Precalculus 3: Trigonometry[1]

Mathematics from high school to university

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A short table of contents

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An extremely detailed table of contents; the videos (titles in green) are numbered

In blue: problems solved on an iPad (the solving process presented for the students; active problem solving)

In red: solved problems demonstrated during a presentation (a walk-through; passive problem solving)

In dark blue: *Read along with this section*: references for further reading and more practice problems in the *Precalculus* book by Carl Stitz and Jeff Zeager:

- *Precalculus*: Carl Stitz, Ph.D., Lakeland Community College; Jeff Zeager, Ph.D., Lorain County Community College; version from July 4, 2013:
  1. Chapter 10: Foundations of Trigonometry (pp. 693–880)
  2. Chapter 11: Applications of Trigonometry (pp. 8801–1068).

This book is added as resources to Video 1, with kind permission of Professor Carl Stitz.

S1 Introduction to the course

You will learn: what is awaiting in this course, and what you are going to learn.

1 Introduction to the course. Extra material: this list with all the movies and problems.

Extra material: an article listing main trigonometric identities, where to find them in the course and in the Precalculus book, how to memorize them, and what they are applied for.

Extra material: the book named above (*Precalculus*, chapters 10 and 11, pp. 693–1068).

2 What is trigonometry?

3 Place of trigonometry in mathematics and sciences.

4 Warning: inverse versus reciprocal.

5 One, three, six, nine, twelve functions.

6 Various ways of defining the sine function; trigonometric vs circular.

7 What you are going to learn and why.

8 Proportions and their crucial role in this course.

Problem 1: Let $a, b, c, d > 0$. Compute x in the following four cases (each case is a new problem to solve):

$$\frac{x}{b} = \frac{c}{d}, \quad \frac{a}{x} = \frac{c}{d}, \quad \frac{a}{x} = \frac{x}{d}, \quad \frac{30}{180} = \frac{x}{\pi}.$$

Lemma 1: Let $a, b, c, d > 0$. Then $\frac{a}{b} = \frac{c}{d} \Leftrightarrow \frac{a}{c} = \frac{b}{d}$.

Lemma 2: Let $a, b, c, d > 0$. Then $\frac{a}{a+b} = \frac{c}{c+d} \Leftrightarrow \frac{b}{a+b} = \frac{d}{c+d}$.

S2 Crash course in Euclidean geometry

You will learn: everything you need to know about geometry in order to feel comfortable with the new content in this course: geometrical concepts such as straight lines, straight line segments, angles, triangles (acute, right, obtuse), polygons, circles (inscribed, circumscribed), congruence rules for triangles (SSS, SAS, ASA), similar triangles, Thales' theorem, Pythagorean theorem, congruence rules for right triangles (HA, HL, LL), measuring angles, measuring distances, computing area of squares and triangles, isometries in the plane (symmetries, rotations, translations).

9 How to use this section.

10 Euclidean geometry: Primitive notions, axioms, definitions, and theorems.

11 Angles: definition, usual notation, measures, kinds of angles.

* Angles: zero, acute, right, obtuse, straight, reflexive.
* Angles: $360°$, $180°$, $90°$, $30°$, $60°$.
* Angles in the Cartesian coordinate system (angles in standard position, coterminal angles, quadrantal angles): see Section 7.

12 Angle-related terminology.

* Pairs of angles (relations between their measures): congruent, supplementary, complementary.
* Configurations of angles (how they are situated relative each other): vertical, adjacent, linear pair.
* Alternate angles: see Video 15.

13 Triangles: definition and usual notation; congruent triangles (SSS, SAS, ASA).

14 **Optional**: Euclidean distance and triangle inequality.

15 Fifth postulate and alternate angles; parallelograms.

Theorem and Corollary: Two parallel lines are intersected by a third line,
forming four interior angles and four exterior angles.

1. Same side interior angles are supplementary (prove from the Fifth Postulate).
2. Alternate interior angles are equal (a corollary).
3. Alternate exterior angles are equal (a corollary).
4. Same side exterior angles are supplementary (a corollary).
5. Corresponding angles are equal (a corollary).

Exercise (will be used in Video 20): Prove that in each parallelogram the parallel sides are of the same length; distance between two parallel lines is constant at all points.

16 The sum of angles in a triangle, and some consequences.

Some statements derived from the Theorem about alternate angles:

1. In each triangle, the sum of measures of all the angles is $180°$.
2. A triangle can have maximally one angle greater than or equal to the right angle; three types of triangles (acute, right, obtuse).
3. Each exterior angle of a triangle is equal to the sum of the two remote interior angles.

17 The concept of area; area of a rectangle.

18 Area of a triangle; the height (altitude).

19 Isometries: line and point symmetries, rotations, translations.

Extra material: notes from the iPad.

20 Thales' theorem and similar triangles.

21 Right triangles and Pythagorean theorems.

Some theorems and examples:

1. Pythagorean theorem with proof. Conclusion: The hypotenuse is the longest side in a right triangle.
2. The contrapositive to Pythagorean theorem. Example: Triangle with side lengths 2, 4 and 5 is not right.
3. The converse to Pythagorean theorem. Example: All the $3k : 4k : 5k$ triangles (for $k \in \mathbb{R}^+$) are right.
4. Two congruence rules (for right triangles only): HA, HL, and LL.

22 Important facts about equilateral and isosceles triangles.

Three problems solved during this lecture:

1. A triangle is isosceles iff one of its heights divides the corresponding side in half.
2. A triangle is isosceles iff two of its angles are equal to each other.

3. How to compute the height in an equilateral triangle with side a?

Extra material: notes from the iPad, with solutions and proofs.

23 Back to Euclidean distance: how to compute it in the coordinate system.

24 Circles: a geometrical definition and equation in the coordinate system.

25 Inscribed and central angles.

Inscribed Angle Theorem: An angle inscribed in a circle is half of the central angle that subtends the same arc on the circle.

Thales' Theorem: The angle subtended by a diameter is always $90°$, i.e., a right angle.

Extra material: notes with proof of the theorem.

26 Circumscribed and inscribed circles for triangles; bisectors.

27 When it is possible to circumscribe/inscribe a circle around/in a quadrilateral.

28 The sum of angles in a polygon; the interior angle in a regular polygon.

Problem: What is the sum of all the interior angles in any convex polygon with n sides? Verify the obtained formula for the interior angle in a regular polygon with n sides:

$$\alpha_n = 180° - \frac{360°}{n}$$

with help of the solution to the current problem.

Extra material: notes with solved problem.

S3.14159... The magnificent number π

You will learn: about the number π: its meaning for the circles and disks, and some basic (geometrical) approximation methods.

29 Length and area: not always easy.

30 A very early approximation.

31 Archimedes and approximations by perimeters of regular polygons.

Extra material: notes from the iPad.

32 What about the area of a disk? An intuitive answer.

33 What about the area of a disk? A hardcore answer (Eudoxus).

S4 Trigonometric functions of acute angles: the geometric approach

You will learn: the geometric definition of six trigonometric functions, why there are *six* of them, and how we can know that they are well defined as functions of (acute) angles; first (very basic) relationships between these functions.

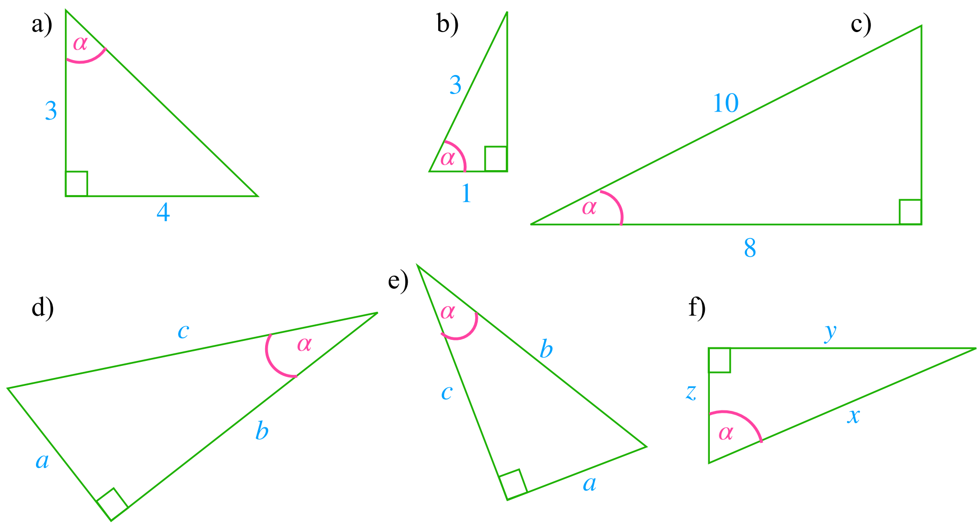
34 Where to find geometry for this section.

35 Six possible ratios and why they describe functions; Cofunction Identities.

Extra material: an article listing main trigonometric identities, where to find them in the course and in the Precalculus book, how to memorize them, and what they are applied for.

36 Six trigonometric functions, Exercise 1.

Exercise 1: In each of the six triangles (see the picture on the next page), determine the length of the opposite



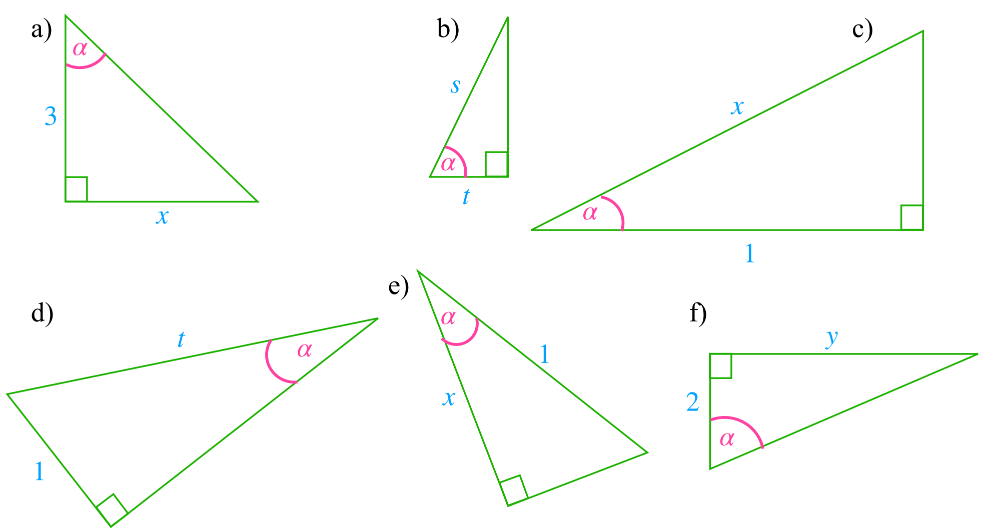
Figur 1: Illustration to Exercise 1 in Video 36.

and the adjacent legs for the angle α, and write expressions for the six trigonometric functions for α.

Extra material: notes with solved Exercise 1.

37 Six trigonometric functions, Exercise 2.

Exercise 2: In each of the six triangles (see the picture below), determine the length of the opposite and the adjacent legs for the angle α, and write expressions for the six trigonometric functions for α.



Figur 2: Illustration to Exercise 2 in Video 37.

Extra material: notes with solved Exercise 2.

38 Some special triangles, Exercise 3.

Exercise 3: Let $0 < x < 1$ be given. Sketch a right triangle having one acute angle α such that:

$$1.\ \ \sin\alpha = x, \qquad 2.\ \ \cos\alpha = x, \qquad 3.\ \ \tan\alpha = x.$$

In all these cases, compute the values of the two other trigonometric functions for α.

Extra material: notes with solved Exercise 3.

39 One half of a square: the 45-45-90 triangles, Exercise 4.

Exercise 4: Using Pythagorean theorem in a square, compute the values of sine, cosine, and tangent of 45°.

Extra material: notes with solved Exercise 4.

40 The Pythagorean Identities, Exercise 5.

Exercise 5: Using Pythagorean theorem, show that for each acute angle α we have

1. $\sin^2\alpha + \cos^2\alpha = 1$
2. $1 + \tan^2\alpha = \sec^2\alpha$
3. $1 + \cot^2\alpha = \csc^2\alpha$.

Use some of the identities to show that for each acute angle α we have

$$(\sec\alpha - \tan\alpha)(\sec\alpha + \tan\alpha) = 1, \quad \frac{\sin\alpha}{1-\cos\alpha} = \frac{1+\cos\alpha}{\sin\alpha}.$$

Moreover, for each acute angle $\alpha \neq 45°$ (why this restriction?):

$$\frac{\sec\alpha}{1-\tan\alpha} = \frac{1}{\cos\alpha - \sin\alpha}.$$

Extra material: notes with solved Exercise 5.

41 Starting plotting the sine function.

Some basic properties of the sine function for acute angles:

1. The function has only positive values, between 0 and 1
   (close to 0 for very small angles, close to 1 for almost right angles).
2. The function is increasing.
3. The function seems continuous.

42 It is enough to plot the sine in order to get the cosine.

Some basic properties of the cosine function for acute angles:

1. The function has only positive values, between 0 and 1
   (close to 1 for very small angles, close to 0 for almost right angles).
2. The function is decreasing.
3. The function seems continuous.

43 Some remarks about the tangent function.

Some basic properties of the tangent function for acute angles:

1. The function has only positive values; values close to 0 for very small angles,
   value 1 for 45°, and arbitrarily large values for angles close to the right angle.
2. The function is increasing.

S5 Computing exact values of trigonometric functions

You will learn: how to derive the exact values of trigonometric functions for angles: 15, 18, 30, 36, 45, 54, 60, 72, 75, and 22.5 degrees using *geometric* methods; we will also derive, also using just geometry, some trigonometric formulas valid for acute angles (but later, in the second half of the course, you will learn that *all* of them are valid just for *any* angle, so they are really worth learning); these formulas will be then used for computing values of trigonometric functions for some angles (knowing the values for some other angles). We will, step by step, create the graph of the sine and cosine functions for acute angles.

44 Stage 0: what we already know about the sine curve (0, 45, 90).

45 Equilateral triangle and the 30-60-90 triangles, Problem 1.

Problem 1: Derive the exact values of the trigonometric functions for the angles of 30 and 60 degrees.

Extra material: notes with solved Problem 1.

46 Derivation of the sum identity for sine, using geometry; Method 1.

Let $0^\circ < \alpha,\ \beta,\ \alpha+\beta < 90^\circ$. Show by a geometrical illustration that $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$.

Lemma: The area A of a parallelogram with side lengths a and b, and the acute angle α is $A = ab\sin\alpha$.

47 Derivation of the sum identities for sine and cosine, using geometry; Method 2.

Let $0^\circ < \alpha,\ \beta,\ \alpha+\beta < 90^\circ$. Show by a geometrical illustration that $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$ and $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

Show by a computation that, with the same assumptions:

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}.$$

Extra material: notes from the iPad.

48 Conclusion: Double Angle Formulas and Power Reduction Formulas.

Let $0^\circ < \alpha < 45^\circ$. Use the result from the previous video to show that $\sin 2\alpha = 2\sin\alpha\cos\alpha$, $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$, and

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}.$$

Extra material: notes from the iPad.

Extra material: an article showing how Power Reduction Formulas can be used for computing some integrals.

49 Half Angle Formulas, Problem 2.

Problem 2: Compute sine and cosine of 15 degrees, and then of 75 degrees.

Extra material: notes with solved Problem 2.

50 Applications of the sum identities, Problem 3.

Problem 3: Compute sine and cosine of 75 degrees, and then of 15 degrees (the same as in Video 49, but in another order).

Extra material: notes with solved Problem 3.

51 A geometric way of computing sine and cosine of 18 degrees (72 degrees), Problem 4.

Problem 4: Compute sine and cosine of 18 degrees, and then of 72 degrees.

Extra material: notes with solved Problem 4.

52 A geometric way of computing sine and cosine of 54 degrees (36 degrees), Problem 5.

Problem 5: Use the same triangle as in Video 51 to compute sine and cosine of 54 degrees, and then of 36 degrees. Confirm the results for 36 degrees with help of the previous lecture (sine and cosine of 18 degrees) and double angle formulas. Conclusion: The diagonal of a regular pentagon with side 1 is Φ, the Golden Ratio.

Extra material: notes with solved Problem 5.

53 Sine and cosine of triple angles, Problem 6.

Problem 6: Use the sum identities and formulas for sine and cosine of double angles to derive the formulas for sine and cosine of a triple angle (right now for angles with measure less than 30 degrees; but the same formula will hold for all angles).

Extra material: notes with solved Problem 6.

54 Computing sine of 18 degrees using the formulas for double and triple angles, Problem 7.

Problem 7: Compute sine of 18 degrees using the formulas for double and triple angles.

Extra material: notes with solved Problem 7.

55 Tangent of 22.5 degrees by elegant geometry, Problem 8.

Problem 8: Determine tangent of 22.5 degrees using the unit square and its diagonal in a creative way.

Extra material: notes with solved Problem 8.

56 Sine and cosine of 22.5 degrees by half angle formulas, Problem 9.

Problem 9: Determine sine and cosine of 22.5 degrees using the half angle formulas, and confirm the value of tangent of 22.5 degrees from the previous video.

57 A cool trick for computing sine and cosine of half of an angle with known values, Problem 10.

Problem 10: Determine sine and cosine of 27 degrees using the double angle formulas, Pythagorean identity, and the result obtained in Video 52.

58 Same trick one more time, Problem 11.

Problem 11: Determine sine and cosine of 15 degrees using the double angle formulas, Pythagorean identity, and the result obtained in Video 45.

Extra material: notes with solved Problem 11.

59 Different looks of sine and cosine of 15 degrees, Problem 12.

Problem 12: Verify that the three expressions for sine and cosine of 15 degrees describe the same numbers:

$$\sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4} = \frac{\sqrt{2-\sqrt{3}}}{2}, \quad \cos 15^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4} = \frac{\sqrt{2+\sqrt{3}}}{2}.$$

Extra material: notes with solved Problem 12.

60 An identity to prove, Problem 13.

Problem 13: Let $0^\circ < \alpha < 45^\circ$. Show (both by formula manipulations and by a geometrical illustration) that

$$\tan \alpha = \frac{\sin 2\alpha}{1 + \cos 2\alpha}.$$

61 Tangent half-angle formulas, Problem 14.

Problem 14: Let $0^\circ < \alpha < 90^\circ$. Show (both by formula manipulations and by a geometrical illustration) that

$$\sin \alpha = \frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}, \qquad \cos \alpha = \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}, \qquad \tan \alpha = \frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}}.$$

Extra material: notes with solved Problem 14.

62 Experimenting with double angle formulas, Problem 15.

Use the double angles formulas to compute:

1. $\sin 60^\circ$
2. $\cos 60^\circ$
3. $\sin 90^\circ$
4. $\cos 90^\circ$
5. $\sin 120^\circ$
6. $\cos 120^\circ$

The last four only theoretically! We have only defined trigonometric functions for *acute* angles.

Extra material: notes with solved Problem 15.

63 Summary Section 5.

S6 An introduction to inverse trigonometric functions and to solving triangles

You will learn: the geometrical meaning of inverse trigonometric functions (arcsin, arccos and arctan) for acute angles, and how to use them in simple problem solving (more advanced problem solving with triangles comes later in the course).

64 Inverse trigonometric functions: arcsine, arccosine, arctangent.

Examples: $\arcsin\frac{1}{2} = 30°$, $\arcsin\frac{\sqrt{3}}{2} = 60°$, $\arccos\frac{\sqrt{3}}{2} = 30°$, $\arccos\frac{\sqrt{5}+1}{4} = 36°$, $\arctan\sqrt{3} = 60°$.

65 What does it mean to solve a triangle?

66 Solving right triangles, with or without approximations.

Exercise: Solve the right triangles given:

1. $l_1 = 3,\ l_2 = 4.$ 2. $x = 3,\ y = 4.$ 3. $\alpha = 30°,\ h = 6.$ 4. $\alpha = 30°,\ x = 6.$ 5. $\alpha = 30°,\ x = 6,\ y = 4.$

Extra material: notes with solved Exercise.

67 More (elementary) problem solving.

Exercise: Find the angle α between the positive x-axis and the ray starting in the origin and passing through the point $(4, 2)$. Show three methods for finding this angle (using one of the three trigonometric functions at a time). Verify, both using exact values and the approximations, that $\sin^2\alpha + \cos^2\alpha = 1$ and $\tan\alpha = \sin\alpha/\cos\alpha$.

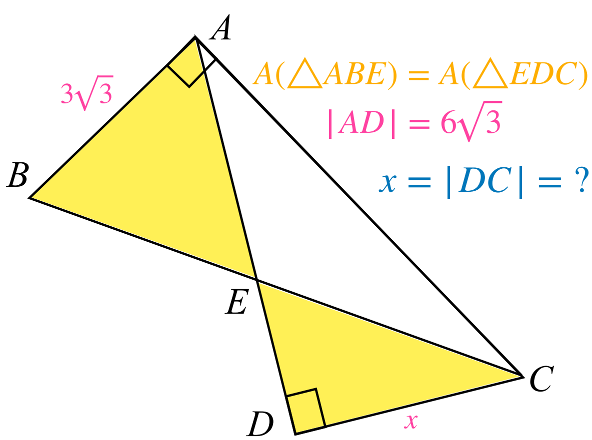
Exercise: If $\theta = 12°$ and the side adjacent to θ has length 4, how long is the hypotenuse?

Exercise: If $\theta = 15°$ and the hypotenuse has length 10, how long is the side opposite θ?

Extra material: notes with solved Exercise.

68 A difficult problem.

Problem: See the image.



Figur 3: Illustration to the problem in Video 68.

Extra material: notes with solved problem.

S7 From degrees to radians: why and how

You will learn: the definition of radian; how to calculate degrees to radians and back, using proportions; the values of the most common angles in radians; angles in the Cartesian coordinate system.

Read along with this section: **Precalculus book**: Section 10.1 (pp. 694–716); this time you can read the entire section and solve all the problems you want to.

69 Why we need a new way of measuring angles.

70 What is radian and how to work with it.

71 The angles in radians to remember, and how to do it.

Converting to radians: 0, 15, 18, 30, 36, 45, 54, 60, 72, 75, 90, 180, 270, 360, and 22.5 degrees.

72 Degree to radian (and back) conversion, exercises.

Converting to radians (and back): you get $10 + 10$ quiz questions (with answers) from the Precalculus book.

73 Angles in the coordinate system: some terminology (continuation after Video 11).

You get some quiz questions (with answers) from the Precalculus book.

* Angles in standard position,
* Coterminal angles,
* Quadrantal angles.

S8 Trigonometric (circular) functions of any angle: the unit circle and circular motion

You will learn: two ways of expanding the trigonometric functions sine and cosine (defined geometrically, for acute angles, in Section 3) to any angles (or, actually, to any real number):

1. a **static** one: $\cos t = x$, $\sin t = y$, where (x, y) are the coordinates of the intersection point between the unit circle and the terminal side for the angle of t radians, in standard position (obviously functions $\mathbb{R} \to \mathbb{R}$ as each point has exactly one pair of Cartesian coordinates),
2. a **dynamic** one: a point is moving along the unit circle starting in the point $(1, 0)$ for $t = 0$, and continuing counterclockwise until the point on the circle where the length of the path from the beginning to this point is t; the coordinates of this point define the cosine and the sine functions as follows: $x = \cos t$ and $y = \sin t$ (obviously functions $\mathbb{R} \to \mathbb{R}$ as each point has exactly one pair of Cartesian coordinates).

In order to construct these functions, we will wrap the number axis on the unit circle, which is a really cool operation.

Read along with this section: **Precalculus book**: Section 10.2 (pp. 717–743); this section of the book can be read along with this section, and the next one. Observe that my approach is different than the approach of the Authors: I started with geometry, they started with the functional approach. This means that the part 10.2.1 *Beyond the unit circle* will not really be necessary for us, as we treated all the ratio-related issues much earlier in the course (Sections 2 and 4). You can wait with the exercises from the book until the next section.

74 What we have by now.

75 What we want to have.

76 Wrapping the number axis (the t-axis) around the unit circle.

77 Definition of circular functions sine and cosine, approach 1.

Confirm the experimental results from Video 62:

1. $\sin 60° = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$
2. $\cos 60° = \cos \frac{\pi}{3} = \frac{1}{2}$
3. $\sin 90° = \sin \frac{\pi}{2} = 1$
4. $\cos 90° = \cos \frac{\pi}{2} = 0$
5. $\sin 120° = \sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2}$
6. $\cos 120° = \cos \frac{2\pi}{3} = -\frac{1}{2}$

using the new definitions of sine and cosine.

78 Definition of circular functions sine and cosine, approach 2.

S9 Basic properties of six trigonometric (circular) functions; graphing

You will learn: the definition of the other circular functions (tangent, and the three reciprocals) defined with help of sine and cosine; basic properties following immediately from the definitions and symmetries of the unit circle: the domain and range for all these functions, Reference Angles Identities, monotonicity in intervals, being even or odd, periodicity (a new concept, not introduced in Precalculus 1), the graphs; basic relationships between these functions: the Pythagorean Identity, cofunction identities. You will also learn the etymology of the names sine, tangent, and secant.

Read along with this section: **Precalculus book**: Section 10.2 (pp. 717–743); you have probably already seen some parts of this section while studying our Section 8. Observe that my approach is different than the approach of the Authors: I started with geometry, they started with the functional approach. This means that the part 10.2.1 *Beyond the unit circle* will not really be necessary for us, as we treated all the ratio-related issues much earlier in the course (Sections 2 and 4). Section 10.3 (pp. 744–769) and some parts of Section 10.5 (pp. 790–818): only about the graphs of the six circular functions, **not yet** about their transformations. You can also omit the exercises about solving trigonometric equations, as we will deal with them in Section 13.

79 From sine and cosine as functions of one real variable to the six circular functions.

We complete the table from Video 71 with the values of the sine, cosine, and tangent for: 0, 90, 180, 270, and 360 degrees.

Reciprocal and Quotient Identities follow effortlessly from the definitions of the circular functions.
The Pythagorean Identities and other identities proven in Video 40 hold, with some restrictions in some cases (see the last slide).

80 The signs of the circular functions in different quadrants, Exercise 1.

Exercise 1: Determine the signs of the following numbers: $\sin 2464$, $\cos 120.5$, $\cos 120.5°$, $\sin\sqrt{78}$, and $\sin\sqrt{19}$.

Extra material: notes with solved Exercise 1.

81 Reference angles and their wonderful applicability.

The Reference Angle Theorem. The Cofunction Identities (as in Video 35) are proven and illustrated.

82 Exact values of sine and cosine of various angles, Exercise 2.

A quick rule for remembering the sine of 0, 30, 45, 60, and 90 degrees.

Exercise 2: Determine the exact values of the cosine and sine of the angles:

$$\alpha = \tfrac{7\pi}{6},\ \ \alpha = -\tfrac{3\pi}{4},\ \ \alpha = \tfrac{23\pi}{6},\ \ \alpha = \tfrac{10\pi}{3},\ \ \alpha = 117\pi.$$

In all the cases: 1. ignore the even multiples of π, 2. sketch the coterminal angle which is left after step 1,
3. determine the reference angle, 4. compute the sine and cosine of this reference angle, 5. determine the sign of the sine and cosine for the given quadrant, 6. formulate your answer.

Extra material: notes with solved Exercise 2.

83 Pythagorean triples. Finding values of cofunctions, Exercise 3.

Exercise 3: Determine the cofunction in each of the cases:

a) Determine $\sin\alpha$ if α is in Q3 and $\cos\alpha = -\frac{3}{5}$.
b) Determine $\cos\alpha$ if α is in Q4 and $\sin\alpha = -\frac{7}{25}$.
c) Determine $\cos\alpha$ if α is in Q2 and $\sin\alpha = \frac{5}{13}$
d) Determine $\sin\alpha$ if $2\pi < \alpha < \frac{5\pi}{2}$ and $\cos\alpha = \frac{\sqrt{10}}{10}$.

A general method: use the Pythagorean Identity to compute the square of the cofunction (in case of Pythagorean triples: you see the answer at once!), determine the sign of this cofunction for the given quadrant, formulate your answer.

Extra material: notes with solved Exercise 3.

84 Periodic functions.

The fractional part function $\{x\} = x - \lfloor x \rfloor$ as an example of a periodic function.

85 The sine function, its properties and graph.

86 The cosine function, its properties and graph.

87 The tangent function, its properties and graph.

88 The cotangent function, its properties and graph.

89 The cosecant function, its properties and graph.

90 The secant function, its properties and graph.

91 The names of the circular functions.

Here you get all the graphs of the six circular functions, and an illustration for one of the Pythagorean identities (the one involving the tangent and the secant).

92 Circular functions, Exercise 4.

Exercise 4: Find the exact value or state that it is undefined:
$\tan(117\pi)$, $\csc(3\pi)$, $\cot\frac{7\pi}{6}$, $\cot\frac{3\pi}{4}$, $\sec(-7\pi)$, $\tan\frac{31\pi}{2}$, $\csc(-\frac{\pi}{3})$, $\cot(-5\pi)$.

Extra material: notes with solved Exercise 4.

93 Circular functions, Exercise 5.

Exercise 5: Find the five remaining circular functions if:

1. $\csc\alpha = \frac{25}{24}$, α in Quadrant 1.
2. $\tan\alpha = -2$, α in Quadrant 4.
3. $\tan\alpha = \sqrt{10}$, $\pi < \alpha < \frac{3\pi}{2}$.

Extra material: notes with solved Exercise 5.

S10 Trigonometric identities; graph transformations

You will learn: good news for them who were afraid they were wasting their time in Section 5: everything done back there will be reused here! The only topic which must be redone is the derivation of the Sum Identities for sine and cosine, as the derivations done in Section 5 were geometrical and restricted to acute angles. All the other formulas (the double angle formulas, the power reduction formulas, half angle formulas, tangent half angle formulas, triple angle formulas) were proven by formula manipulation, so they are valid also in the new situation. Two new groups of formulas (sum to product, and product to sum formulas). The Sum Identities will be used for graph transformations, which will also be discussed in this section. The terminology related to sinusoids will be introduced (period, phase, amplitude).

Read along with this section: **Precalculus book**: Section 10.4 (pp. 770–789) (without the non-basic identities, i.e., the ones without names; these less known identities will be discussed in Section 12) and Section 10.5 (pp. 790–818).

94 So much will be recycled!

After proving the Sum Identities in Videos 95 and 96, we will automatically get the following identities, which were derived from the Sum Identities (restricted for acute angles) in Section 5. We don't have to assume any more that the angles are acute, but sometimes we have to add some restrictions, which are necessary for some circular functions (if their domain is not the entire $\mathbb{R}$):

1. **The sum identity for tangent**, as proven in Video 47, but this time in a general case, and with some restrictions:
$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}, \quad \alpha,\ \beta,\ \alpha+\beta \neq \frac{\pi}{2} + k\pi\ \ (k\in\mathbb{Z}).$$
2. **The Double Angle Formulas** as proven in Video 48, but now in a general case:
$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$
$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha = 2\cos^2\alpha - 1$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} \quad \alpha,\ 2\alpha \neq \frac{\pi}{2} + k\pi, \ \ (k \in \mathbb{Z}).$$

3. **The Power Reduction Formulas** as proven in Video 48, but now in a general case:

$$\sin^2\alpha = \frac{1-\cos 2\alpha}{2}, \qquad \cos^2\alpha = \frac{\cos 2\alpha + 1}{2}.$$

4. **Half Angle Formulas** as proven in Video 49, but now in a general case (**and with some modification!**):

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}, \qquad \cos\frac{\alpha}{2} = \pm\sqrt{\frac{\cos\alpha+1}{2}}, \qquad \tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}},$$

where the sign depends on the quadrant in which the terminal side of $\alpha/2$ lies. (In the last formula, $\alpha \neq (2k+1)\pi,\ k \in \mathbb{Z}$, so that the tangent of $\alpha/2$ is defined; at the same time the denominator in the last formula is different from zero.)

5. **Tangent Half Angle Formulas** as proven in Video 61 (computationally; the graphical proof was just for acute angles), but now in a general case:

$$\sin\alpha = \frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}, \qquad \cos\alpha = \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}, \qquad \tan\alpha = \frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}},$$

where $\alpha \neq (2k+1)\pi,\ k \in \mathbb{Z}$, so that the tangent of $\alpha/2$ is defined; for the last formula we have to make more restrictions: $\alpha \neq \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$ (so that $\tan\alpha$ is well defined) and $\alpha \neq \frac{\pi}{4} + k\frac{\pi}{2},\ k \in \mathbb{Z}$ (so that $\tan^2\alpha \neq 1$ and the denominator in the last formula is different from zero).

6. **Multiple Angle Identities** as proven in Video 53, but now in a general case:

$$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha, \qquad \cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha.$$

We will come back to these formulas in Section 14 (Complex numbers).

All these formulas can now be used for the circular functions of any variable in the domain.

95 Derivation of the sum identity for the cosine in a general case.

Extra material: notes from the iPad.

96 Derivation of the sum identity for the sine in a general case.

Extra material: notes from the iPad.

97 Derivation of the difference identity for the cosine in a general case.

Extra material: notes from the iPad.

98 Derivation of the difference identity for the sine in a general case.

Prove the difference identity for tangent:

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}, \qquad \alpha,\ \beta,\ \alpha-\beta \neq \frac{\pi}{2} + k\pi \ \ (k \in \mathbb{Z}).$$

Extra material: notes from the iPad.

99 Graph transformations, an introduction.

A repetition from Precalculus 1: Problem from Video 98.

100 Sinusoids: some terminology.

- Vertical shift / Baseline,
- Amplitude,
- Period,
- Phase shift.

101 Some basic transformations of the sinusoid.

Some examples: Sketch the following curves, and determine their baseline, vertical shift, amplitude, period, and phase shift: $y = \sin x,\ y = \sin x + 2,\ y = \sin x - 3,\ y = 3\sin x,\ y = -3\sin x,\ y = 3\sin x + 1,\ y = \sin 2x,$ $y = \sin\frac{x}{2},\ y = \sin(x + \frac{\pi}{2}) = \cos x.$

102 Non-sinusoidal circular functions: to study from the book.

An old (unrelated) example: draw the graph of $f(x) = 5\sin 3x + 7$.

103 How to scale any pair of real numbers into the sine and cosine of the same angle.

104 A linear combination of the sine and the cosine.

Rewrite the following linear combination of the functions sine and cosine $f(x) = a\sin x + b\cos x$ (where a and b are not both equal to 0) as a scaled sine function with some phase shift. What are the amplitude and the shift?

Extra material: notes from the iPad.

105 A linear combination of the sine and the cosine, Exercise 1.

Exercise 1: Rewrite the formulas to show that the curves are sinusoids:

a) $f(x) = \dfrac{3}{2}\cos 2x - \dfrac{3\sqrt{3}}{2}\sin 2x + 6$

b) $f(x) = -\sin x + \cos x - 2$

c) $f(x) = 3\sqrt{3}\sin 3x - 3\cos 3x$.

Extra material: notes with solved Exercise 1.

106 Find max and min, Exercise 2.

Exercise 2: Find the maximum and the minimum values of $f(x) = 5\cos x + 12\sin x + 12$.

Extra material: notes with solved Exercise 2.

107 Even Odd Identities, Exercise 3.

Exercise 3: Use the Even Odd Identities to show that $\tan(-t^2+1) = -\tan(t^2-1)$ and $\cos(-\frac{\pi}{4}-5t) = \cos(5t+\frac{\pi}{4})$.

108 Sum and Difference Identities, Exercise 4.

Exercise 4: Find the exact values of:

a) $\sec 165°$

b) $\cot 255°$

c) $\csc\dfrac{5\pi}{12}$.

Extra material: notes with solved Exercise 4.

109 Sum and Difference Identities, Exercise 5.

Exercise 5: If $\csc\alpha = 3$, where $0 < \alpha < \dfrac{\pi}{2}$, and β is a Quadrant II angle with $\tan\beta = -7$, find

$$\cos(\alpha+\beta),\quad \sin(\alpha+\beta),\quad \tan(\alpha+\beta).$$

Extra material: notes with solved Exercise 5.

110 Applications of Double Angle Formulas, Exercise 6.

Example: Use the Power Reduction Formulas to rewrite $\sin^2\alpha\cos^2\alpha$ as a sum that does not involve any powers of sine or cosine greater than 1.

Exercise 6: Use Half Angle Formulas to find exact values of $\tan 112.5°$ and $\cos\dfrac{7\pi}{12}$.

Extra material: notes with solved Exercise 6.

111 Applications of Double Angle Formulas, Exercise 7.

Exercise 7: Knowing that $\sin\alpha = -\frac{4}{5}$ and $\pi < \alpha < \frac{3\pi}{2}$, find the exact values of

$$\sin 2\alpha, \quad \cos 2\alpha, \quad \tan 2\alpha, \quad \sin\frac{\alpha}{2}, \quad \cos\frac{\alpha}{2}, \quad \tan\frac{\alpha}{2}.$$

Extra material: notes with solved Exercise 7.

112 Product To Sum Formulas.

Show that for all real numbers $\alpha,\ \beta$ we have

$$\cos\alpha\cos\beta = \tfrac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$$
$$\sin\alpha\sin\beta = \tfrac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$$
$$\sin\alpha\cos\beta = \tfrac{1}{2}[\sin(\alpha-\beta)+\sin(\alpha+\beta)].$$

Extra material: notes from the iPad, with derivation of the formulas above.

113 Product To Sum Formulas, Exercise 8.

Exercise 8: Problems 74–79 from page 785 in the book.

114 Sum To Product Formulas.

Show that for all real numbers $\alpha,\ \beta$ we have

$$\cos\alpha+\cos\beta = 2\cos\tfrac{\alpha+\beta}{2}\cos\tfrac{\alpha-\beta}{2}$$
$$\cos\alpha-\cos\beta = -2\sin\tfrac{\alpha+\beta}{2}\sin\tfrac{\alpha-\beta}{2}$$
$$\sin\alpha\pm\sin\beta = 2\sin\tfrac{\alpha\pm\beta}{2}\cos\tfrac{\alpha\mp\beta}{2}.$$

Extra material: notes from the iPad, with derivation of the formulas above.

115 Sum To Product Formulas, Exercise 9.

Exercise 9: Write the given sums as products:

a) $\sin 2\alpha - \sin 7\alpha$
b) $\sin\alpha + \cos\alpha$
c) $\sin 87^\circ - \sin 59^\circ - \sin 93^\circ + \sin 61^\circ$.

Extra material: notes with solved Exercise 9.

116 Monotonicity of the sine in the first quadrant.

Prove that $f(x) = \sin x$ is increasing on $[0,\ \frac{\pi}{2}]$, using the Sum/difference To Product Formula. Use this first statement and the fact that the sine is an odd function to prove that $f(x) = \sin x$ is increasing on $[-\frac{\pi}{2},\ \frac{\pi}{2}]$. (Remember: all odd functions have value 0 in 0.)

Extra material: notes with the proof.

117 Monotonicity of the cosine in the first quadrant.

Conclusion: the tangent is increasing on $(-\frac{\pi}{2},\ \frac{\pi}{2})$.

118 A preparation for arcsine, Exercise 10.

Exercise 10: Suppose α is an angle in Quadrant 1, with $\sin\alpha = x$. Verify the following formulas:

$$\cos\alpha = \sqrt{1-x^2}, \qquad \sin 2\alpha = 2x\sqrt{1-x^2}, \qquad \cos 2\alpha = 1-2x^2.$$

Extra material: notes with solved Exercise 10.

119 A preparation for arctangent, Exercise 11.

Exercise 11: Suppose α is an angle in Quadrant 1, with $\tan\alpha = x$. Verify the following formulas:

$$\cos\alpha = \frac{1}{\sqrt{x^2+1}}, \qquad \sin\alpha = \frac{x}{\sqrt{x^2+1}}, \qquad \sin 2\alpha = \frac{2x}{x^2+1}, \qquad \cos 2\alpha = \frac{1-x^2}{x^2+1}.$$

Extra material: notes with solved Exercise 11.

S11 Inverse trigonometric functions, their properties and graphs

You will learn: about the inverse trigonometric functions arcsine, arccosine, and arctangent (the inverse to their reciprocals can be studied from the Precalculus book: pages 824–833; this is *not* covered in our course), their properties, graphs, and some interesting compositions with the trigonometric (circular) functions.

Read along with this section: **Precalculus book**: Section 10.6 (pp. 819–826, selected exercises on pages 841–847). Inverse functions to the reciprocal circular functions are optional, *not* a part of this course, but they can be studied from the book: pp. 824–833.

120 How to get around the no-injection issue.

121 Arcsine: its properties and graph.

The graphs of $y = \arcsin x$, $y = \sin(\arcsin x)$, and $y = \arcsin(\sin x)$.

122 Arccosine: its properties and graph.

The graphs of $y = \arccos x$, $y = \cos(\arccos x)$, and $y = \arccos(\cos x)$.

123 Arctangent: its properties and graph.

The graphs of $y = \arctan x$, $y = \tan(\arctan x)$, and $y = \arctan(\tan x)$.

124 Why such names?

125 The link between inverse trigonometric functions and trigonometric equations.

126 Inverse trigonometric functions, Exercise 1.

Exercise 1: Two groups of easy problems (1–25 from page 841, and 57–71 from page 842):

a) Find the exact value or state that it is undefined: $\sin(\arcsin \frac{5}{4})$, $\cos(\arccos \pi)$, $\tan(\arctan 3\pi)$.

b) Find exact values of $\arcsin\left(-\frac{\sqrt{3}}{2}\right)$, $\arccos\left(-\frac{\sqrt{3}}{2}\right)$, and $\arctan(-1)$.

Extra material: notes with solved Exercise 1.

127 Inverse trigonometric functions, Exercise 2.

Exercise 2: Find the domains of the given functions:

a) $f(x) = \arcsin(5x)$

b) $f(x) = \arccos\left(\dfrac{3x-1}{2}\right)$

c) $f(x) = \arcsin(2x^2)$

d) $f(x) = \arctan(4x)$

e) $f(x) = \arccos(1/(x^2-4))$.

Extra material: notes with solved Exercise 2.

128 Inverse trigonometric functions, Exercise 3.

Exercise 3: Find the exact value or state that it is undefined:

$$\arcsin\left(\sin\frac{11\pi}{6}\right),\ \arcsin\left(\sin\frac{4\pi}{3}\right),\ \arccos\left(\cos\frac{3\pi}{2}\right),\ \arccos\left(\cos\frac{5\pi}{4}\right),\ \arctan\left(\tan\frac{\pi}{2}\right),\ \arctan\left(\tan\frac{2\pi}{3}\right).$$

Extra material: notes with solved Exercise 3.

129 Inverse trigonometric functions, Exercise 4.

Exercise 4: Compute the values of:

$$\sin\left(\arcsin\left(\frac{5}{13}\right)+\frac{\pi}{4}\right),\quad \tan\left(\arctan(3)+\arccos\left(-\frac{3}{5}\right)\right),\quad \sin(2\arctan(2)),\quad \sin\left(2\arcsin\left(\frac{x\sqrt{3}}{3}\right)\right).$$

Extra material: notes with solved Exercise 4.

S12 More identities

You will learn: how to prove trigonometric identities.

Read along with this section: **Precalculus book**: Section 10.3 (pp. 745–765) with exercises 82–128 on pages 762–763, and Section 10.4 (pp. 770–789) with exercises 26–38 on page 783 and exercises 59–73 on page 784.

130 Identities we have seen until now.

131 How to prove identities.

132 Identities, Exercise 1.

Exercise 1: Prove the following identities:

a) $\dfrac{\csc\alpha}{1+\cot^2\alpha} = \sin\alpha, \quad \dfrac{\tan\alpha}{\sec^2\alpha - 1} = \cot\alpha,$

b) $\sec^{10}\alpha = \left(1+\tan^2\alpha\right)^4 \sec^2\alpha, \quad \tan^3\alpha = \tan\alpha\sec^2\alpha - \tan\alpha.$

Assume all quantities are defined.

Extra material: notes with solved Exercise 1.

133 Identities, Exercise 2.

Exercise 2: Prove the following identities:

a) $\tan 2\alpha = \dfrac{1}{1-\tan\alpha} - \dfrac{1}{1+\tan\alpha}$

b) $\csc 2\alpha = \frac{1}{2}(\cot\alpha + \tan\alpha)$

c) $\tan\alpha + \tan\beta = \dfrac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}.$

Assume all quantities are defined.

Extra material: notes with solved Exercise 2.

134 Identities, Exercise 3.

Exercise 3: Verify the identities for $h \neq 0$:

a) $\dfrac{\sin(t+h)-\sin t}{h} = \cos t\left(\dfrac{\sin h}{h}\right) + \sin t\left(\dfrac{\cos h - 1}{h}\right)$

b) $\dfrac{\cos(t+h)-\cos t}{h} = \cos t\left(\dfrac{\cos h - 1}{h}\right) - \sin t\left(\dfrac{\sin h}{h}\right).$

Extra material: notes with solved Exercise 3.

135 Identities, Exercise 4.

Exercise 4: Show that for all $x \neq 0$. For the second one, we need an additional restriction for x. Formulate it.

$$\frac{\cos x - 1}{x} = -\frac{x}{2}\cdot\frac{\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}, \qquad \frac{\cos x - 1}{x} = -\frac{\sin x}{x}\cdot\frac{\sin x}{\cos x + 1}.$$

Extra material: notes with solved Exercise 4.

136 Identities, Exercise 5.

Exercise 5: Verify the following identities:

a) $\sin 4\alpha = 4\sin\alpha\cos^3\alpha - 4\sin^3\alpha\cos\alpha$

b) $\cos 4\alpha = 8\cos^4\alpha - 8\cos^2\alpha + 1$

c) $\cos 8\alpha = 128\cos^8\alpha - 256\cos^6\alpha + 160\cos^4\alpha - 32\cos^2\alpha + 1.$

Extra material: notes with solved Exercise 5.

137 Identities, Exercise 6.

Exercise 6: Verify the following identities:

a) $8\sin^4\alpha = \cos 4\alpha - 4\cos 2\alpha + 3$

b) $8\cos^4\alpha = \cos 4\alpha + 4\cos 2\alpha + 3$.

Extra material: notes with solved Exercise 6.

138 Identities, Exercise 7.

Exercise 7: Prove the following identities:

a) $(\sin x + \cos x)^2 = 1 + \sin 2x$

b) $\sin^4 x + \cos^4 x = 1 - \frac{1}{2}\sin^2 2x = \frac{3}{4} + \frac{1}{4}\cos 4x$

c) $\sin^6 x + \cos^6 x = 1 - \frac{3}{4}\sin^2 2x = \frac{5}{8} + \frac{3}{8}\cos 4x$.

Extra material: notes with solved Exercise 7.

139 Identities, Exercise 8.

Exercise 8: Show that $\sin(\arcsin(x) + \arccos(x)) = 1$ for all $-1 \leqslant x \leqslant 1$. Explain the result geometrically for $0 < x < 1$. Explain on the graph of the arcsine and arccosine what the result means.

Extra material: notes with solved Exercise 8.

S13 Trigonometric equations

You will learn: how to solve some basic types of trigonometric equations, how to write the series of solutions, and how to interpret both equations and their solution sets graphically. The following types of equations (or: methods of solving equations) are discussed:

1. The very basic types of trigonometric equations: $\sin x = a$, $\cos x = a$, $\tan x = a$,
2. Using sum or difference identities for sine and cosine,
3. Factorization: Sum-To-Product Formulas,
4. Factorization of polynomials,
5. Using the Product-To-Sum Formulas,
6. Reducing the degree of trigonometric functions,
7. Solution method by Universal Substitution: tangent of half argument,
8. Homogenous equations,
9. Combinations of the methods above.

Read along with this section: **Precalculus book**: Section 10.7 (pp. 857–879). Simple trigonometric equations (as discussed in Video 144) are shown on pages 737 (for sine and cosine) and 760 (for tangent).

140 What is a trigonometric equation or inequality.

141 What is the difference between a trigonometric equation and identity?

142 Functional equations and inequalities and their graphical illustrations.

Example 0: An illustration of functional equations $f(x) = g(x)$ and inequalities $f(x) < g(x)$ etc. No solution to $f(x) = g(x)$ is the same as no intersections between the graphs $y = f(x)$ and $y = g(x)$, and is the same as no zeros for $h(x) = f(x) - g(x)$.

143 Inequalities and equations you can solve without any computations.

Example 1: Solve the following equations and inequalities:

a) $\sin^2 3x + 1 > 0$,

b) $\sin^2 7x + 6 < 0$,
c) $\sin x = 0$,
d) $\cos x = 1$,
e) $\sin x + \cos x = 3$.

144 The very basic types of trigonometric equations, Problem 1.
Problem 1: Solve $\sin^2 x = \frac{3}{4}$ (the last one in the lecture) and:

$$a\sin(nx+\varphi)+b=c, \quad a,b,c,\varphi\in\mathbb{R}, \quad a\neq 0, \quad n\in\mathbb{N}^+$$
$$a\cos(nx+\varphi)+b=c, \quad a,b,c,\varphi\in\mathbb{R}, \quad a\neq 0, \quad n\in\mathbb{N}^+$$
$$a\tan(nx+\varphi)+b=c, \quad a,b,c,\varphi\in\mathbb{R}, \quad a\neq 0, \quad n\in\mathbb{N}^+$$

Extra material: notes with solved Problem 1.

145 Using sum or difference identities for sine and cosine, Problem 2.
Problem 2: Solve the equations $\sqrt{2}\sin x + \sqrt{2}\cos x = -1$ and $\sin 6x \cos x = -\cos 6x \sin x$.
Extra material: notes with solved Problem 2.

146 Factorization: Sum-To-Product Formulas, Problem 3.
Problem 3: Solve the equations $\sin x + \sin 5x = 0$ and $\cos 3x + \sin 2x - \sin 4x = 0$.
Extra material: notes with solved Problem 3.

147 Factorization of polynomials, Problem 4.
Problem 4: Solve the equations $\cos 2x = 1 - \sin x$ and $2\sin^2 x + \sin x - 1 = 0$.
Extra material: notes with solved Problem 4.

148 Using the Product-To-Sum Formulas, Problem 5.
Problem 5: Solve the equations $\cos 3x \cos 6x = \cos 4x \cos 7x$ and $\sin 5x \cos 3x = \sin 6x \cos 2x$.
Extra material: notes with solved Problem 5.

149 Reducing the degree of trigonometric functions, Problem 6.
Problem 6: Solve the equation $\sin^2 x + \sin^2 3x = 1$.
Extra material: notes with solved Problem 6.

150 Universal Substitution: tangent of half argument, Problem 7.
Problem 7: Solve the equations $2\sin 2x + 3\tan x = 5$ and $\tan x \tan 2x = \tan x + \tan 2x$.
Extra material: notes with solved Problem 7.

151 Homogenous equations, Problem 8.
Problem 8: Solve the equation $6\sin^2 x + \sin x \cos x - \cos^2 x = 2$.
Extra material: notes with solved Problem 8.

152 Combinations of the methods above, Problem 9.
Problem 9: Solve the equation $\frac{1}{2}(\tan^2 x + \cot^2 x) = 1 + \frac{2}{\sqrt{3}}\cot 2x$.
Extra material: notes with solved Problem 9.

153 A difficult one, Problem 10.
Problem 10: Solve the equation $2(\sin^6 x + \cos^6 x) - 3(\sin^4 x + \cos^4 x) = \cos 2x$.
Extra material: notes with solved Problem 10.

154 Another difficult one, Problem 11.
Problem 1: Solve the equation $\cot^4 x = \cos^3 2x + 1$.
Extra material: notes with solved Problem 11.

S14 Some applications of trigonometry

Read along with this section: **Precalculus book**: Chapter 11 (pp. 881–1068).

You will learn: Including applications would make this course twice as large, so I will just concentrate on the most common applications. The lectures will not have the same level of detail as the lectures in the previous sections, but by now, you are probably able to read and understand Chapter 11 in the Precalculus book on your own, so do it and ask me questions if needed. I will address the following topics in this section: slopes of straight lines in the coordinate system; The Law of Cosines as a generalization of Pythagorean Theorem; a sine-based formula for the area of a triangle; The Law of Sines; Heron's Formula; solving oblique triangles; vectors in the plane (or in the 3-space) and angles between them; rotations and their matrices; complex numbers: rectangular and polar form, multiplication of complex numbers and an explanation of how its geometry is determined by the Sum Identities for the sine and cosine, de Moivre's formula for taking powers of complex numbers, roots of unity.

155 Tangent and slopes of straight lines in the 2D coordinate system.

156 How to compute the angle between two lines in the 2D coordinate system.

Derivation of the formula for the angle between two straight lines: $y = m_1x + b_1$ and $y = m_2x + b_2$. Two such lines are orthogonal if $m_1m_2 = -1$.

157 The Law of Cosines as a generalization of Pythagorean Theorem.

If $\triangle ABC$ is any triangle, then:

- $|\sphericalangle BCA| < 90^\circ \quad \Rightarrow \quad |CB|^2 + |AC|^2 > |BA|^2$,
- $|\sphericalangle BCA| = 90^\circ \quad \Rightarrow \quad |CB|^2 + |AC|^2 = |BA|^2$,
- $|\sphericalangle BCA| > 90^\circ \quad \Rightarrow \quad |CB|^2 + |AC|^2 < |BA|^2$.

Formulas for the cosine of the angles in a triangle, given the lengths of the sides: a, b, c:

$\cos\alpha = \dfrac{b^2 + c^2 - a^2}{2bc}, \qquad \cos\beta = \dfrac{a^2 + c^2 - b^2}{2ac}, \qquad \cos\gamma = \dfrac{a^2 + b^2 - c^2}{2ab}.$

158 A sine-based formula for the area of a triangle.

Derivation of the formula for the area of a triangle with side lengths a, b, c and corresponding angles α, β, γ: $A = \frac{1}{2}ab\sin\gamma = \frac{1}{2}bc\sin\alpha = \frac{1}{2}ac\sin\beta$.

159 The Law of Sines.

Derivation of The Law of Sines for triangle with side lengths a, b, c and corresponding angles α, β, γ:

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R,$$

where R is the radius of the circumscribed circle on the triangle.

160 Heron's Formula.

161 Solving oblique triangles.

162 Vectors in the plane (or in the 3-space) and angles between them.

163 The dot product and projection.

164 Rotations and their matrices.

165 Complex numbers: rectangular and polar form.

166 Multiplication of complex numbers and the Sum Identities for the sine and cosine.

Multiply two complex numbers in polar form

$$z_1 = |z_1|(\cos\alpha + i\sin\alpha) \quad \text{and} \quad z_2 = |z_2|(\cos\beta + i\sin\beta).$$

Write the product $z = z_1 z_2$ in a polar form. Draw some conclusions about multiplication of complex numbers in polar form. Use this conclusion to confirm the formula from Video 164:

$$R_\alpha(x, y) = (x\cos\alpha - y\sin\alpha,\ \ x\sin\alpha + y\cos\alpha),$$

where R_α denotes the rotation about the angle α around the origin in the plane.

Extra material: notes from the iPad.

167 De Moivre's formula for taking powers of complex numbers.

Derivation of de Moivre's Formula for natural powers of complex numbers:

$(|z|(\cos\alpha + i\sin\alpha))^n = |z|^n(\cos n\alpha + i\sin n\alpha)$.

Exercise: Verify the following identities using de Moivre's Formula:

a) $\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$

b) $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$.

Extra material: notes with solved Exercise.

168 Roots of unity.

169 Harmonic motion: reading recommendations.

S15 Sneak peek into trigonometry in Calculus

You will learn: This section will give you some pointers to applications of trigonometry in Calculus. The purpose is *not* to teach you this stuff, but rather to give you an idea about how the skills gained during this course will help you in the future Calculus class. The topics mentioned here are:

- the limit of $\sin x/x$ in zero, and its importance in Calculus,
- the slope of a straight line and its importance for Differential Calculus,
- differentiability of the sine and cosine: which formulas to use,
- the derivatives (with examples of the sine, cosine, tangent, arcsine, and arctangent) and their role in finding extremums and for determining intervals of monotonicity,
- classes of functions (C^0, C^1, C^2, ...) and some fun trigonometric examples,
- a word about Fourier and spirographs, Euler's formula, and Euler's identity.
- trigonometric functions in solutions of differential equations,
- polar coordinates in the plane,
- cylindrical and spherical coordinates,
- parametric curves,
- Power Reduction Formulas and integration,
- Trigonometric substitutions in integrals.

170 The limit of $\sin x/x$ in zero, and its importance in Calculus.

Estimation of the expression $\sin x/x$ for $x \in (0, \pi/2)$ (approximation below by $\cos x$, and above by 1), and a visual illustration of one of the most important limits of an indeterminate form in zero:

$$\lim_{x\to 0}\frac{\sin x}{x} = 1.$$

This limit will be used in Video 172 for motivation that the derivative of the sine is cosine.

171 The slope of a straight line and its importance for Differential Calculus.

172 Differentiability of the sine and cosine: which formulas to use.

The derivative of sine is cosine; the derivative of cosine is negative sine; applications of various identities.

Extra material: notes from the iPad.

173 The derivatives of sine, cosine, tangent, arcsine and arctangent; some nice visuals.

The derivative of tangent is computed using the quotient rule.

174 Interesting examples of sine-based functions.

The following examples are presented: $\sin \frac{1}{x}$, $x \sin \frac{1}{x}$, $x^2 \sin \frac{1}{x}$ $(x \neq 0)$.

175 A word about Fourier, spirographs, Euler's formula, and Euler's identity.

176 Trigonometric functions in solutions of differential equations.

177 Polar coordinates in the plane.

1. Equation $r = c$ for any positive constant c describes a circle with centre in the origin.
2. Equation $\theta = \theta_0$ for any $\theta_0 \in [0,\ 2\pi)$ describes a half-line without the origin.
3. $\{(x, y) \in \mathbb{R}^2;\ x^2 + y^2 \leqslant 4\} = \{(r, \theta) \in \mathbb{R}^2;\ \ 0 \leqslant r \leqslant 2,\ 0 \leqslant \theta \leqslant 2\pi\}$, i.e., a disk in the Cartesian coordinate system describes as a rectangle in the polar coordinate system.

178 Cylindrical and spherical coordinates in the 3-space.

1. Equation (in spherical coordinates) $R = c$ for a constant $c > 0$ describes a sphere with centre in the origin.
2. Equation (in spherical coordinates) $\theta = \theta_0$ for any $\theta_0 \in [0,\ 2\pi)$ describes a vertical half-plane with edge along the z-axis, without this edge.
3. Equation (in spherical coordinates) $\phi = \phi_0$ describes a cone, the xy-plane or a half-line, depending on ϕ_0.

179 Parametric curves.

Three parametrizations of the unit circle:

a) $\vec{r}(t) = (\cos t,\ \sin t),\ \ t \in [0,\ 2\pi)$,

b) $\vec{r}(t) = (\cos 2t,\ \sin 2t),\ \ t \in [0,\ \pi)$,

c) $\vec{r}(t) = (\cos 5t,\ \sin 5t),\ \ t \in [0,\ 2\pi/5)$.

180 Power Reduction Formulas and integration.

181 Trigonometric substitutions in integrals.

S16 Problem-solving: varia

You will learn: This section gives you a *Smörgåsbord* of problems to solve; the difficulty level varies, and, as the problems are *not* linked to specific sections, you will have to decide on your own what method to choose. Generally, the problems and exercises in the previous sections were on a basic level (with some minor exceptions), and the problems in this section are somewhat harder. Originally, I planned to assign badges **Basic**, **Medium**, or **Hard** to each problem, but then I thought: Each problem you *can't* solve is hard; each problem you *can* solve is simple (for you). So I changed my mind, and the problems are just presented to you without any labels.

182 How to use this section.

183 Problem 1.

Problem 1: Compute the following:

$$\sin^2 25^\circ + \sin^2 65^\circ + \sin 75^\circ, \quad \tan 30^\circ - \sin 30^\circ, \quad \frac{\sin^2 38^\circ + \cos^2 38^\circ - 1}{\sin^2 52^\circ + \cos^2 52^\circ + 1}.$$

Extra material: notes with solved Problem 1.

184 Problem 2.

Problem 2: Compute $\dfrac{2\sin^2 50^\circ - 1}{2\cot 95^\circ \cdot \cos^2 175^\circ}$.

Extra material: notes with solved Problem 2.

185 Problem 3.

Problem 3: If $\sin\alpha = \frac{1}{4}$ and $\frac{\pi}{2} < \alpha < \pi$, then $\cos\alpha$ is equal to:

a) $\dfrac{\sqrt{15}}{4}$,

b) $-\dfrac{\sqrt{15}}{4}$,

c) $\pm\dfrac{\sqrt{15}}{4}$,

d) another answer.

Extra material: notes with solved Problem 3.

186 Problem 4.

Problem 4.1: If $\tan\alpha = p$ and $\frac{\pi}{2} < \alpha < \pi$, then $\sin\alpha$ is equal to:

a) $\dfrac{-p}{\sqrt{1+p^2}}$,

b) $\dfrac{p}{\sqrt{1+p^2}}$,

c) $\pm\dfrac{p}{\sqrt{1+p^2}}$,

d) another answer.

Problem 4.2: If $\cos\alpha = p$ and $\pi < \alpha < \frac{3\pi}{2}$, then $\tan\alpha$ is equal to:

a) $\dfrac{\sqrt{1-p^2}}{p}$,

b) $\dfrac{\sqrt{1-p^2}}{|p|}$,

c) $\pm\dfrac{\sqrt{1-p^2}}{p}$,

d) another answer.

Extra material: notes with solved Problem 4.

187 Problem 5.

Problem 5: If α is an angle in a triangle, and $\tan\alpha = 7$, then:

a) $0 < \alpha < \dfrac{\pi}{6}$,

b) $\dfrac{\pi}{3} < \alpha < \dfrac{\pi}{2}$,

c) such an angle does not exist,

d) none of the above.

Extra material: notes with solved Problem 5.

188 Problem 6.

Problem 6: Determine the max and the min for $f(x) = \dfrac{1}{\cos^4 x + \sin^4 x}$ and for $g(x) = \dfrac{1}{\cos^6 x + \sin^6 x}$.

Extra material: notes with solved Problem 6.

189 Problem 7.

Problem 7: Verify the following identities using de Moivre's Formula:

a) $\sin 4\alpha = 4\sin\alpha\cos^3\alpha - 4\sin^3\alpha\cos\alpha$

b) $\cos 4\alpha = 8\cos^4\alpha - 8\cos^2\alpha + 1$.

Extra material: notes with solved Problem 7.

190 Problem 8.

Problem 8: Derive the following sum identities; assume that all the quantities are well defined:

$$\sin(\alpha+\beta+\gamma) = \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\beta\cos\gamma + \cos\alpha\cos\beta\sin\gamma - \sin\alpha\sin\beta\sin\gamma$$

$$\cos(\alpha+\beta+\gamma) = \cos\alpha\cos\beta\cos\gamma - \cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\beta\sin\gamma - \sin\alpha\sin\beta\cos\gamma$$

$$\tan(\alpha+\beta+\gamma) = \frac{\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma}{1 - \tan\alpha\tan\beta - \tan\beta\tan\gamma - \tan\gamma\tan\alpha}.$$

Extra material: notes with solved Problem 8.

191 Problem 9.

Problem 9: Prove the following sum identities for cotangent; assume that all the quantities are well defined:

$$\cot(\alpha+\beta) = \frac{\cot\alpha\cot\beta - 1}{\cot\alpha + \cot\beta}, \qquad \cot(\alpha+\beta+\gamma) = \frac{\cot\alpha + \cot\beta + \cot\gamma - \cot\alpha\tan\beta\cot\gamma}{1 - \cot\alpha\cot\beta - \cot\beta\cot\gamma - \cot\gamma\cot\alpha}.$$

Extra material: notes with solved Problem 9.

192 Problem 10.

Problem 10: use the results from the previous two problems to prove the following (assume that all the quantities are well defined):

a) **The triple tangent identity**: If $\alpha+\beta+\gamma=\pi$ then $\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\cdot\tan\beta\cdot\tan\gamma$.

b) **The triple cotangent identity**: If $\alpha+\beta+\gamma=\pi$ then $\cot\frac{\alpha}{2} + \cot\frac{\beta}{2} + \cot\frac{\gamma}{2} = \cot\frac{\alpha}{2}\cdot\cot\frac{\beta}{2}\cdot\cot\frac{\gamma}{2}$.

Show that $\tan 50^\circ + \tan 60^\circ + \tan 70^\circ = \tan 50^\circ\cdot\tan 60^\circ\cdot\tan 70^\circ$.

Extra material: notes with solved Problem 10.

193 Problem 11.

Problem 11: Prove **Heron's formula**: Suppose a, b and c denote the lengths of the three sides of a triangle. Let s be the semiperimeter of the triangle, i.e., $s = \frac{1}{2}(a+b+c)$. Then the area A of the triangle is:

$$A = \sqrt{s(s-a)(s-b)(s-c)}.$$

Extra material: notes with solved Problem 11.

194 Problem 12.

Problem 12: Let

$$P = \sin 10^\circ + \sin 20^\circ + \sin 30^\circ + \cdots + \sin 180^\circ, \qquad Q = \cos 10^\circ + \cos 20^\circ + \cos 30^\circ + \cdots + \cos 180^\circ$$

Which of the following is true?: $A.\ P=0$ $\quad B.\ Q=0$ $\quad C.\ Q=-1$ $\quad D.\ P^2+Q^2=1$.

195 Problem 13.

Problem 13: Identify the curve given by the following parametrization: $\vec{r}(t) = (\cos 2t,\ \sin t)$, $\ t\in[0,\,2\pi]$.

Extra material: notes with solved Problem 13.

196 Problem 14.

Problem 14: Let $n\in\mathbb{N}^+$. Compute $\sin x + \sin 2x + \sin 3x + \cdots + \sin nx$.
(This is the problem mentioned in Video 113.)

Extra material: notes with solved Problem 14.

197 Problem 15.

Problem 15: Show that the number $\sin 10°$ is irrational.

Extra material: notes with solved Problem 15.

198 Problem 16.

Problem 16: Compute $\tan\frac{x}{2}$ if $4\sin x + 3\cos x = 3$.

Extra material: notes with solved Problem 16.

199 Problem 17.

Problem 17: Determine $x+y$ knowing that $(1+\tan x)(1+\tan y) = 2$.

Extra material: notes with solved Problem 17.

200 Problem 18.

Problem 18: One of the angles of a right triangle satisfies the equation

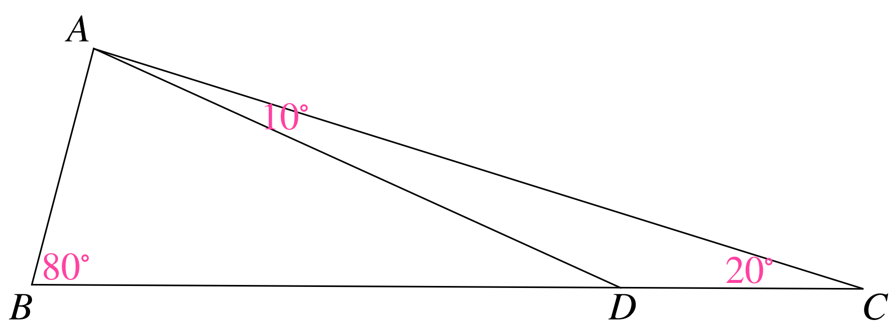
$$\sin^3 x + \sin x \sin 2x - 3\cos^3 x = 0.$$

Prove that the triangle is isosceles.

Extra material: notes with solved Problem 18.

201 Problem 19.

Problem 19: Given $\triangle ABC$ with angles as in the picture. Show that $AB = DC$.

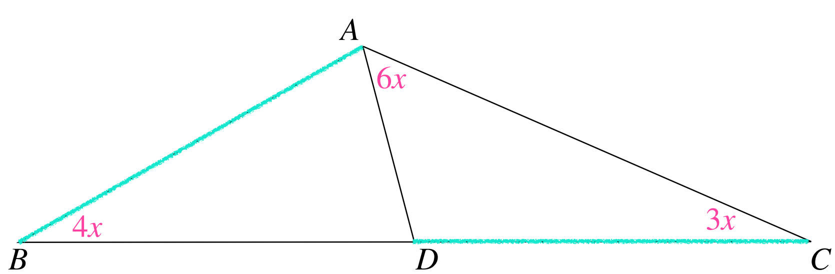


Figur 4: Illustration to the problem in Video 201.

Extra material: notes with solved Problem 19.

202 Problem 20.

Problem 20: Given $\triangle ABC$ with angles as in the picture. Moreover, $AB = DC$. Determine x.

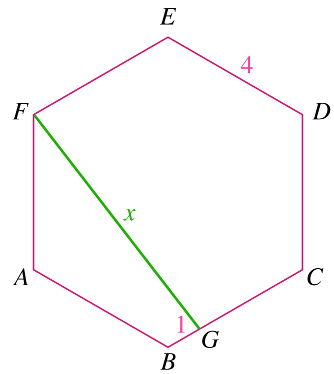


Figur 5: Illustration to the problem in Video 202.

Extra material: notes with solved Problem 20.

203 Problem 21.

Problem 21: Given regular hexagon $ABCDEF$ with side length 4. The length of BG (see the picture) is 1. Determine x (the length of FG).



Figur 6: Illustration to the problem in Video 203.

Extra material: notes with solved Problem 21.

204 Problem 22.

Problem 22: Given a parallelogram with side lengths a and b where $0 < b < a$. Moreover, the diagonals of this parallelogram have lengths b and $2b$. Determine the ratio $\frac{a}{b}$.

Extra material: notes with solved Problem 22.

205 Problem 23.

Problem 23: Given a rectangular cuboid with edges of lengths a, b, and c. If the angles between these edges and the diagonal of the cuboid are equal to α, β, and γ, show that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. If the angles between the faces and the diagonal of the cuboid are equal to λ, μ, and ν, show that $\cos^2\lambda + \cos^2\mu + \cos^2\nu = 2$.

206 Problem 24: one of the nicest problems ever, with two really cool solutions.

Problem 24: Prove that $\cos\frac{\pi}{7} \cdot \cos\frac{2\pi}{7} \cdot \cos\frac{3\pi}{7} = \frac{1}{8}$.

Two proofs are given: one using trigonometric identities, one geometrical.

Extra material: notes with solved Problem 24.

207 Problem 25: my favourite problem with many ways of solving it.

Problem 25: Show that $\arctan 1 + \arctan 2 + \arctan 3 = \pi$, or, equivalently, that $\operatorname{arccot} 1 + \operatorname{arccot} 2 + \operatorname{arccot} 3 = \frac{\pi}{2}$.

Five proofs are given: all of them use trigonometry in one way or another.

Extra material: notes with solved Problem 25.

208 Precalculus 3, Wrap-up.

S17 Extras

You will learn: about all the courses we offer, and where to find discount coupons. You will also get a glimpse into our plans for future courses, with approximate (very hypothetical!) release dates.

B Bonus lecture.

Extra material 1: a pdf with all the links to our courses, and coupon codes.

Extra material 2: a pdf with an advice about optimal order of studying our courses.

Extra material 3: a pdf with information about course books, and how to get more practice.