Precalculus 2: Polynomials and rational functions[1]

Mathematics from high school to university

Hania Uscka-Wehlou

A short table of contents

[1] Recorded May–June 2022. Published on `www.udemy.com` on 2022-06-XX.

An extremely detailed table of contents; the videos (titles in green) are numbered

In blue: problems solved on an iPad (the solving process presented for the students; active problem solving)

In red: solved problems demonstrated during a presentation (a walk-through; passive problem solving)

In magenta: additional problems solved in written articles (added as resources).

In dark blue: *Read along with this section*: references for further reading and exercises in *Prerequisites* book and in the *Precalculus* book by Carl Stitz and Jeff Zeager.

Books to read along with the course, with more practice problems:

1. *Precalculus Prerequisites, a.k.a. Chapter* 0: Carl Stitz, Ph.D., Lakeland Community College; Jeff Zeager, Ph.D., Lorain County Community College; version from August 16, 2013.
2. *Precalculus*: Carl Stitz, Ph.D., Lakeland Community College; Jeff Zeager, Ph.D., Lorain County Community College; version from July 4, 2013.

These books are added as resources to Video 1, with kind permission of Professor Carl Stitz.

S3 Powers, expressions, and polynomials

You will learn: about powers with natural, integer, and rational exponents and the computation rules holding for them (the product rules, the quotient rules, the power rule); basic terminology concerning polynomials (term, degree, monomial, binomial, trinomial, monic polynomial); polynomial arithmetic (addition, subtraction, scaling, multiplication), composition of polynomials.

Read along with this section: **Prerequisites book**: Section 0.2 (pp. 26–37; you can skip the exercises with fractional exponents (roots) this time: they come back in the course about exponential functions: Precalculus 4), Section 0.5 (pp. 60—63, Ex. p. 68; skip the exercises about division, by now), **Precalculus book**: Section 5.1 (just compositions of polynomials).

12 Powers with positive natural exponents.

Exercise 1: Compute: $2^3,\ (-3)^3,\ (-\frac{3}{2})^3,\ (-4)^2,\ 10^5$.
Write as a power with a natural exponent: $9,\ -125,\ 27,\ \frac{4}{25},\ 64,\ -1$.

Extra material: notes with solved Exercise 1.

13 Two product rules.

Exercise 2: Explain why the product rules are true; use $n = 3$ and $m = 4$.
Compute: $2^{10} \cdot 5^{10},\ (\frac{1}{2})^{100} \cdot 2^{100},\ (\frac{2}{3})^{10} \cdot (\frac{3}{4})^{10},\ x^3 \cdot x^7,\ x^3yz^7 \cdot x^2y^5z^3$.

Extra material: notes with solved Exercise 2.

14 Before we continue: Why we need more rules for powers.

15 Two quotient rules.

Exercise 3: Explain why the quotients rules are true; use $n = 5$ and $m = 3$.
Compute: $\frac{50^{10}}{25^{10}},\ \frac{2^{100}}{2^{97}}$.

Extra material: notes with solved Exercise 3.

16 The power rule.

Exercise 4: Explain why the power rule is true; use $n = 3$ and $m = 2$.
Compute: $(x^3)^6,\ (x^2 + 1)^2,\ (x^4 + 3x)^2$.

Extra material: notes with solved Exercise 4.

17 How we expand the definition of powers to other exponents (than just the positive natural ones).

18 Zero powers.

Extra material: notes with the explanation why $a^0 = 1$ for all $a \neq 0$.

19 Powers with negative integer exponents.

Extra material: notes with the explanation why $a^{-n} = \frac{1}{a^n}$ for all $a \neq 0$ and $n \in \mathbb{N}^+$.

20 Powers with rational exponents.

Extra material: notes with the explanation why $a^{1/n} = \sqrt[n]{a}$ for all $a \in \mathbb{R}$ and $n \in \mathbb{N}^+ \setminus \{1\}$. If n is even, then we only can take $a \geqslant 0$. Consequently, $a^{m/n} = \sqrt[n]{a^m}$ for each $m \in \mathbb{Z}$ (and $a > 0$ in some cases).

21 An exercise for computing powers.

Exercise 5: Compute: $2^{-5},\ 27^{1/3},\ 16^{-1/2},\ \left(\frac{4}{9}\right)^{-1/2},\ (-81)^{1/2},\ 8^{2/3}$.

Extra material: notes with solved Exercise 5.

22 Back to precedence rules.

Exercise 6: Compute: $(-1)^6 + 5 \cdot (-1)^3 + 4 \cdot (-1)^2 - (-1)^1 + 2$.

Extra material: notes with solved Exercise 6.

23 Definition of *univariate* polynomials, and some basic terminology.

24 Polynomials, a terminology exercise.

Exercise 7: What is the degree and the coefficients of the following polynomials? Which of these polynomials are *monic*? For these which are not monic, represent them as a product of a monic polynomial and a real number.

a) $f(x) = x + 2$

b) $f(x) = -4x^5 - 7x^4 - x$

c) $f(x) = \pi x^3 - \sqrt{3}x^2 - 5$

d) $f(x) = 3x^7 - \sqrt{3}x^5 - 13x^2 + \pi$

e) $f(x) = x^{100}$

f) $f(x) = (x^2 + 3)^2$

Extra material: notes with solved Exercise 7.

25 The domain of polynomial functions.

Exercise 8: Polynomial or not?

a) $f(x) = 3x^7 - \sqrt{3}x^5 - 13x^2 + \pi$

b) $f(x) = 3x^7 - \sqrt{3x^5} - 13x^2 + \pi$

c) $f(x) = \frac{x^3+x}{x^2+1}$

d) $f(x) = x^2 + \sin x$

e) $f(x) = x^2 + \left(\sin \frac{\pi}{6}\right) \cdot x + 17$

f) $f(x) = \frac{x^2+2}{x}$

g) $f(x) = \frac{x^2+2x}{x}$

h) $f(x) = \sqrt{\sin x}$

i) $f(x) = 3x^5 - 5x^4 - 9x^{-1} + 6$

Extra material: notes with solved Exercise 8.

26 Evaluation of polynomials.

Exercise 9: Evaluate $p(x) = x^6 + 5x^3 + 4x^2 - x + 2$ at $x = -1$, $x = 0$, and $x = 1$.

Extra material: notes with solved Exercise 9.

27 Addition of polynomials, like terms.

Exercise 10: Add (and subtract) following polynomials:

a) $p(x) = 5x^3 - 7x^2 - x$ and $q(x) = x^5 - x^3 + 5$

b) $p(y) = 7y^4 - 5y^2 - 7$ and $q(y) = 3y^4 + 2y^3 + y^2$

c) $p(t) = -3t^7 + t^6 - 11t^4 - 9t^3 + 18t^2 - 1$ and $q(t) = (t^3 - 2t)^2$.

Extra material: notes with solved Exercise 10.

28 Multiplication of polynomials. The distributive law.

Exercise 11: Perform the following operations:

a) the scaling $5(4x^3 - 3x^2 - 8x + 3)$

b) polynomial multiplication $(x^2 + 4)(x - 1)$

c) $(2x^2 + x - 1)(x^3 - 2x)$.

Extra material: notes with solved Exercise 11.

29 Another exercise for multiplication.

Exercise 12: Prove the following formulas:

a) $(a - b)(a + b)(a^2 + b^2) = a^4 - b^4$

b) $(a^2 + \sqrt{2}ab + b^2)(a^2 - \sqrt{2}ab + b^2) = a^4 + b^4$.

Extra material: notes with solved Exercise 12.

30 Composition of polynomials; Pascal's triangle.

Exercise 13: Verify the following formulas:

a) $(a+b)^2 = a^2 + 2ab + b^2$

b) $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

c) $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$.

Generalize c): $(a_1 + \cdots + a_n)^2$ for any natural $n \geqslant 2$.

Extra material: notes with solved Exercise 13.

31 Composition of polynomials, an exercise.

Exercise 14: Perform the compositions $p \circ q$ and $q \circ p$:

a) $p(x) = x^2 + 2, \quad q(x) = 3x^2 - x + 5$

b) $p(x) = 2x^3, \quad q(x) = 4x^2 - 5$.

Compute $(p \circ q)(1)$ and $(q \circ p)(1)$ in both cases.

Extra material: notes with solved Exercise 14.

32 Monic monomials and their importance.

33 **Advanced**: Some theory about the values of polynomial functions.

Exercise 15: Analyze the following polynomials in both infinities:

a) $p(x) = 5x^3 - 7x^2 - x$

b) $p(x) = -x^3 + 2x^2$

c) $p(x) = x^4 + x^3 - 11x^2 - 9x + 18$

d) $p(x) = -x^4 + 5x^2 + 10$.

Extra material: notes with solved Exercise 15.

Extra material: An article with some theorems and proofs:

a) Theorem 1: About existence of non-zero values of polynomials.

b) Corollary 1: An upper bound for zeros of polynomials.

c) Corollary 2: Values of polynomials can be arbitrarily large if $a_n > 0$ and $n \geqslant 1$.

d) Corollary 3: About the zero polynomial.

e) Corollary 4: About equality of polynomials.

f) Theorem 2: A better bound for the zeros of a polynomial (Cauchy's Bound).

34 How various operations on polynomials affect the degree; the zero polynomial.

S4 Linear equations and systems of equations

You will learn: how to solve $n \times n$ systems of linear equations and why you need it for your works with polynomials and rational functions.

Read along with this section: **Precalculus book**: Chapter 8 (Sections 8.1 and 8.2 should be enough for now). If you want to see the entire theory on the topic (which you *do not* need for Calculus), you can watch my course *Linear Algebra and Geometry* 1.

35 Linear and non-linear equations.

Example: Which of the following equations are linear. Explain:

$2x + 3y + 5z - 7t = 0, \quad \sqrt{3}x - \pi y = 7, \quad \sqrt{x^2+1} = 7, \quad \sin x - \cos x = 0, \quad (\sin \frac{\pi}{6})x - \pi^2 y - \sqrt{7}z = 0,$
$2x + \sin y = 5, \quad x + 2t = 17, \quad \ln x - 7 = 0, \quad (\ln 2)x - \sqrt{\pi}y + \frac{1}{z} = 0, \quad xy = 7, \quad 2x + 6y = 5x - 7y + z.$

36 Linear polynomials and straight lines.

37 Systems of linear equations.

Example: solve the following system of linear equations

$$\begin{cases} 2x + 3y = -1 \\ 4x + y = 3 \end{cases}$$

using three methods: the geometrical method; elimination of one variable by expressing it with help of another one from one of the equations, and plugging it in in the second equation; elimination of one variable by using opposite coefficients.

38 Solving systems of linear equations (Calculus), Problem 1.

Problem 1: Find values of a, b and c such that the graph of the polynomial $p(x) = ax^2 + bx + c$ passes through the points $(1, 2)$, $(-1, 6)$, and $(2, 3)$.

Extra material: notes with solved Problem 1.

39 Solving systems of linear equations (Calculus), Problem 2.

Problem 2: Find the cubic polynomial whose graph passes through the points

$$(-1, -2), \quad (0, -4), \quad (1, 0), \quad (2, 16).$$

Extra material: notes with solved Problem 2.

40 Undetermined coefficients and *Ansatz*, Problem 3.

Problem 3: The cubic polynomial $p(x) = x^3 + 3x^2 - 4$ from the previous video has a zero $x_0 = 1$. Represent $p(x)$ as a product of the monomial $x - 1$ and a second-degree polynomial.

Extra material: notes with solved Problem 3.

41 Overdetermined systems.

42 Inconsistent overdetermined system, Problem 4.

Problem 4: Show that the polynomial $p(x) = 2x^4 - 3x^3 + 4x^2 - 5x + 6$ is **not** divisible by $d(x) = x^2 - 3x + 1$.

Extra material: notes with solved Problem 4.

43 **Future**: Undetermined coefficients and partial fractions, Problem 5.

Problem 5: Find constants A and B such that

$$\frac{1}{x^2 - 1} = \frac{A}{x - 1} + \frac{B}{x + 1}$$

for all $x \in \mathbb{R} \setminus \{1, -1\}$.

Extra material: notes with solved Problem 5.

44 **Future**: Undetermined coefficients and partial fractions, Problem 6.

Problem 6: Find constants A, B and C such that

$$\frac{2x + 1}{x(x^2 + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}$$

for all $x \in \mathbb{R} \setminus \{0\}$.

Extra material: notes with solved Problem 6.

S5 Second degree polynomials

You will learn: solving quadratic equations by using qualified guesses for factoring, completing the square, and the quadratic formula; plotting parabolas by finding the coordinates of the vertex and transforming the parabolas $y = x^2$ and $y = ax^2$ to this vertex; Vieta's formula with proof and some applications.

Read along with this section: **Prerequisites book**: Section 0.7 *Quadratic equations* (pp. 83–95). If you need to repeat some high-school stuff, you can also read Section 0.6 *Factoring* (pp. 70–82) in the **Prerequisites book**.

45 From linear to quadratic.

46 The most important parabola.

Exercise 1: Some properties of $p(x) = x^2$:

a) the function is increasing for $x > 0$,

b) the function is decreasing for $x < 0$,

c) the function is even,

d) the graph of p is symmetric along the y-axis,

e) the range is $[0, +\infty)$,

f) the polynomial has one double zero $x_0 = 0$.

Extra material: notes with the solution of Exercise 1 (and the proof that you can multiply sidewise inequalities between positive numbers; and that $f(x) = x^n$ for natural $n \geqslant 2$ are increasing for $x > 0$).

47 How to plot graphs of other second-degree polynomials using transformations of the standard parabola.

Exercise 2: Plot the graphs of:

a) $f(x) = x^2$,

b) $f(x) = x^2 - 4$,

c) $f(x) = x^2 - 2x + 1$,

d) $f(x) = x^2 + 4x + 4$,

e) $f(x) = x^2 + 6x + 4$

f) $f(x) = x^2 + 6x - 40$.

48 The method of completing the square.

Exercise 3: Complete the square in the following cases:

a) x^2,

b) $x^2 + 2x + 1$,

c) $x^2 + 6x + 10$,

d) $x^2 + 6x - 40$,

e) $x^2 + x - 1$.

Extra material: notes with the solution of Exercise 3.

49 Completing the square for finding the vertex.

Exercise 4: Plot the graph of $f(x) = x^2 + 6x - 40$ and $g(x) = -x^2 + 4x + 7$.

50 The square root of a number versus a root of a quadratic equation.

Extra material: notes from the iPad.

51 Completing the square for finding the zeros.

Exercise 5: Solve the following equations by completing the square:

a) $x^2 = 0$,

b) $x^2 + 2x + 1 = 0$,

c) $x^2 + 6x + 10 = 0$,

d) $x^2 + 6x - 40 = 0$,

e) $x^2 + x - 1 = 0$.

Show an illustration for d), explaining the geometrical name of the method.

Extra material: notes with the solution of Exercise 5.

52 Completing the square for factoring polynomials.

53 Solving quadratic equations; derivation of the quadratic formula.

54 Quadratic formula or completing the square?

Exercise 6: Complete the square in the following cases:

a) $4x^2 - 7x - 2$,

b) $6x^2 + x - 1$.

Plot the corresponding parabolas and find their x-intercepts by solving the corresponding quadratic equations. Verify your results using the quadratic formula and the formula for the vertex.

Extra material: notes with the solution of Exercise 6.

55 Second degree polynomials as products of two linear polynomials.

Example 1: Four illustrations of second-degree polynomials as products of two first-degree polynomials:

a) $-x^2 + 4x - 3 = (x-1)(-x+3)$,

b) $x^2 - 4x + 3 = (x-1)(x-3)$,

c) $-x^2 - x + 6 = (-x-3)(x-2)$,

d) $x^2 + x - 6 = (x+3)(x-2)$.

56 Vieta's formulas.

Example 2: Given a quadratic equation with $x_1 = 1$, find x_2:

a) $17x^2 + 4x - 21 = 0, \quad x_2 = -\frac{21}{17}$,

b) $12x^2 - 5x - 7 = 0, \quad x_2 = -\frac{7}{12}$.

Example 3: Factoring polynomials (reverse F.O.I.L.):

a) $x^2 - 7x + 6 = (x-1)(x-6)$ because $1 \cdot 6 = 6,\ 1 + 6 = -(-7) = 7$,

b) $x^2 - 3x + 2 = (x-1)(x-2)$ because $1 \cdot 2 = 2,\ 1 + 2 = -(-3) = 3$.

57 Vieta's formulas, Problem 1.

Problem 1:

a) Real numbers x_1 and x_2 are the roots of the quadratic equation $x^2 + bx - 1 = 0$. Evaluate $x_1^3 + x_2^3$.

b) For what value of a parameter a is the ratio of the roots of quadratic equation $x^2 + ax - 16 = 0$ equal to -4?

Extra material: notes with the solution of Problem 1.

58 Confirmation of Vieta's formulas with help of the quadratic formula.

Extra material: notes from the iPad.

59 Rational zeros of second-degree polynomials with integer coefficients.

Example 4: Find all the rational zeros of the following polynomials:

a) $x^2 + x - 2$,

b) $x^2 + x - 6$,

c) $6x^2 + x - 1$,

d) $4x^2 - 7x - 2$,

e) $x^2 + x - 5$.

Extra material: notes with the solution of part e).

60 Complex numbers, a brief introduction.

61 Complex numbers and their arithmetic, an exercise.

Exercise 7: Compute the following:

a) the conjugate of $(2-3i) + (4+5i)$,

b) the conjugate of $(3-2i)(4+i)$,

c) i^n for $n = 1, 2, 3, \ldots,$

d) the conjugate of $(2+i)^3$.

Extra material: notes with the solution of Exercise 7.

62 Solving quadratic equations with negative discriminant.

Exercise 8: Solve the following equations:

a) $x^2 + 1 = 0,$

b) $x^2 + 4 = 0,$

c) $x^2 - 2x + 5 = 0,$

d) $x^2 + x + 4 = 0.$

Extra material: notes with the solution of Exercise 8.

63 The sign of product is the same as the sign of quotient (of the same non-zero factors).

64 Summary 1: everything about quadratic polynomials with some coefficient equal to zero.

65 Summary 2: everything about quadratic polynomials with only non-zero coefficients.

66 Two test questions, pair 1.

Test 1: Let $b, c \in \mathbb{R}$ and $c < 0$. The equation $x^2 + bx + c = 0$

a) has two different real roots,

b) has only one real root,

c) has no real roots,

d) one cannot say.

Test 2: Let $a, b, c \in \mathbb{R}$ and $c < 0$. The equation $ax^2 + bx + c = 0$

a) has two different real roots,

b) has only one real root,

c) has no real roots,

d) one cannot say.

Extra material: notes from the iPad.

67 Two test questions, pair 2.

Test 1: Let $p \in \mathbb{R}$. If the equation $x^2 + px + 7 = 0$ has two real roots then

a) both roots are positive,

b) both roots are negative,

c) one root is positive and one is negative,

d) one cannot say.

Test 2: Let $p \in \mathbb{R}$. If the equation $x^2 + px - 7 = 0$ has two real roots then

a) both roots are positive,

b) both roots are negative,

c) one root is positive and one is negative,

d) one cannot say.

Extra material: notes from the iPad.

68 Two test questions, pair 3.

Test 1: Let $b, c \in \mathbb{R}$ and $c \neq 0$. If the equation $x^2 + bx + c = 0$ has two real roots with **the same** sign then

a) $b > 0$,

b) $c > 0$,

c) $bc > 0$,

d) one cannot say.

Test 2: Let $b, c \in \mathbb{R}$ and $c \neq 0$. If the equation $x^2 + bx + c = 0$ has two real roots with **different** signs then

a) $b > 0$,

b) $c > 0$,

c) $bc > 0$,

d) one cannot say.

Extra material: notes from the iPad.

69 Two test questions, pair 4.

Test 1: Let $a, b, c \in \mathbb{R}$ and $a \neq 0$.
If $ax^2 + bx + c < 0$ for all $x \in \mathbb{R}$ then

a) $b^2 - 4ac < 0$,

b) $b^2 - 4ac = 0$,

c) $b^2 - 4ac > 0$,

d) one cannot say.

Test 2: Let $a, b, c \in \mathbb{R}$ and $a \neq 0$.
If $ax^2 + bx + c < 0$ for all $x \in \mathbb{R}$ then

a) $a > 0$,

b) $ac > 0$,

c) $c > 0$,

d) one cannot say.

Extra material: notes from the iPad.

70 Quadratic equations in disguise, Problem 2.

Problem 2: Solve the equation $x^4 + 4x^2 + 16 = 0$ in real numbers, and in complex numbers.

Extra material: notes with solved Problem 2.

71 Quadratic equations in disguise, Problem 3.

Problem 3: Solve the equation $10x = 7x^3 - x^5$ in real numbers, and in complex numbers.

Extra material: notes with solved Problem 3.

72 Quadratic equations in disguise, Problem 4.

Problem 4: Solve the equation $(x + 1)(x + 2)(x + 3)(x + 4) = 50$ in real numbers.

Extra material: notes with solved Problem 4.

73 Quadratic equations in disguise, Problem 5.

Problem 5: Solve the equation $x^4 - 2px^3 + p^2x^2 - q^2 = 0$ in real and complex numbers, given that $p, q \in \mathbb{R}$ and $q > \frac{1}{4}p^2$.

Extra material: notes with solved Problem 5.

S6 Factoring polynomials is the same as finding zeros of polynomials

You will learn: polynomial divisibility; polynomial division, various methods: long division (two different notations), division with help of undetermined coefficients, Ruffini–Horner Scheme for division by monic binomials; consequences of the Fundamental Theorem of Algebra; Vieta's formulas; methods of finding rational zeros of polynomials with integer coefficients; Cauchy's Bound for zeros.

Read along with this section: **Prerequisites book**: Subsection 0.5.2 *Polynomial long division* (pp. 64–69). **Precalculus book**: Section 3.2 *The Factor Theorem and the Remainder Theorem* (pp. 257–268), Section 3.3 *Real Zeros of Polynomials* (pp. 269–286), and Section 3.4 *Complex Zeros and the Fundamental Theorem of Algebra* (pp. 287–300).

74 Tricks and formulas for factoring polynomials.

75 Factoring by tricks and formulas (sum or difference of powers), Problem 1.

Problem 1: Solve in real numbers:

a) $x^4 - 16 = 0$

b) $x^5 - x = 0$

c) $(x^2 + x)^4 - 1 = 0$

d) $x^3 - 8 = 0$.

Extra material: notes with solved Problem 1.

76 Factoring by tricks and formulas (Pascal), Problem 2.

Problem 2: Solve in real numbers:

a) $x^3 + 3x^2 + 3x + 1 = 0$

b) $x^3 - 6x^2 + 12x - 8 = 0$

c) $x^5 - 6x^4 + 12x^3 - 8x^2 = 0$

d) $x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1 = 0$

e) $x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1 = 0$.

Extra material: notes with solved Problem 2.

77 Factoring by smart grouping, Problem 3.

Problem 3: Solve in real numbers:

a) $x^5 + x - 2x^4 - 2 = 0$

b) $x^5 - 3x^3 - 8x^2 + 24 = 0$

c) $x^3 - 3\sqrt{5}x^2 + 5x - 15\sqrt{5} = 0$

d) $2x^5 - x^4 - 2x^3 + x^2 - 4x + 2 = 0$.

Extra material: notes with solved Problem 3.

78 Factoring by substitution, Problem 4.

Problem 4: Solve in real numbers:

a) $x^4 + x^2 - 20 = 0$ b) $x^6 + 2x^3 - 15 = 0$

c) $(x^2 + 2x - 5)^2 + 2(x^2 + 2x - 5) - 5 = x$.

Extra material: notes with solved Problem 4.

79 Factoring by grouping and substitution, Problem 5.

Problem 5: Solve in real numbers:

a) $2x^5 - x^4 - 2x^3 + x^2 - 4x + 2 = 0$

b) $5(x - \frac{3}{2})^3 + 2(x - \frac{3}{2})^2 + 5(x - \frac{3}{2}) + 2 = 0$.

Extra material: notes with solved Problem 5.

80 Polynomial division, the theorem.

Extra material: an article with the proof of the existence-and-uniqueness theorem for polynomial division.

81 Methods of polynomial division.

82 Method 1, Example 1.1.

Example 1.1: Divide $p(x) = x^4 + 3x^3 + x^2 + 7x - 1$ by $d(x) = x^2 + 2$.

83 Method 2: long division, Example 2.1.

Example 2.1: Long division: $p(x) = x^4 + 3x^3 + x^2 + 7x - 1$ by $d(x) = x^2 + 2$.

84 Method 3: undetermined coefficients, Example 3.1.

Example 3.1: Divide: $p(x) = x^4 + 3x^3 + x^2 + 7x - 1$ by $d(x) = x^2 + 2$.

Extra material: notes with solved Example 3.1.

85 How to verify the division, Example 1.

Example 1: Verify the result of division of $p(x) = x^4 + 3x^3 + x^2 + 7x - 1$ by $d(x) = x^2 + 2$.

Extra material: notes with solved Example 1.

86 Method 3: undetermined coefficients, Example 3.2.

Example 3.2: Divide $p(x) = 2x^4 - 3x^3 + 4x^2 - 5x + 6$ by $d(x) = x^2 - 3x + 1$. We know from Video 42 that it will be division with remainder.

Extra material: notes with solved Example 3.2.

87 Method 2: long division, Example 2.2.

Example 2.2: Divide $p(x) = 2x^4 - 3x^3 + 4x^2 - 5x + 6$ by $d(x) = x^2 - 3x + 1$.

Extra material: notes with solved Example 2.2.

88 Method 1, Example 1.2.

Example 1.2: Divide $p(x) = 2x^4 - 3x^3 + 4x^2 - 5x + 6$ by $d(x) = x^2 - 3x + 1$.

Extra material: notes with solved Example 1.2.

89 More polynomial division, Problem 6.

Problem 6: Divide:

a) $p(x) = 3x^3 + 2x^2 - 5x + 7$ by $d(x) = x^2 + 3$

b) $p(x) = x^5 - 1$ by $d(x) = x - 1$ by long division

c) $p(x) = x^3 - 5x^2 + 8x - 4$ by $d(x) = x - 1$ by Method 1

d) $p(x) = x^4 - 2x^3 + 4x^2 - 6x + 8$ by $d(x) = x - 1$.

Extra material: notes with solved Problem 6.

90 Why division by monic binomials is very important.

91 The Remainder Theorem.

Extra material: notes with proof of the theorem (in the presentation).

92 The Factor Theorem.

Extra material: notes with proof of the theorem.

93 The Remainder Theorem, Problem 7.

Problem 7: Find the remainders in division of $9x^4 - 12x^3 - 12x^2 + 31x - 30$ by $x - 1$ and by $x + 2$.

Extra material: notes with solved Problem 7.

94 The Remainder Theorem, Problem 8.

Problem 8: Find the values of $a \in \mathbb{R}$ for which $p(x) = 2ax^3 - 4x^2 + ax - 2a$ has remainder -8 in division by $x - 2$.

Extra material: notes with solved Problem 8.

95 The Remainder Theorem, Problem 9.

Problem 9: Some polynomial has remainder 1 in division by $x-1$, and remainder 2 in division by $x-2$. Find the remainder in division of this polynomial by $(x-1)(x-2)$.

Extra material: notes with solved Problem 9.

96 The Remainder Theorem, Problem 10.

Problem 10: Some polynomial has remainder 1 in division by $x-1$, remainder 2 in division by $x-2$, and remainder 3 in division by $x-3$. Find the remainder in division of this polynomial by $(x-1)(x-2)(x-3)$.

Extra material: notes with solved Problem 10.

97 The Remainder Theorem, Problem 11.

Problem 11: Find the values of $a, b, c \in \mathbb{R}$ for which $p(x) = x^4 - x^3 + ax^2 + bx + c$ is divisible by by $x-1$, $x+2$ and $x-3$.

Extra material: notes with solved Problem 11.

98 Polynomial division, Problem 12.

Problem 12: Find the values of $a, b \in \mathbb{R}$ for which $p(x) = x^5 + 3x^3 + ax^2 + bx + 3$ gives remainder $r(x) = x+1$ in division by $d(x) = x^2 + 2$.

Extra material: notes with solved Problem 12.

99 Polynomial division, Problem 13.

Problem 13: Find the remainder in division of $p(x) = x^{100} - x^{98} + x^{96} - x^{94} + \cdots + x^4 - x^2 + 1$ by $d(x) = x^2 - 1$.

Extra material: notes with solved Problem 13.

100 Ruffini–Horner Scheme for division by monic binomials.

101 Ruffini–Horner Scheme, Problem 14.

Problem 14: Divide $3x^4 - 8x^3 + 7x^2 + 2$ by $x+1$.

Extra material: notes with solved Problem 14.

102 Ruffini–Horner Scheme, Problem 15.

Problem 15: Divide x^5 by $x-1$.

Extra material: notes with solved Problem 15.

103 Rational zeros of polynomials with integer coefficients, Problem 16.

Problem 16: Find all the rational zeros of the following polynomials:

a) $2x^3 + x^2 + x - 1$,

b) $x^5 + x + 1$,

c) $x^3 + 2x^2 - 4x + 5$

d) $x^3 - 6x^2 + 11x - 6$.

104 Factoring polynomials, Problem 17.

Problem 17: Factor $p(x) = x^3 - 2x^2 - 5x + 6$.

Extra material: notes with solved Problem 17.

105 Factoring polynomials, Problem 18.

Problem 18: Factor $p(x) = x^3 - 5x^2 + 8x - 4$.

Extra material: notes with solved Problem 18.

106 Factoring polynomials, Problem 19.

Problem 19: Factor $p(x) = x^4 + x^3 - 11x^2 - 9x + 18$.

Extra material: notes with solved Problem 19.

107 Factoring polynomials, Problem 20.

Problem 20: Factor $p(x) = x^5 - 5x^4 + 7x^3 - 2x^2 + 4x - 8$.

Extra material: notes with solved Problem 20.

108 Vieta's formulas for cubic polynomials; The Binomial Theorem as a special case.

Example: How the Binomial Theorem for $n = 3$ is a special case of Vieta's formulas for cubic polynomials.

109 Vieta's formulas for cubic polynomials, Problem 21.

Problem 21: Let's denote the three real solutions of the equation $x^3 - 3x + 1 = 0$ by x_1, x_2, x_3. Construct a cubic equation with roots x_1^2, x_2^2, x_3^2.

Extra material: notes with solved Problem 21.

110 Cauchy Bound for zeros of a polynomial.

Example: Cauchy's Bound is decent for the polynomial $p(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$, but less good for the polynomial $p(x) = x^4 + x^3 - 11x^2 - 9x + 18$ from Video 106.

111 The Fundamental Theorem of Algebra and its consequences.

112 The only irreducible over $\mathbb{R}$ polynomials have degree 1 or 2.

Example: Factor $p(x) = x^4 + 1$ over $\mathbb{R}$.

113 The number of real zeros of polynomials with real coefficients.

Example: Possible number of real zeros for polynomials, depending on degree:

a) $ax + b$, $a \neq 0$: always **one** zero,

b) $x^2 + x + 1$: **no** zeros, $x^2 + 2x + 1$: a **double** zero, $x^2 + 2x - 3$: **two** different zeros,

c) $(x-1)(x^2+x+1)$: **one** zero, $(x-1)(x-2)(x-3)$: **three** zeros, $(x-1)^2(x-2)$: **three** zeros, a double and a single, $(x-1)^3$: a **triple** zero,

d) $(x^2+x+1)^2$: **no** zeros, $(x-1)(x-2)(x^2+x+1)$: **two** zeros, $(x-1)^2(x^2+x+1)$: **double** zero, $(x-1)^2(x-3)^2$: **two double** zeros, $(x-1)(x-2)(x-3)(x-4)$: **four** different zeros.

114 Difficult factorisation, Problem 22.

Problem 22: Factor as far as you can $p(x) = x^9 + x^4 - x - 1$.

Extra material: notes with solved Problem 22.

115 Difficult factorisation, Problem 23.

Problem 23: Factor as far as you can $p(x) = x^5 + x + 1$.

Extra material: notes with solved Problem 23.

116 Factorisation over $\mathbb{R}$ and $\mathbb{C}$, Problem 24.

Problem 24: Factor the polynomial $x^4 - 5x^3 - 2x^2 + 46x - 60$ in linear factors, and over $\mathbb{R}$.

Extra material: notes with solved Problem 24.

117 Factorisation over $\mathbb{R}$ and $\mathbb{C}$, Problem 25.

Problem 25: Factor the polynomial $x^4 - x^3 + 5x^2 - 4x + 4$ in linear factors, and over $\mathbb{R}$, given that $2i$ is one zero of the polynomial.

Extra material: notes with solved Problem 25.

118 Factorisation over $\mathbb{R}$ and $\mathbb{C}$, Problem 26.

Problem 26: Factor the polynomial $6x^4 - 23x^3 + 25x^2 + 9x - 5$ in linear factors, and over $\mathbb{R}$, given that $2 - i$ is one zero of the polynomial.

Extra material: notes with solved Problem 26.

Extra material: an article with more solved problems on factoring polynomials.

- ⋆ **Extra problem 1**: Factor the polynomial $x^3 - 5x^2 + 11x - 15$ in linear factors.
- ⋆ **Extra problem 2**: Factor the polynomial $p(x) = x^4 + 3x^2 + 6x + 10$ into polynomials of degree 1, given that the complex number $1 + 2i$ is a zero of the polynomial.

S7 Factoring polynomials: school versus reality

You will learn: that the reality is not as nice as school.

119 What kind of polynomials you will get at school.

120 Polynomials living and roaming freely in nature, Problem 1.

Problem 1: Construct a polynomial of degree 5, with only one real root: the square root of 17.

Extra material: notes with solved Problem 1.

121 Polynomials living and roaming freely in nature, Problem 2.

Problem 2: Construct a polynomial of degree 7, with only three real roots: $\sqrt[7]{\pi}$, $\sin 2022$, and $\ln 2$.

Extra material: notes with solved Problem 2.

122 Polynomials living and roaming freely in nature, Problem 3.

Problem 3: Construct a polynomial of degree 8, with no real roots.

Extra material: notes with solved Problem 3.

S8 Polynomial equations and inequalities

You will learn: solve polynomial equations and inequalities by factoring polynomials and analysing the signs (with help of the table or a sketch); you will also gain a geometrical understanding of the solution sets (graphically). Factoring of polynomials is omitted in this section, because this was the topic of the previous section, but at school you will have to factor polynomials in order to solve polynomial equations and inequalities.

Read along with this section: **Precalculus book**: pp. 277–279; Exercises 35–54 on page 281.

123 What is a polynomial equation or inequality.

124 Inequalities you can solve without any computations.

Exercise 1: Solve the following inequalities:

a) $x^2 + 1 > 0$,

b) $-x^2 - 4 < 0$,

c) $6x^6 + 2x^4 + 5x^2 + 1 > 0$,

d) $x^2 + 4 < 0$.

125 Functional equations and inequalities and their graphical illustration.

Example 0: An illustration of functional equations $f(x) = g(x)$ and inequalities $f(x) < g(x)$ etc. No solution to $f(x) = g(x)$ is the same as no intersections between the graphs $y = f(x)$ and $y = g(x)$, and is the same as no zeros for $h(x) = f(x) - g(x)$: illustrated by $f(x) = x^2 + 1$ and $g(x) = -x^2$.

126 How it looks graphically and computationally, Example 1.

Example 1: $x^2 + x - 6 > x + 3$.

127 How it looks graphically and computationally, Example 2.

Example 2: $-x^2 - x + 6 > x + 2$.

128 How it looks graphically and computationally, Example 3.

Example 3: $8 - x^2 = -x^2 + 2x$, $8 - x^2 < -x^2 + 2x$, $8 - x^2 > -x^2 + 2x$.

129 How it looks graphically and computationally, Example 4.

Example 4: $x^2 - 2x = -x^2 - 2x + 2$, $x^2 - 2x < -x^2 - 2x + 2$, $x^2 - 2x > -x^2 - 2x + 2$.

130 How it looks graphically, Example 5.

Example 5: $x^3 - 2x^2 - 5x + 6 > -x^3 + 5x^2 - 2x - 8$.

131 How to solve polynomial equations, the method.

Example 6.0: Solve the following:

a) $8x^5-17x^4+(\sqrt{3}+1)x^3+(13+\pi)x^2+47x+\sqrt{17}+13 = 8x^5-17x^4+\sqrt{3}x^3+(15+\pi)x^2+52x+\sqrt{17}+7$

b) $8x^5-17x^4+(\sqrt{3}+1)x^3+(13+\pi)x^2+47x+\sqrt{17}+13 > 8x^5-17x^4+\sqrt{3}x^3+(15+\pi)x^2+52x+\sqrt{17}+7$

c) $8x^5-17x^4+(\sqrt{3}+1)x^3+(13+\pi)x^2+47x+\sqrt{17}+13 < 8x^5-17x^4+\sqrt{3}x^3+(15+\pi)x^2+52x+\sqrt{17}+7$

132 Stacking factor method with a table, Example 6.1.

Example 6.1: Solve the following:

a) $x^3-2x^2-5x+6=0$

b) $x^3-2x^2-5x+6>0$

c) $x^3-2x^2-5x+6<0$

knowing (from Video 104) that $x^3-2x^2-5x+6=(x+2)(x-1)(x-3)$.

133 Stacking factor method with a coordinate system, Example 6.2.

Example 6.2: Solve the following:

a) $x^3-2x^2-5x+6=0$

b) $x^3-2x^2-5x+6>0$

c) $x^3-2x^2-5x+6<0$

knowing (from Video 104) that $x^3-2x^2-5x+6=(x+2)(x-1)(x-3)$.

134 Monic monomials in plus infinity and why the exact numbers don't matter for the big picture.

Extra material: notes from the iPad.

135 An inequality solved with both methods, Problem 1.

Problem 1: Solve this inequality, both graphically and with the stacking factor method: $x^4-5x^2+4\geqslant 0$.

Extra material: notes with solved Problem 1.

136 An inequality solved with both methods, Problem 2.

Problem 2: Solve this inequality, both graphically and with the stacking factor method:

$$(x^2+2x-3)(x-\tfrac{1}{2})^2(x-2)\leqslant 0.$$

Extra material: notes with solved Problem 2.

137 The quick graph method, Problem 3.

Problem 3: Solve the following inequalities:

a) $(x-1)(x-2)^2(x^2+x+1)<0$

b) $-6(x-3)(x-5)^2(-x^2-x-1)<0$

c) $(x-7)(x+8)(x^2+4x+20)(x^2-6x+10)(x^2+1)\leqslant 0$.

Extra material: notes with solved Problem 3.

138 The quick graph method, Problem 4.

Problem 4: Solve the following inequalities:

a) $(8-2x^2)(3x-1)(x^2+x+1)^3\geqslant 0$

b) $(x^2-3x+2)^2(-2x+3)\leqslant 0$

c) $(x-3)^2x(x+4)^3>0$.

Extra material: notes with solved Problem 4.

139 The quick graph method, Problem 5.

Problem 5: Solve the inequality $x^6-15x^4+8x^3+51x^2-72x+27>0$

Extra material: notes with solved Problem 5.

140 The quick graph method, Problem 6.

Problem 6: Solve the inequality $x^5 - 2x^4 - 4x^3 + 4x^2 - 5x + 6 \geqslant 0$.

Extra material: notes with solved Problem 6.

141 Why I really dislike sign diagrams.

142 Back to a pessimistic promise from Video 92, Problem 7.

Problem 7: Create a polynomial p which has the desired characteristics:

a) The solutions to $p(x) = 0$ are $x = \pm 3,\ x = -2$, and $x = 4$.

b) The leading term of $p(x)$ is $-x^5$.

c) The point $(-2, 0)$ is a local maximum on the graph of $y = p(x)$.

Extra material: notes with solved Problem 7.

S9 *Intermezzo*: Some topics from Calculus

You will learn: what it means that a function is continuous and that polynomials are continuous functions; the concept of the derivative; compute the derivatives of polynomials; why the curves of polynomials are rounded while intersecting the x-axis in multiple zeros of the polynomials; limits in the infinities and infinite limits.

143 What it means that a function is continuous.

144 **Future**: All polynomials are continuous functions.

145 A naive thinking about continuity is OK for polynomials.

146 **Future**: Time to deal with derivatives.

147 **Future**: Derivatives of polynomials are polynomials.

Exercise: Derivatives of polynomials:

0) $p(x) = a_0$

a) $p(x) = a_1x + a_0, \quad a_1 \neq 0$

b) $p(x) = x^2 + x - 2$

c) $p(x) = x^3 - 2x^2 - 5x + 6$

d) $p(x) = x^4 + x^3 - 11x^2 - 9x + 18$.

Extra material: notes with solved Exercise.

148 **Future**: The product rule for derivatives and what it says about the graph around multiple zeros.

149 Back to monic monomials.

150 A formalisation of infinite limits in the infinities.

151 **For rational functions**: A formalisation of finite limits in the infinities.

152 **For rational functions**: A formalisation of infinite limits in a point.

153 Various types of limits and where to find them: a summary.

S10 Plotting (sketching) polynomials

You will learn: how to sketch graphs of polynomial functions: how to establish the domain, the range, the x- and y-intercepts, the intervals of monotonicity (increasing, decreasing), and local extremums (max, min).

Read along with this section: **Precalculus book**: Section 3.1 *Graphs of Polynomials* (pp. 235–256); Exercises on page 246.

154 Graphs of polynomials, what to think about.

155 Graphs of polynomials, Example 1.

Example 1: Plot the polynomials:

a) $p(x) = a_1x + a_0, \quad a_1 \neq 0$

b) $p(x) = x^2 + x - 2$

c) $p(x) = x^3 - 2x^2 - 5x + 6$

d) $p(x) = x^4 + x^3 - 11x^2 - 9x + 18$.

156 Graphs of polynomials, Problem 1.

Problem 1: Sketch the graph of $p(x) = x^3 - 4x^2 + 4x$.

Extra material: notes with solved Problem 1.

157 Graphs of polynomials, Problem 2.

Problem 2: Sketch the graph of $p(x) = x^3 - 3x - 2$.

Extra material: notes with solved Problem 2.

158 Graphs of polynomials, Problem 3.

Problem 3: Sketch the graph of $p(x) = x^4 - 2x^2 - 3$.

Extra material: notes with solved Problem 3.

S11 More advanced future topics on polynomials

You will learn: in what other domains you will enjoy your gained knowledge about polynomials; I will *not* teach you about this topics, I will just give you some information on where to find them.

159 Algebra: the ring of polynomials $\mathbb{R}[x]$.

160 Calculus 2: Taylor polynomials.

161 Calculus 3: Multivariable polynomials.

162 Linear Algebra: Characteristic polynomials and eigenvalues.

163 Linear ODE: Characteristic polynomials.

C2 Rational functions

S12 Rational functions and their domains

You will learn: the definition of rational functions; how to determine their domains, their zeros, and y-intercepts.

164 What is a rational function.

165 The domain problem.

Example: Determine the domain of $f(x) = (x^2 - 1)/(x - 1)$ and of $f(x) = \frac{1}{x^2} - \frac{2}{x-7}$.

166 Factoring polynomials for getting the domains and zeros, Problem 1

Problem 1: Determine the domains, the zeros, and the y-intercepts of the following rational functions

$$f_1(x) = \frac{x+3}{x-1}, \quad f_2(x) = \frac{x^2 - x + 1}{x^2 + x + 1}, \quad f_3(x) = \frac{x^2 + x - 2}{x^2 + 2x + 1}, \quad f_4(x) = \frac{x(x-1)}{x^3 + x^2 - 2x}.$$

Extra material: notes with solved Problem 1.

S13 Rational equations and inequalities

You will learn: add, subtract, multiply and divide rational expressions; solve rational equations and inequalities and understand the link between rational and polynomial equations and inequalities.

Read along with this section: **Prerequisites book**: Subsection 0.8 *Rational Expressions and Equations* (pp. 96–110). **Precalculus book**: Section 4.1 *Introduction to Rational Functions* (pp. 301–319), Section 4.3 *Rational Inequalities and Applications* (pp. 342–357).

167 The same arithmetic as for regular fractions.

168 Rational expressions in Precalculus 1.

169 Addition of rational functions, Problem 1.

Problem 1: Perform the following operation:

$$\frac{3}{x^2+x}+\frac{4}{x^2-x}.$$

Extra material: notes with solved Problem 1.

170 Addition of rational functions, Problem 2.

Problem 2: Perform the following operation:

$$\frac{1}{x^2-1}+\frac{1}{x^2+2x+1}-2.$$

Extra material: notes with solved Problem 2.

171 Subtraction of rational functions, Problem 3.

Problem 3: Perform the following operation:

$$\frac{4x}{x-4}-\frac{-x}{x+2}.$$

Extra material: notes with solved Problem 3.

172 Multiplication of rational functions, Problem 4.

Problem 4: Perform the following operation:

$$\frac{x^2-3x}{3x+6}\cdot\frac{x+2}{x^2-9}.$$

Extra material: notes with solved Problem 4.

173 Division of rational functions, Problem 5.

Problem 5: Perform the following operation:

$$\frac{2x}{x^2+x-2}\Big/\frac{6x^3}{x^3+2x^2+2x+4}.$$

Extra material: notes with solved Problem 5.

174 What is a rational equation or inequality.

175 Never ever multiply both sides of an inequality by a variable expression (without a careful consideration).

176 How to solve rational equations and inequalities.

177 How to solve rational equations and inequalities, Problem 6.

Problem 6: Solve the equations and inequalities:

$$x^2-4x+3=0,\quad \frac{x-1}{x-3}<0,\quad \frac{x-3}{x-1}<0,\quad \frac{x-3}{x-1}\leqslant 0,\quad \frac{x-3}{x-1}=\frac{x-1}{x-3}.$$

Extra material: notes with solved Problem 6.

178 How to solve rational equations and inequalities, Problem 7.

Problem 7: Solve the equations and inequalities:

$$\frac{1}{x^2} = \frac{2}{x-7}, \quad \frac{1}{x^2} < \frac{2}{x-7}, \quad \frac{1}{x^2} > \frac{2}{x-7}.$$

Extra material: notes with solved Problem 7.

179 How to solve rational equations and inequalities, Problem 8.

Problem 8: Solve the inequality:

$$\frac{(4x^2-4x+1)(2-x-x^2)}{(x^2-4)(x+3)} \geqslant 0.$$

Compare to Problem 2 in Video 136.

Extra material: notes with solved Problem 8.

180 How to solve rational equations and inequalities, Problem 9

Problem 9: Solve the inequality:

$$\frac{(x+2)(x-1)}{(x-3)(x^2+1)} \geqslant 0.$$

Compare to Problem 6 in Video 140.

Extra material: notes with solved Problem 9.

S14 Asymptotes

You will learn: horizontal and vertical asymptotes (intuitively; the concepts come back in the Calculus class).

Read along with this section: **Precalculus book**: Section 4.1 *Introduction to Rational Functions* (pp. 301–319).

181 Vertical asymptotes.

182 Does every excluded point give a vertical asymptote?

Example 1: Vertical asymptotes for:

$$f_1(x) = \frac{1}{x^2-1}, \qquad f_2(x) = \frac{x^2-1}{x-1}, \qquad f_3(x) = \frac{x^2-1}{x+1}, \qquad f_4(x) = \frac{x-1}{x^2-1}.$$

Draw the graphs of f_2 and f_3. The other two will be plotted in the next section (f_4 in Video 188; f_1 comes back in Videos 190, 197, and 203).

Extra material: notes with solved Example 1.

183 Vertical asymptotes, Problem 1.

Problem 1: What are the vertical asymptotes of:

$$f(x) = \frac{x+4}{2x-5}, \qquad g(x) = \frac{5x+4}{2x-1}.$$

Extra material: notes with solved Problem 1.

184 Horizontal asymptotes.

185 Does every rational function have a horizontal asymptote?

Example 2: Horizontal asymptotes for:

$$f_1(x) = \frac{1}{x^2-1}, \qquad f_2(x) = \frac{x^2}{x-1}, \qquad f_3(x) = \frac{x^3-1}{x^2+1}, \qquad f_4(x) = \frac{x-1}{x^2-1}.$$

Extra material: notes with solved Example 2.

186 Horizontal asymptotes, Problem 2.

Problem 2: What are the horizontal asymptotes of:

$$f(x) = \frac{x+4}{2x-5}, \qquad g(x) = \frac{5x+4}{2x-1}.$$

Extra material: notes with solved Problem 2.

S15 Plotting (sketching) rational functions

You will learn: to sketch some simple graphs of rational functions using graph transformations of $y = 1/x$ and $y = 1/(x^2+1)$; understand the link to polynomial division and polynomial / rational equations and inequalities.

Read along with this section: **Precalculus book**: Section 4.2 *Graphs of Rational Functions* (pp. 320–341).

187 What to think about while making a graph.

188 Plots by graph transformations of the best known hyperbola, Problem 1.

Problem 1: Sketch the graphs of:

$$f_1(x) = \frac{1}{x}, \qquad f_2(x) = \frac{2}{x}, \qquad f_3(x) = -\frac{1}{x}, \qquad f_4(x) = \frac{x-1}{x^2-1} \qquad f_5(x) = \frac{x+1}{x-1}.$$

Extra material: notes with solved Problem 1.

189 More graph transformations, this time with completing the square, Problem 2.

Problem 2: Sketch the graphs of:

$$f_1(x) = \frac{1}{x^2+1}, \quad f_2(x) = \frac{1}{x^2+4x+5}, \quad f_3(x) = \frac{5}{x^2-6x+10}, \quad f_4(x) = -\frac{1}{x^2+1}, \quad f_5(x) = \frac{3-2x+x^2}{2-2x+x^2}.$$

Extra material: notes with solved Problem 2.

190 Plotting rational functions, Problem 3.

Problem 3: Sketch the graph of f_1 from Video 182:

$$f_1(x) = \frac{1}{x^2-1}.$$

Extra material: notes with solved Problem 3.

191 Plotting rational functions, Problem 4.

Problem 4: Sketch the graph of:

$$f(x) = \frac{x+4}{2x-5}, \qquad g(x) = \frac{5x+4}{2x-1}.$$

Extra material: notes with solved Problem 4.

192 Plotting rational functions, Problem 5.

Problem 5: Sketch the graph of:

$$f(x) = \frac{x^2-2x-3}{x^2-x-2}.$$

Extra material: notes with solved Problem 5.

S16 Partial fraction decomposition

You will learn: how to perform partial fraction decomposition of rational functions.

Read along with this section: **Precalculus book**: Section 8.6 *Partial Fraction Decomposition* (pp. 628–636).

193 A grey box: Partial Fraction Decomposition.

194 Partial fraction decomposition, Problem 1.

Problem 1: Find the form needed to begin the process of partial fraction decomposition in

$$\frac{8x^6+5x^5}{(x+8)(x^2+4x+17)^3}, \qquad \frac{x^9}{(x^4-1)^2}, \qquad \frac{6x^{11}-8x^9+7x^8+5x^3-20}{(x-7)^3(x^2+x+1)^2}.$$

Extra material: notes with solved Problem 1.

195 Why just a constant numerator in fractions containing powers of linear factors.

Extra material: notes from the iPad.

196 Why just a first-degree numerator in fractions containing powers of second-degree factors.

Extra material: notes from the iPad.

197 Partial fraction decomposition, Problem 2.

Problem 2: Perform partial fraction decomposition of the following rational function:

$$\frac{1}{x^2-1}.$$

Extra material: notes with solved Problem 2.

198 Partial fraction decomposition, Problem 3.

Problem 3: Perform partial fraction decomposition of the following rational function:

$$\frac{x-1}{x^2+3x+2}.$$

Extra material: notes with solved Problem 3.

199 Partial fraction decomposition, Problem 4.

Problem 4: Perform partial fraction decomposition of the following rational function:

$$\frac{x^3+6x^2+12x+5}{x^2+3x+2}.$$

Extra material: notes with solved Problem 4.

200 Partial fraction decomposition, Problem 5.

Problem 5: Perform partial fraction decomposition of the following rational function:

$$\frac{4}{x^3-x^5}.$$

Extra material: notes with solved Problem 5.

201 Partial fraction decomposition, Problem 6.

Problem 6: Perform partial fraction decomposition of the following rational function:

$$\frac{x^2+3x-2}{(x-1)(x^2+x+1)^2}.$$

Extra material: notes with solved Problem 6.

S17 More advanced future topics on rational functions

You will learn: about significant terms for polynomials near zero and in the infinity: the huge difference between computing indefinite limits of rational functions in zero (like in Taylor approximations) and in the infinity (for plotting graphs of rational functions, finding asymptotes, etc); importance of partial fraction decomposition for integrating rational functions. I will *not* teach you this stuff, I will only prepare you for some future topics and motivate why you should study rational functions.

202 **Future**: The quotient rule for derivatives.

203 **Future**: The derivative of a rational function is a rational function.

Exercise: Compute the derivative of

$$f(x) = \frac{1}{x^2-1}$$

and revisit Problem 3 in Video 190.

Extra material: notes with solved Exercise.

204 **Future**: The antiderivative of a rational function is **not** necessarily a rational function.

205 **Future**: Partial fraction decomposition makes it possible to integrate rational functions.

Extra material: notes with solved Problem 3.

Extra material: an article with some solved problems on application of partial fraction decomposition in Calculus 2 (integrals of rational functions). If you don't feel ready for Calculus (integrals), read just the partial fraction decomposition: it ends always with **We get thus** and the decomposition of the given rational function into partial fractions; this final result is always written in red; everything what comes later is Calculus and it is outside the scope of this course, so it can be omitted.

- ⋆ **Extra problem 1**: Compute
$$\int \frac{x}{x^2+3x+2}\,dx.$$
- ⋆ **Extra problem 2**: Compute
$$\int \frac{x^3-5x-7}{x^2-x-2}\,dx.$$
- ⋆ **Extra problem 3**: Compute
$$\int \frac{1}{x^2+8x+12}\,dx.$$
- ⋆ **Extra problem 4**: Compute
$$\int \frac{9+x}{9-x^2}\,dx.$$
- ⋆ **Extra problem 5**: Compute
$$\int \frac{3x+11}{x^2-x-6}\,dx.$$
- ⋆ **Extra problem 6**: Compute
$$\int \frac{2x+1}{x(x^2+1)}\,dx.$$
- ⋆ **Extra problem 7**: Compute
$$\int \frac{1-x}{x^2+4x+3}\,dx.$$
- ⋆ **Extra problem 8**: Compute
$$\int_{-1}^{0} \frac{x-x^2-3}{(x^2+1)(x-2)}\,dx.$$

206 **Future**: Formal proof for vertical and horizontal asymptotes for $f(x) = 1/x$.

Extra material: notes from the iPad.

207 **Future**: Significant terms and limits of indeterminate rational expressions in the infinity.

Problem 1: Evaluate the following limits:

$$\lim_{x\to\infty} \frac{4x^5-5x^3+x^2-7}{8x^5-9x^4-x^3+4x^2-9}, \qquad \lim_{x\to\infty} \frac{5x^3+6x^2-3}{8x^2-5}, \qquad \lim_{x\to\infty} \frac{-x^7-8x^2-1}{5x^{10}-3x^4-x}.$$

Extra material: notes with solved Problem 1.

208 **Future**: Significant terms and limits of indeterminate rational expressions in zero.

Problem 2: Evaluate the following limits:

$$\lim_{x\to 0} \frac{4x^2-3}{3x^2-9}, \qquad \lim_{x\to 0} \frac{12x^2+6x}{4x^2-5x}, \qquad \lim_{x\to 0} \frac{x^5-3x^4}{-x^2+7x}, \qquad \lim_{x\to 0} \frac{-x^2+7x}{x^5-3x^4}, \qquad \lim_{x\to 0} \frac{x^5-3x}{-x^5+7x^3}.$$

Extra material: notes with solved Problem 2.

S18 Some words about power functions and algebraic functions

You will learn: the definition and examples of power functions and algebraic functions.

209 What is a power function; some examples.

Example 1: Graphs of the following power functions:

$$f_1(x) = x^n, \qquad f_2(x) = x^{-n} = \frac{1}{x^n}, \qquad f_3(x) = x^{1/2} = \sqrt{x}, \qquad f_4(x) = x^{1/3} = \sqrt[3]{x}, \qquad f_5(x) = x^{2/3}.$$

210 What is an algebraic function; some examples.

Example 2: Graphs of the following algebraic functions:

$$f_1(x) = \sqrt{1-x^2}, \qquad f_2(x) = \frac{1}{\sqrt{1-x^2}}, \qquad f_3(x) = \frac{x^2+4x-5}{\sqrt{1+x^2}}.$$

211 Precalculus 2, Wrap-up.

S19 Extras

You will learn: about all the courses we offer, and where to find discount coupons. You will also get a glimpse into our plans for future courses, with approximate (very hypothetical!) release dates.

B Bonus lecture.

Extra material: a pdf with all the links and coupon codes.