notes with solved Problem 2.

158 Exponential equations, Problem 3.

Problem 3: Solve the equation:

$$e^{7x-9} = e^{x^2-x-29}.$$

Extra material: notes with solved Problem 3.

159 Exponential equations, Problem 4.

Problem 4: Solve the equation:

$$\frac{3^x + 2^x}{3^x - 2^x} = 7.$$

Extra material: notes with solved Problem 4.

160 Exponential equations, Problem 5.

Problem 5: Solve the equation: $9^x - 5 \cdot 3^x + 6 = 0$.

Extra material: notes with solved Problem 5.

161 Exponential equations, Problem 6.

Problem 6: Solve the equation: $e^{6x} - e^{3x} - 6 = 0$.

Extra material: notes with solved Problem 6.

162 Exponential equations, Problem 7.

Problem 7: Solve the equation: $9^x - 6^x - 2^{2x+1} = 0$.

Extra material: notes with solved Problem 7.

163 Exponential equations, Problem 8.

Problem 8: Solve the equation: $\dfrac{8^x + 5 \cdot 4^x}{4 - 2^x} = 2$.

Extra material: notes with solved Problem 8.

164 Exponential equations, Problem 9.

Problem 9: Solve the equation: $0.125 \cdot 16^{2x-1} = (\sqrt[3]{0.25})^{3x-6} \cdot 8^{x-1}$.

Extra material: notes with solved Problem 9.

165 Exponential equations, Problem 10.

Problem 10: Solve the equation: $\left(\sqrt{5+2\sqrt{6}}\right)^x + \left(\sqrt{5-2\sqrt{6}}\right)^x = 10$.

Note: Exactly the same method would work for:

$$\left(\sqrt{3+2\sqrt{2}}\right)^x + \left(\sqrt{3-2\sqrt{2}}\right)^x = 6, \quad \left(\sqrt{4+\sqrt{15}}\right)^x + \left(\sqrt{4-\sqrt{15}}\right)^x = 8,$$

$$\left(\sqrt{6+\sqrt{35}}\right)^x + \left(\sqrt{6-\sqrt{35}}\right)^x = 12, \quad \left(\sqrt{7+4\sqrt{3}}\right)^x + \left(\sqrt{7-4\sqrt{3}}\right)^x = 14, \quad \text{etc.}$$

Generalisation: If $a > 1$ (or $0 < b < 1$) and $c \in \mathbb{R}$, then you can determine the number of solutions to $a^x + a^{-x} = c$ (or to $b^x + b^{-x} = c$) for each c, following the substitution method.

Extra material: notes with solved Problem 10.

166 Exponential equations, Problem 11.

Problem 11: Solve the equations: $3 \cdot 4^x + \frac{1}{3} \cdot 9^{x+2} = 6 \cdot 4^{x+1} - \frac{1}{2} \cdot 9^{x+1}$, $\quad \dfrac{4^{(x-1)/2} \cdot 5}{5^x} = \frac{1}{9} \cdot 3^{2x}$.

Extra material: notes with solved Problem 11.

167 Finding inverses to functions defined by exponentials, Problem 12.

Problem 12 (Exercises 71 and 73 on page 430 in the Precalculus book): Find inverse functions to $f(x) = 3^{x+2} - 4$ and to $f(x) = -2^{-x} + 1$.

Extra material: notes with solved Problem 12.

168 More about inverses, Problem 13.

Problem 13 (Problem 48 on page 457 in the Precalculus book): In Example 6.3.4 (pages 454–455), we found that the inverse of $f(x) = \dfrac{5e^x}{e^x + 1}$ was $f^{-1}(x) = \ln\left(\dfrac{x}{5-x}\right)$ but we left a few loose ends for you to tie up.

1. Show that $\left(f^{-1} \circ f\right)(x) = x$ for all x in the domain of f and that $\left(f \circ f^{-1}\right)(x) = x$ for all x in the domain of f^{-1}.
2. Find the range of f by finding the domain of f^{-1}.
3. Let $g(x) = \dfrac{5x}{x+1}$ and $h(x) = e^x$. Show that $f = g \circ h$ and that $(g \circ h)^{-1} = h^{-1} \circ g^{-1}$ which is an illustration of **Theorem** (proven in **Precalculus 1: Basic notions** in Video 189): Composition $f = g \circ h$ of two invertible functions g and h is invertible, and $f^{-1} = (g \circ h)^{-1} = h^{-1} \circ g^{-1}$.

Extra material: notes with solved Problem 13.

169 The inverse to a very special function, Problem 14.

Problem 14: Find the inverse function to $f(x) = \dfrac{e^x - e^{-x}}{2}$. Look back at Video 143. Look forward to Section 11 (Video 221).

Extra material: notes with solved Problem 14.

170 Graphical and analytical solutions of functional inequalities.

Some old examples:

E1 Solve graphically the inequalities: $-9^x - 2\cdot 3^x + 8 < 5$, $-9^x - 2\cdot 3^x + 8 \geqslant 8$ (see V88).

E2 Solve graphically the inequality $\left(\dfrac{1}{2}\right)^{x^2-4x} < 8$ (see V90).

E3 Solve graphically the inequality $f(x) = |-2^{x-5}+1| \geqslant 1$ (see V94).

Example: Two ways of approaching simple inequalities: Solve the inequality: $e^{3x-1} > 6$ applying two methods (monotonicity of exponential functions, and monotonicity of logarithmic functions).

Extra material: notes with solved Example.

171 Exponential inequalities, Problem 15.

Problem 15: Solve the inequality: $0.5^{x^2-3} < \left(\dfrac{1}{4}\right)^x$.

Extra material: notes with solved Problem 15.

172 Exponential inequalities, Problem 16.

Problem 16: Solve the inequalities: $2^{x+1} + 4^x \leqslant 80$ and $3^{-3x} - 9\cdot 3^{-2x} - (\frac{1}{3})^x + 9 > 0$.

Extra material: notes with solved Problem 16.

173 Exponential inequalities, Problem 17.

Problem 17: Solve the inequality: $8\cdot 3^{\sqrt{x}+\sqrt[4]{x}} + 9^{\sqrt[4]{x}+1} > 9^{\sqrt{x}}$.

Extra material: notes with solved Problem 17.

S9 Logarithmic equations and inequalities

You will learn: how to solve logarithmic equations and inequalities, starting with some simple ones, ending with some more complex examples; determining inverse functions by solving logarithmic equations.

Read along with this section: **Precalculus book**: Section 6.4 (pp. 459–468).

174 What is a logarithmic equation or inequality, and how to deal with them.

E1 Empty domain: $\ln x = \ln(-x)$.

E2 Non-empty domain, a solution exists: $\ln(2x) = \ln(x+1)$.

E3 Non-empty domain, no solutions: $\ln x = \ln(x^2+1)$.

E4 Non-empty domain, two solutions: $\ln x = \ln(x^2+1) - 1$.

175 Simple logarithmic equations and inequalities, Problem 1.

Problem 1:

E1 Solve the equation: $\log_3(2x-1) = 2$.

E2 Solve the equation: $\log_{x-3} 16 = 2$.

E3 Solve the equation $\ln(7x-9) = \ln(x^2-x-29)$. (Compare to Problem 3 in V158.)

I1 Solve the inequality $\ln(2x+1) < 3$, using monotonicity of $f(t) = e^t$ and a cancellation property, or using monotonicity of logarithmic functions and a cancellation property.

Extra material: notes with solved Problem 1.

176 Finding inverse functions to functions defined by logarithms, Problem 2.

Problem 2 (Exercises 72 and 74 on page 430 in the Precalculus book): Find inverse functions to $f(x) = \log_4(x-1)$ and to $f(x) = 5\log x - 2$. (Compare to Problem 12 in V167.)

Extra material: notes with solved Problem 2.

177 Logarithmic equations, Problem 3.

Problem 3: Solve the equation: $\ln(x+1) - \ln(x-1) = \ln x$.

Extra material: notes with solved Problem 3.

178 Logarithmic equations, Problem 4.

Problem 4: Solve the equation: $2\ln(e^x - 1) = \ln 2 + \ln(e^x + 3)$.

Extra material: notes with solved Problem 4.

179 Logarithmic equations, Problem 5.

Problem 5: Solve the equation:

$$\log_{0.5}\left(\log_6 \frac{x^2+x}{x+4}\right) = 0.$$

Extra material: notes with solved Problem 5.

180 Logarithmic equations, Problem 6.

Problem 6: Solve the equation:

$$\frac{\log(\sqrt{x-1}+1)}{\log(\sqrt{x-1}+7)} = \frac{1}{2}.$$

Extra material: notes with solved Problem 6.

181 Logarithmic equations, Problem 7.

Problem 7: Solve the equation: $\log_x 64 - 6\log_4 x + 7 = 0$.

Extra material: notes with solved Problem 7.

182 Logarithmic inequalities, Problem 8.

Problem 8: Solve the inequality: $\log_2(x^2 - x) < \log_2 12 - \log_2 6$.

Extra material: notes with solved Problem 8.

183 Logarithmic inequalities, Problem 9.

Problem 9: Solve the inequality:

$$\frac{\log_3 x + 1}{\log_3 x - 1} \leqslant \log_3 9.$$

Extra material: notes with solved Problem 9.

184 Logarithmic inequalities, Problem 10.

Problem 10: Solve the inequality:

$\log_x(x+2) > \log_{1/x} 0.25$.

Another proof of a formula from V115:

If $a > 0$, $a \neq 1$ and $x > 0$, then $\log_{1/a} \frac{1}{x} = \log_a x$.

Extra material: notes with solved Problem 10.

185 Logarithmic inequalities, Problem 11.

Problem 11: Solve the inequality:

$$\log_3 x + \log_{\sqrt{3}} x + \log_{1/3} x < 6.$$

Extra material: notes with solved Problem 11.

S10 Applications of exponential and logarithmic functions

You will learn: changes in percent, and the *change factor* in growth and decay; compound interests and how annuities relate to geometric series; some applications of exponential and logarithmic functions, for example for analysis of growth or decay; logarithmic scale. **This section is different than the earlier sections**, because you will mostly read about the topics (which are very *language intensive*) from the Precalculus book; I will introduce each topic in the videos, and you are welcome to ask questions on QA (under the corresponding videos, so that all the other students can find them on the right place) if you need my assistance.

Read along with this section: **Precalculus book**: entire Chapter 6, but particularly Section 6.5 (pp. 469–493); Chapter 9 (pay particular attention to Section 9.2 on pages 666–669, with exercises on page 671).

186 A different section: more reading from the book.

187 Growth, decay, percent, and change factors.

Example 1: After a price increase of 10% followed by a price decrease of 10%, the final price is:

a) the same as in the beginning,

b) higher than in the beginning,

c) lower than in the beginning,

d) one cannot say.

Example 2: The length of one side of a rectangle has increased by 10%, and the length of the second side of the rectangle has increased by 30%. The area of this rectangle has increased by:

1. 40%, 2. 30%, 3. 143%, 4. none of the above.

Example 3: The lengths of two edges of a rectangular block have **increased** by 10% each, and the length of the third edge of the rectangular block has *decreased* by 20%. The volume of this rectangular block has:

1. increased, 2. remained unchanged, 3. decreased, 4. one cannot say.

Example 4: The lengths of two edges of a rectangular block have *decreased* by 10% each, and the length of the third edge of the rectangular block has **increased** by 20%. The volume of this rectangular block has:

1. increased, 2. remained unchanged, 3. decreased, 4. one cannot say.

188 Geometric series and annuity.

189 Compound interest.

190 **Advanced**: How is the number e getting into the picture?

Lemma (Calculus): Let (a_n) be such a sequence of real numbers that for all n we have $a_n \neq 0$, $a_n > -1$, and moreover $a_n \to 0$ with $n \to \infty$. Then

$$\lim_{n\to\infty} (1 + a_n)^{1/a_n} = e.$$

A consequence: If $r, t \in \mathbb{R}$ and $r \neq 0$, then

$$\left(1 + \frac{r}{n}\right)^{nt} = \left(\left(1 + \frac{r}{n}\right)^{n/r}\right)^{rt} \to e^{rt} \quad \text{with} \quad n \to \infty$$

(If $r < 0$, start the sequence above from such n that $n > |r|$, or, equivalently: $r/n > -1$, so that we don't get negative bases.)

191 Exponential functions and decreasing prices.

192 Exponential functions and coffee getting cold.

193 Radioactive decay and half-life.

Exercise 1 (Problem 14 on page 483 in the Precalculus book): Radioactive isotope Cobalt 60 with initial amount 50 grams, and with half-life of 5.27 years decays according to the formula $A(t) = A_0 e^{kt}$, where A_0 is the initial amount of the material, and k is the decay constant.

(a) Find the decay constant k.

(b) Find a function which gives the amount of the isotope which remains after time t years.

(c) Determine how long it takes for 90% for the material to decay.

Exercise 2: The mass $A(t)$ grams of a radioactive sample after time t years is given by the formula $A(t) = 100e^{-3t}$.

(a) What is the initial mass of radioactive substance in the sample?

(b) Find the half-life of the radioactive substance.

(c) Prove the relationship $hk = -\ln 2$ between the half-life h and the constant k in for the process of radioactive decay described by $A(t) = A_0 e^{kt}$. Formulate then in an explicit way $h = h(k)$ and $k = k(h)$.

Extra material: notes with solved Exercise 2.

194 Uninhibited growth and doubling the population.

Exercise 3: The world population, in billions, t years after 1950, is given by the formula $P(t) = 2.54e^{0.0178t}$.

(a) What was the world population in 1950?

(b) Find, to the nearest year, the time taken for the world population to double.

(c) Prove the relationship $dk = \ln 2$ between the doubling time d and the constant k in for the process of population growth described by $P(t) = P_0 e^{kt}$. Formulate then in an explicit way $d = d(k)$ and $k = k(d)$.

Extra material: notes with solved Exercise 3.

195 The instantaneous rate of change and a promise of derivatives.

196 Logarithmic scale.

197 Logarithmic functions and measuring the magnitude of earthquakes.

Explain the following equation and inequality in terms of the Richter scale for earthquake magnitude:

$$\log\left(\frac{x}{10^{-3}}\right) = 4.7, \qquad \log 5.6 \leqslant \log\left(\frac{x}{10^{-3}}\right) \leqslant 7.1.$$

198 Logarithmic functions and measuring the intensity level of sounds.

199 Logarithmic functions and measuring acidity.

S11 Some more advanced topics

You will learn: some more advanced topics concerning the subjects of the course, like some examples from Calculus (hyperbolic functions; how to demonstrate with help of derivatives that the graphs of exponential functions look like they do; Taylor polynomials for $f(x) = e^x$; some examples with ODE modelled for situations of growth or decay); you will also get to see some more advanced problems which didn't match any of the categories in the previous sections (like mixed equations and inequalities, i.e., mixtures of radical, exponential, and logarithmic equations).

200 About this section.

201 Mixed equations and inequalities, harder problems, Problem 1.

Problem 1: Solve the inequality: $x^2 \cdot 2^x + x \cdot 2^{x-1} > 0$.

Extra material: notes with solved Problem 1.

202 Mixed equations and inequalities, harder problems, Problem 2.

Problem 2: Solve the equation: $27^{\log x} - 7 \cdot 9^{\log x} - 21 \cdot 3^{\log x} + 27 = 0$.

Extra material: notes with solved Problem 2.

203 Mixed equations and inequalities, harder problems, Problem 3.

Problem 3: Solve the inequality: $|x|^{x^2-x-2} < 1$.

Extra material: notes with solved Problem 3.

204 Mixed equations and inequalities, harder problems, Problem 4.

Problem 4: Solve the equation: $x^{\log x} = 100x$.

Extra material: notes with solved Problem 4.

205 Mixed equations and inequalities, harder problems, Problem 5.

Problem 5: Solve the equation: $\sqrt{x^{\log\sqrt{x}}} = 10$.

Extra material: notes with solved Problem 5.

206 Mixed equations and inequalities, harder problems, Problem 6.

Problem 6: Solve the inequality: $(x^2 + x + 1)^x < 1$.

Extra material: notes with solved Problem 6.

207 Mixed equations and inequalities, harder problems, Problem 7.

Problem 7: Solve the inequality: $\left(2^x + 3\cdot 2^{-x}\right)^{\log_2 x^2 - \log_2(x+6)} > 1$.

Extra material: notes with solved Problem 7.

208 Mixed equations and inequalities, harder problems, Problem 8.

Problem 8: Solve the equation: $x^{\log_{25} 81} + 81^{\log_{25} x} = 18$.

Extra material: notes with solved Problem 8.

209 A really cool problem about the domain, Problem 9.

Problem 9: Determine the domain of the function $f(x) = \sqrt{\log(\cos(\pi x))}$.

210 Comparing large powers of numbers, Problem 10.

Problem 10: Which is greater: 2^{2000} or 10^{800}? Compare with Video 50.

Extra material: notes with solved Problem 10.

211 Powers with real exponents; back to the promise from Video 33.

Prove the following statements:

a) Sketch the proof of the fact that if $a > 1$ and $x, y \notin \mathbb{Q}$, then $a^{x+y} = a^x a^y$. (The statement for $0 < a < 1$ follows automatically.)

b) Show that if $f : \mathbb{R} \to \mathbb{R}$ is not identically equal to zero, $D_f = \mathbb{R}$, and $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$, then $f(x) \neq 0$ for all $x \in \mathbb{R}$.

c) Show that if $f : \mathbb{R} \to \mathbb{R}$ is not identically equal to zero, $D_f = \mathbb{R}$, and $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$, then $f(x) > 0$ for all $x \in \mathbb{R}$.

Extra material: notes from the iPad.

212 Four important classes of functions.

E1 All the functions continuous on the entire $\mathbb{R}$ and satisfying the condition $f(x+y) = f(x) + f(y)$ for each pair $x, y \in \mathbb{R}$ are (**linear**) functions defined by $f(x) = cx$ for some $c \in \mathbb{R}$.

E2 All the functions continuous on the entire $\mathbb{R}$ and satisfying the condition $f(x+y) = f(x)f(y)$ for each pair $x, y \in \mathbb{R}$ are the **exponential** functions $f(x) = a^x$ where $a > 0, a \neq 1$, and two constant functions: $g(x) = 0,\ h(x) = 1$.

E3 All the functions continuous on the entire $\mathbb{R}^+$ and satisfying the condition $f(xy) = f(x) + f(y)$ for each pair $x, y \in \mathbb{R}^+$ are the **logarithmic** functions $f(x) = \log_a x$ where $a > 0, a \neq 1$, and $g(x) = 0$.

E4 All the functions continuous on the entire $\mathbb{R}^+$ and satisfying the condition $f(xy) = f(x)f(y)$ for each pair $x, y \in \mathbb{R}^+$ are the **power** functions $f(x) = x^\alpha$ where $\alpha \in \mathbb{R}$, and $g(x) = 0,\ h(x) = 1$.

Extra material: notes from the iPad, with the solution to E2 and E3.

213 Superpowers of the half-life and the doubling time.

- H Exponential **decay** $A(t) = A_0 e^{kt}$ with $k < 0$ is characterised by a **fixed halving time** (*half-life*) h, i.e., if $A(h) = \frac{1}{2}A_0$ then $A(t+h) = \frac{1}{2}A(t)$ for each $t > 0$. (By definition of the half-life h we have $e^{kh} = \frac{1}{2}$.)
- D Exponential **growth** $P(t) = P_0 e^{kt}$ with $k > 0$ is characterised by a **fixed doubling time** d, i.e., if $P(d) = 2P_0$ then $P(t+d) = 2P(t)$ for each $t > 0$. (By definition of the doubling time d we have $e^{kd} = 2$.)

214 The exponential function is its own derivative, an illustration.

215 Derivatives of exponential functions.

216 Exponential functions as solutions to an important class of ODE.

217 Exponential versus logarithmic growth in the light of the derivatives.

218 Taylor expansion and the relationship between the sine, the cosine, and the exponential function $y = e^x$.

219 Power functions, exponential functions... And if both base and exponent are variable?

In case of variable base and exponent, we rewrite: $y = f(x)^{g(x)} = e^{g(x)\ln f(x)}$. An example: $y = x^{1/x}$.

Problem 11: Which is greater: π^e or e^π?

Extra material: notes with solved Problem 11.

220 A problem about 2022 and 2023, with two solutions.

Problem 12: Which is greater: 2022^{2023} or 2023^{2022}?

Extra material: notes with solved Problem 12.

221 Hyperbolic functions.

Problem 13: The following functions are called *hyperbolic cosine* and *hyperbolic sine*:

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}.$$

Show that the hyperbolic cosine is an **even** function, and the hyperbolic sine is an **odd** function. Show that:

a) $\cosh x + \sinh x = e^x$,
b) $\cosh x - \sinh x = e^{-x}$,
c) $\cosh^2 x - \sinh^2 x = 1$.

In Video 169 we showed that $y = \ln(x + \sqrt{x^2+1})$ is the inverse function to the hyperbolic sine. You can use the same method to show that $y = \ln(x + \sqrt{x^2-1})$ is the inverse to the hyperbolic cosine restricted to $x \geqslant 0$ (the inverse is defined on $[1, +\infty)$).

Extra material: notes with solved Problem 13.

222 Precalculus 4, Wrap-up.

S12 Extras

You will learn: about all the courses we offer, and where to find discount coupons. You will also get a glimpse into our plans for future courses, with approximate (very hypothetical!) release dates.

B Bonus lecture.

Extra material 1: a pdf with all the links to our courses, and coupon codes.

Extra material 2: a pdf with an advice about optimal order of studying our courses.

Extra material 3: a pdf with information about course books, and how to get more practice.