Precalculus 1: Basic notions[1]

Mathematics from high school to university

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A short table of contents

A detailed table of contents follows (see next pages).

Books to read along with the course, with more practice problems:

1. *Precalculus Prerequisites, a.k.a. Chapter* 0: Carl Stitz, Ph.D., Lakeland Community College; Jeff Zeager, Ph.D., Lorain County Community College; version from August 16, 2013.
2. *Precalculus*: Carl Stitz, Ph.D., Lakeland Community College; Jeff Zeager, Ph.D., Lorain County Community College; version from July 4, 2013.

These books are added as resources to Video 1, with kind permission of Professor Carl Stitz.

[1] Recorded November 2021–January 2022. Published on `www.udemy.com` on 2022-01-XX.

An extremely detailed table of contents; the videos (titles in green) are numbered

In blue: problems solved on an iPad (the solving process presented for the students; active problem solving)

In red: solved problems demonstrated during a presentation (a walk-through; passive problem solving)

In dark blue: *Read along with this section*: references for further reading and exercises in *Prerequisites* book and in the *Precalculus* book by Carl Stitz and Jeff Zeager (both books are placed as resources to Video 1).

11 Prime factorization, Problem 1.

Problem 1: Factor the numbers 60, 105, and 48.

Extra material: notes with solved Problem 1.

12 How to find prime numbers? Problem 2.

Problem 2: Find all the prime numbers less than 121, using the sieve of Eratosthenes.

Extra material: notes with solved Problem 2.

13 Integer numbers: addition, subtraction, and multiplication.

This lecture contains a repetition about integer numbers and their arithmetic. The main computation rules are listed, motivated, and illustrated with examples.

Example 1: You have earned 1 000 $ and you spent 1 200 $. How much money do you have left?

Example 2: Compute $-1+8-3-4+7+1-8$. Show two different methods of computing this sum.

Example 3: The temperature T rose from $-60°$ Celsius to $+40°$ Celsius. Compute ΔT.

Example 4: Compute the distance between the hot air balloon, which is 200m above the sea level, and the diver, who is 50m under the sea level.

14 Rational numbers as fractions.

This lecture contains a repetition about fractions and their arithmetic. The main computation rules are listed, motivated, and illustrated with examples.

Example 1: Express the improper fraction $\frac{23}{7}$ as a mixed number.

Example 2: What is the least common denominator for the fractions in $\frac{1}{6}-\frac{1}{12}+\frac{1}{9}+\frac{1}{4}+\frac{5}{36}+\frac{1}{2}+\frac{1}{3}$?

Example 3: One fourth of a field is meant for agriculture. Two thirds of this part is for planting strawberries. What part of the entire field will be a strawberry field?

Example 4: How many cans of soda do you need to get 2 liter soda? (One can contains $\frac{1}{3}$ liter.)

Example 5: Arrange the following numbers in increasing order (from the least to the greatest):

$$\frac{1001}{1000};\quad \frac{3}{7};\quad 1;\quad \frac{1}{2};\quad \frac{2}{3};\quad \frac{5}{4};\quad \frac{999}{1000};\quad \frac{1}{70000};\quad \frac{1}{3}.$$

Extra material: notes with solved Example 5.

15 Decimal expansion of rational numbers.

Example 1: How to multiply decimal numbers with various powers of 10: $57.345\cdot 10^2$, $57.345\cdot 10^4$, $57.345\cdot 10^{-5}$.

Example 2: How to compare decimal numbers (which one is larger): 376.12 and 23.789 090; 23.789 096 and 23.790 005 980; 0.000 001 999 970 and 0.000 005 980; 0.012 005 763 and 0.012 005 759 999.

Example 3: A very ineffective method of finding some digits in the decimal expansion of the square root of 2.

16 Finite and periodic decimal expansions.

Example: Some numbers have more than one decimal expansion: $x=1=1.(0)=0.(9)$.

Extra material: notes with solved Example.

17 Finite and periodic decimal expansions, Problem 3.

Problem 3: Represent the numbers $x=0.3$, $y=0.02$, and $z=0.(3)$ as irreducible fractions.

Extra material: notes with solved Problem 3.

18 Finite and periodic decimal expansions, Problem 4.

Problem 4: Represent the number $x=0.(27)$ as an irreducible fraction.

Extra material: notes with solved Problem 4.

19 Finite and periodic decimal expansions, Problem 5.

Problem 5: Represent the number $x=0.14(9)$ as an irreducible fraction.

Extra material: notes with solved Problem 5.

20 The mysterious irrational numbers.

Example: Show that the set of irrational numbers is **not** closed under the four arithmetical operations.

Extra material: notes from the iPad.

21 The distributive law, Problem 6.

Problem 6: Show that for all real numbers a and b holds $(a \pm b)^2 = a^2 \pm 2ab + b^2$. These formulas are known as *square of the sum* and *square of the difference*, and they form a basis for the method of *completing the square*.

Extra material: notes with solved Problem 6.

22 Arithmetic, Problem 7.

Problem 7: Show that the number $(\sqrt{2} - \sqrt{8})^2$ is rational.

Extra material: notes with solved Problem 7.

23 The distributive law, Problem 8.

Problem 8: Show that for all real numbers a and b holds $a^2 - b^2 = (a - b)(a + b)$. This formula is known as *difference of two squares* and is used for solving quadratic equations (for derivation of the formula for solutions).

Extra material: notes with solved Problem 8.

24 Order of operations (precedence rules), Problem 9.

Problem 9: Compute $3 + [6(11 - 4 + 1)]/8 \cdot 2$, $(-1)^2 - 1^2 - (-2)$, and $\frac{-5\cdot(-6)-7\cdot(-3)}{3\cdot 5+2}$.

Extra material: notes with solved Problem 9.

25 The distributive law and precedence rules don't contradict each other.

Example: shows that some computational experience is important:

$$17\left(\frac{5}{283} - \frac{2}{175}\right) + 13 - \frac{85}{283} + 4 + \frac{34}{175}.$$

Extra material: notes from the iPad.

26 Arithmetic, Problem 10.

Problem 10: Express the following as an irreducible fraction:

$$\frac{\frac{2}{3} - \frac{5}{12}}{\frac{1}{8} + \frac{2}{9}}.$$

Extra material: notes with solved Problem 10.

27 Arithmetic, Problem 11.

Problem 11: Simplify the expression as far as possible. Here $a, b \neq 0$:

$$\frac{(a+b)^2 - (a-b)^2}{5ab}.$$

Extra material: notes with solved Problem 11.

28 Arithmetic, Problem 12.

Problem 12: Simplify the expression as far as possible. What restrictions do you have to impose on a and b?:

$$\left(\frac{1}{a} + \frac{1}{b}\right) \cdot \frac{a^4 b^8}{2a^2 - 2b^2}.$$

Extra material: notes with solved Problem 12.

29 **Future**: Domain of a function of two variables, Problem 13.

Problem 13: Consider $z = f(x, y)$ defined beneath. Determine its domain. Is it possible to expand this function to a continuous function over a larger domain?:

$$f(x,y) = \frac{\frac{1}{x^2} - \frac{1}{y^2}}{\frac{3}{x} - \frac{3}{y}}.$$

Extra material: notes with solved Problem 13.

30 **Future**: Values of a function of two variables, Problem 14.

Problem 14: Compute $f(-1,-1)$, $f(1,-1)$, and $f(0,-1)$ if $f(x,y) = 3xy - x^2 - 3y^3 + x - 12$.

Extra material: notes with solved Problem 14.

31 Arithmetic, Problem 15.

Problem 15: Show that the following formula holds for all real numbers a and b and for each positive integer n:

$$a^n - b^n = (a-b)\sum_{k=0}^{n-1} a^{n-1-k}b^k = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + a^2b^{n-3} + ab^{n-2} + b^{n-1}).$$

This formula will be very important in Calculus 1.

Extra material: notes with solved Problem 15.

32 **Future**: the derivative of a polynomial, Problem 16.

Problem 16: Derive the formula for the derivative of $f(x) = x^n$ (where n is any positive integer), using the formula from the previous video.

Extra material: notes with solved Problem 16.

S4 Absolute value and distances

You will learn: Cartesian coordinate system: the axes, the unit, the origin, the coordinates of points, coordinates after reflections about the axes and the origin; absolute value as the distance from a real number to zero; absolute value for measuring distances; distances in abstract metric spaces.

Read along with this section: if you want to, you can read subsections 1.1.2 and 1.1.3 in the Precalculus book and solve some problems about the distances, midpoints, and symmetries in subsection 1.1.4 (starting with problem 20). I don't go through the midpoint formula because we will not need it for our Calculus courses; you can leave it out if you want to. Exercises 1–19 can also be left for later (our Section 5).

33 Distance is necessary for Calculus.

34 Distance between real numbers; absolute value.

Properties of absolute value:

1. $|x| \geqslant 0$ for all $x \in \mathbb{R}$, and $|x| = 0$ if and only if $x = 0$,
2. $|x| = |-x|$ and $|x| \geqslant x$ for all $x \in \mathbb{R}$,
3. $|xy| = |x| \cdot |y|$ for all $x, y \in \mathbb{R}$,
4. $|x^n| = |x|^n$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$,
5. $|\frac{x}{y}| = |x|/|y|$ for all $x, y \in \mathbb{R}$, where $y \neq 0$,
6. $|x| = |y|$ if and only if $x = y$ or $x = -y$,
7. $|x + y| \leqslant |x| + |y|$ for all $x, y \in \mathbb{R}$.

Extra material: notes from the iPad.

35 Cartesian coordinate system in $\mathbb{R}^2$, reflections about axes.

36 Pythagorean Theorem and Euclidean distance between points in the plane.
Explanation why the distance formula covers all the possible cases.
Extra material: notes from the iPad.

37 What is triangle inequality and why is it essential for Calculus?

38 **Advanced**: Distances in metric spaces.
Example 1: Discrete metric on any non-empty set.
Example 2: The set of real numbers with the distance defined by the absolute value: $d(x, y) = |x - y|$.
Example 3: Euclidean distance in the plane: $d_2((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.
Extra material: notes from the iPad: the proof of triangle inequality in $\mathbb{R}$ and the three examples above.

39 **Advanced**: Taxi cab distance / Manhattan distance.
Example: Manhattan distance in the plane: $d_1((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$.
Extra material: notes from the iPad.

40 **Advanced**: Max distance in the plane.
Example: Infinity distance in the plane: $d_\infty((x_1, y_1), (x_2, y_2)) = \max(|x_1 - x_2|, |y_1 - y_2|)$.
Extra material: notes from the iPad.

41 **Advanced**: Open ball and deleted (punctured) neighbourhood.
Example 1: Open balls and punctured neighbourhoods in $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$ with Euclidean distances.
Example 2: A neighbourhood of 3/4, not centered in this point.
Example 3: A punctured neighbourhood of 0, not centered in 0.

42 **Advanced**: Unit circles can be strange: in Taxi cab metric.
Example: Draw the unit circle (in the Taxi cab metric) centered in the origin.

43 **Advanced**: Unit circles can be strange: in max metric.
Example: Draw the unit circle (in the max metric) centered in the origin.

44 **Future**: Why is distance necessary for Calculus?

S5 Equations and inequalities

You will learn: different ways of looking at equations and inequalities (as something to be solved, or as something what describes certain sets), with focus on linear equations and inequalities containing absolute value. Solution sets as subsets of $\mathbb{R}$ or $\mathbb{R}^2$.

Read along with this section: Sections 0.1, 0.3, and 0.4 in the Prerequisites book (pp.3–17, pp.38–59). Subsection 1.1.1 in the Precalculus book (pp.1–5, plus exercises 1–19 on pp.14–15).

45 Equations: unknowns, solutions (roots), solution sets.

46 Inequalities: variables, solution sets.
Extra material: notes from the iPad.

47 What does it mean to solve an equation or inequality, to verify a solution.
Example 1: Verify that $x = 1$ is a root of the equation $x^2 + 2x - 3 = 0$ and that $x = 0$ satisfies the inequality $x^2 + 2x - 3 < 0$.
Extra material: notes with solved Example 1.

48 Examples of equations with one unknown and where to find them.

49 Operations which do not change the solution set of an equation.

Example 2: Solve the equation $3(x-1) = x+5$.

Extra material: notes with solved Example 2.

50 Inconsistent equations and false roots.

Example 3: Show that the equation $6x-2 = 3(2x+1)-4$ is inconsistent.

Example 4: Show that the following equation is inconsistent:

$$\frac{2}{x+1} = 4 - \frac{2x}{x+1}.$$

Extra material: notes with solved Example 3.

51 Linear equations, Problem 1.

Problem 1: Solve the equation

$$\frac{x}{2}\left(\frac{3}{4} - 4x\right) = \left(\frac{x}{4} - \frac{1}{2}\right)\left(\frac{1}{2} - 8x\right).$$

Extra material: notes with solved Problem 1.

52 Linear equations with absolute value, Problem 2.

Problem 2: Solve the equations:

a) $|x-1| = 2$,
b) $|x-1| + |x-5| = 2$,
c) $|x-5| - |x-1| = 2$.

Extra material: notes with solved Problem 2.

53 Linear equations with absolute value, Problem 3.

Problem 3: Solve the equation $1 + |x+1| - |x-2| = x$.

Extra material: notes with solved Problem 3.

54 Linear equations with absolute value, Problem 4.

Problem 4: Solve the equation $1 + |x+1| - |x-2| = x+2$.

Extra material: notes with solved Problem 4.

55 Linear equations with absolute value, Problem 5.

Problem 5: Solve the equation $1 + |x+1| - |x-2| = x+3$.

Extra material: notes with solved Problem 5.

56 Linear equations with absolute value, Problem 6.

Problem 6: Solve the equation $1 + |x+1| - |x-2| = 2x$.

Extra material: notes with solved Problem 6.

57 **Future**: Examples of equations and their solution sets.

58 Equations of straight lines in the plane: slope and intercept.

59 Equations of straight lines in the plane, Problem 7.

Problem 7: Determine an equation for the straight line through the points $(-3, 4)$ and $(1, -4)$.

Extra material: notes with solved Problem 7.

60 Operations which do not change the solution set of an inequality.

61 Careful with multiplying inequalities by variable expressions.

62 Linear inequalities, Problem 8.

Problem 8: Solve the inequality $2(1-x)+5 \leqslant 3(2x-1)$.

Extra material: notes with solved Problem 8.

63 How to handle inequalities with absolute value.

64 Linear inequalities with absolute value, Problem 9.

Problem 9: Solve the inequality $|3-2x| \leqslant 5$.

Extra material: notes with solved Problem 9.

65 Linear inequalities with absolute value, Problem 10.

Problem 10: Solve the inequality $|2x-3| > 7$.

Extra material: notes with solved Problem 10.

66 Linear inequalities with absolute value, Problem 11.

Problem 11: Solve the inequality $|x-2| - |x-1| < x - \frac{3}{2}$.

Extra material: notes with solved Problem 11.

67 **Future**: Equations and inequalities in one variable, in Calculus 1 and 2.

68 **Future**: Equations and inequalities in two variables, in Calculus 3.

69 **Advanced**: A strange circle from Videos 39 and 42, Problem 12.

Problem 12: Draw the set $D = \{(x,y);\ |x| + |y| \leqslant 1\}$ in the plane.

70 What about systems of linear equations?

Example: solve the following system of linear equations

$$\begin{cases} 2x + 3y = -1 \\ 4x + y = 3 \end{cases}$$

using three methods: the geometrical method; elimination of one variable by expressing it with help of another one from one of the equations, and plugging it in in the second equation; elimination of one variable by using opposite coefficients.

S6 Functions

You will learn: about functions: various ways of defining functions; domain, range, graph; x- and y-intercepts; surjections, injections, bijections, inverse functions; increasing and decreasing (monotone) functions; bounded functions; arithmetic operations on functions; compositions of functions; odd and even functions; transformations of graphs.

Read along with this section: Chapters 1 and 5 (pp.43–150, pp.359–416) in the Precalculus book. For now, don't worry about the concept *relation*; we come back to functions as relations in our Section 10. You can also read the relevant parts of sections 2.1 and 2.2 in the Precalculus book (pp.151–187).

71 Function as a relation between variables: independent and dependent.

Example 1: A person is walking during one hour, with constant speed of s equal to 6 km/h. Describe the distance d the person has walked as a function of the time t.

72 Function as a relation between variables; graph and vertical-line test.

Example 2: Describe the area A and the perimeter P of a disk as functions of the radius r. Plot $y = A(r)$ and $y = P(r)$ and illustrate the vertical-line test for them.

Extra material: notes from the iPad.

73 Function as a relation between variables; implicit versus explicit.

Example 3: The implicit relation $x^2 + y^2 = 1$ between variables x and y is not a function $y = f(x)$. Show that the graph does not satisfy the vertical-line test.

74 Examples of functions of one (or more!) variable and where to find them.

75 How to think about sets, their unions, intersections, and differences.

Example 4: Let $f : X \to Y$. Write the definition of its graph as the set of all the points $(x, f(x))$ where $x \in D_f$.

76 Functions: domain, codomain, range.

77 Various ways of describing functions: 1. with a formula.

Example 5: Determine the domain, codomain, range for $f : (-1, 1] \to \mathbb{R}$ defined as $f(x) = -2x + 1$. Also indicate the x- and y-intercepts for f.

Extra material: notes with solved Example 5.

78 Determining the domain: three typical issues to remember.

Issue 1: don't divide by zero.

Issue 2: no even degree roots of negative numbers.

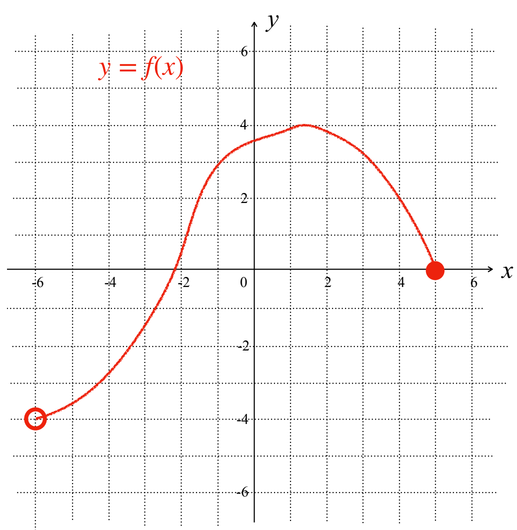
Issue 3: no logarithms of non-positive numbers.

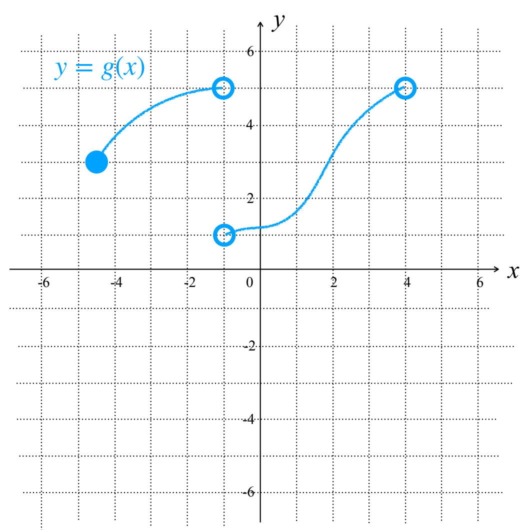
Example 6: Determine the domain, codomain, range for $f(x) = \frac{1}{x}$ and for $f(x) = \sin x$.

Extra material: notes from the iPad.

79 Various ways of describing functions: 2. with a picture.

Example 7: Determine the domain and the range of the functions f and g; also indicate the x- and y-intercepts for f and g:





Extra material: notes with solved Example 7.

80 Various ways of describing functions: 3. with a table.

Example 8: Determine the domain, codomain, range for $f : X \to Y$, where $X = \{a, b, c, d, e, f\}$, $Y = \{1, 2, 3, 4\}$, and f is defined by the following table:

a	b	e	f
1	1	4	2

Try to illustrate this function using Venn diagrams. Mark the domain and the range as subsets of the departure set and the target set respectively.

81 Various ways of describing functions: 4. with a verbal description.

Example 9: Social-security number (in Sweden).

Example 10: Assigning the birth mother to each child (in an ancient village).

Extra material: notes with solved Examples 9 and 10.

82 Surjections, injections, bijections. How to detect injections.

Case 1: $f : \mathbb{R} \to \mathbb{R}$ given by a formula: verify if from $f(x_1) = f(x_2)$ follows $x_1 = x_2$. Ex.5.

Case 2: $f : \mathbb{R} \to \mathbb{R}$ given by its graph: horizontal line test. Ex.7.

Case 3: f given by a table: see if all the values in the second row are different from each other. Ex.8.

Case 4: f given by a verbal description: verify if you can go back to the arguments from the values. Ex.9&10.

Extra material: notes from the iPad.

83 Inverse functions.

Example 11: Determine the inverse to the function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = 2x + 3$.

Example 12: Determine the inverse to the function from V77 (Ex.5): $f : (-1, 1] \to \mathbb{R}$ defined as $f(x) = -2x + 1$.

Extra material: notes with solved Examples 11 and 12.

84 Monotone functions: increasing, decreasing, non-decreasing, non-increasing.

Example 13: Ceiling $\lceil x \rceil$ and floor $\lfloor x \rfloor$ functions are non-decreasing.

Example 14: $f(x) = \frac{1}{x}$ for $x > 0$ is decreasing, but it is not decreasing as a function on $\mathbb{R} \setminus \{0\}$.

85 Bounded functions: bounded above and bounded below, maximum, minimum.

Example 15: $f(x) = \sin x$ is a bounded function.

Example 16: $f(x) = x$ is not bounded.

Example 17: $f(x) = x^2$ is bounded below, but not bounded above.

Extra material: notes with solved Examples 15, 16, and 17.

86 How to scale, add, subtract, multiply, and divide functions.

Example 18: Plot $f(x) = 3 \sin x$ and $f(x) = \sin x + \cos 2x$.

Example 19: Let $f(x) = x + 5$ and $g(x) = x^2 - 1$. Write the formulas for $f + g$, $f - g$, fg, $\frac{f}{g}$, and $5g$.

Extra material: notes with solved Example 19.

87 What is a composition of functions?

Example 20: Analyze the compositions $f_1(x) = \ln(\sin x^2)$, $f_2(x) = \ln(\sin^2 x)$, and $f_3 = \sin(\ln x^2)$.

Example 21: Analyze the following functions: which of them are composed functions, which not:

a) $f(x) = x^3 + \sqrt{x}$ b) $f(x) = \sqrt{x^2 + 4}$ c) $f(x) = x^5 \sin x$ d) $f(x) = \dfrac{3x - 1}{x^2 + 4}$ e) $f(x) = e^{\sqrt{x^4 + x^2}}$.

Extra material: notes with solved Example 21.

88 **Future**: Why is it important to identify the order of functions in a composition?

Example 22: Differentiate $f_1(x) = \ln(\sin x^2)$, $f_2(x) = \ln(\sin^2 x)$, and $f_3 = \sin(\ln x^2)$.

Extra material: notes with solved Example 22.

89 Even and odd functions. **Future**: integration by inspection.

Example 23: Sine and cosine; polynomials with all the terms of odd / even degree.

90 Transformations of graphs.

Example 24: Let $f(x) = x^2$. Use its graph to produce the graphs of

$$x^2 + 3, \quad x^2 - 2, \quad |x^2 - 2|, \quad (x - 1)^2, \quad (x + 1)^2, \quad 2x^2, \quad -x^2, \quad x^2 + 2x - 3.$$

Extra material: notes with solved Example 24.

91 Various operations on functions, Problem 1.

Problem 1: Given $f(x) = 7x + 6$ and $g(x) = 4 - x$. Compute the following:

a) $(f + g)(x)$ b) $(3g - 2f)(x)$ c) $(fg)(x)$ d) $(g \circ f)(x)$ e) $(f \circ g)(x)$ f) $(f \circ f)(x)$.

Extra material: notes with solved Problem 1.

92 Inverse function, its domain and range, Problem 2.

Problem 2: Determine the domain of f and find f^{-1}. What is the domain of f^{-1}? Determine the x- and y-intercepts for both functions, and verify that $f \circ f^{-1} = \mathrm{Id}$ (the identity function $\mathrm{Id}(x) = x$ for all $x \in \mathbb{R}$) for all x for which both functions are defined:

$$f(x) = \frac{x+4}{2x-5}.$$

Extra material: notes with solved Problem 2.

93 Two similar functions with different domains, Problem 3.

Problem 3: Compare the following two functions:

$$f(x) = \frac{x^2 - 1}{x - 1}, \qquad g(x) = x + 1.$$

Plot their graphs, determine their domains and ranges. Any comments?

94 Piecewise functions, Problem 4.

Problem 4: Determine the domain and range for f:

$$f(x) = \begin{cases} -x\,, & -2 \leqslant x \leqslant -1 \\ x^2\,, & -1 \leqslant x \leqslant 1 \\ x\,, & 1 \leqslant x \leqslant 2 \end{cases}$$

Plot the function. Is the function: monotone, bounded, even, odd, invertible?

Extra material: notes with solved Problem 4.

95 Piecewise functions, Problem 5.

Problem 5: Determine the domain and range for f:

$$f(x) = \begin{cases} x\,, & -2 \leqslant x < -1 \\ x^2\,, & -1 \leqslant x \leqslant 1 \\ -x + 4\,, & 1 < x < 2 \end{cases}$$

Plot the function. Is the function: monotone, bounded, even, odd, invertible?

Extra material: notes with solved Problem 5.

96 Piecewise functions, Problem 6.

Problem 6: Determine the domain and range for f:

$$f(x) = \begin{cases} 3\,, & -2 \leqslant x < -1 \\ x^2\,, & -1 < x < 1 \\ 2\,, & 2 \leqslant x < 3 \end{cases}$$

Plot the function. Is the function: monotone, bounded, even, odd, invertible?

Extra material: notes with solved Problem 6.

97 Functions with absolute value, Problem 7.

Problem 7: Plotting absolute values:

a) Illustrate the solution to the problems in Video 64 and Video 65 by plotting the graph of $g(x) = |2x - 3|$ and graphically solving inequalities $|3 - 2x| \leqslant 5$ and $|2x - 3| > 7$.
b) Confirm graphically the solutions to the equations from Videos 53–56, by plotting the function

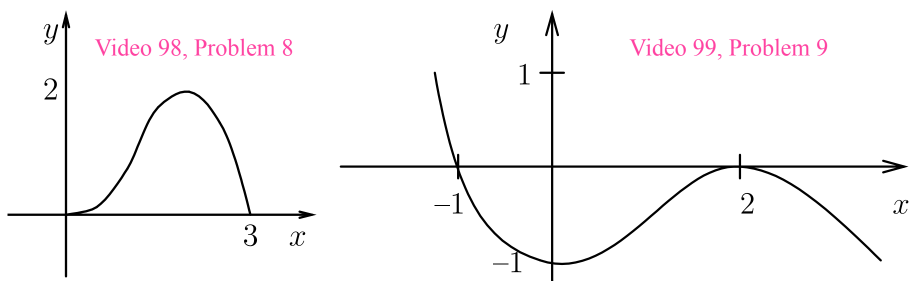
$$f(x) = 1 + |x + 1| - |x - 2|$$

and by examining the intersections of its graph with the graphs of $g_1(x) = x$, $g_2(x) = x + 2$, $g_3(x) = x + 3$, and $g_4(x) = 2x$.

Extra material: notes with solved Problem 7.

98 Transformations of graphs, Problem 8.

Problem 8: Given function f with the graph as in the picture. The domain of f is $[0, 3]$ and the range of f is $[0, 2]$.



Sketch the graphs of the following functions, and determine their domains and ranges:

a) $f(x)+2$, b) $f(x)-3$, c) $f(x+1)$, d) $f(x-2)$, e) $f(-x)$, f) $-f(x)$, g) $2f(x)$, h) $f(2x)$, i) $f(x/2)$.

Extra material: notes with solved Problem 8.

99 Transformations of graphs, Problem 9.

Problem 9: Given function f with the graph as in the picture. Sketch the graphs of the following functions:

a) $f(x) - 1$, b) $f(x - 1)$, c) $\frac{1}{2}f(x)$, d) $f(2x)$, e) $-f(x)$, f) $f(-x)$, g) $|f(x)|$.

Extra material: notes with solved Problem 9.

100 Heaviside step function, Problem 10.

Problem 10: **Heaviside step function**: This function is defined as:

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geqslant 0 \end{cases}$$

Draw the graphs of the following functions:

a) $f(x) = H(x)$, b) $f(x) = H(x-2)$, c) $f(x) = H(x-2)-H(x-3)$, d) $f(x) = (3-x)(H(x-2)-H(x-3))$.

This function (also called **The Unit Step Function**) is important in engineering. It describes an abrupt change (for example: a voltage is switched on or off in an electrical circuit at a specified value of time x).

101 Even and odd functions, Problem 11.

Problem 11: Which of the following functions are even, which are odd, and which are neither even nor odd:

a) $f(x) = x^4$, b) $f(x) = x^3$, c) $f(x) = x^2 - 3x + 1$, d) $f(x) = x(e^x - e^{-x})$, e) $f(x) = x\sqrt{x}$.

Extra material: notes with solved Problem 11.

102 Domain and range of a composed function, Problem 12.

Problem 12: Let $g(x) = x + 1$ and $h(x) = \sin\sqrt{x}$. Determine the domain and range of $f(x) = h(g(x))$.

Extra material: notes with solved Problem 12.

103 **Advanced**: Domain and range, Problem 13.

Problem 13: Determine D_f and R_f if

$$f(x) = \frac{1}{\ln(\sin x)}.$$

104 **Advanced**: A complicated function, Problem 14.

Problem 14: Let

$$f(x) = (x + ((3x)^5 - 2)^{-1/2})^{-6}$$

Draw a chart showing step by step how this function is constructed.

Extra material: notes with solved Problem 14.

105 **Advanced, Future**: The derivative of the complicated function from V104, Problem 15.

Problem 15: Compute the derivative of the function from the previous video.

Extra material: notes with solved Problem 15.

106 Even and odd functions, Problem 16.

Problem 16: Let f be an odd function, and g be an even function. Show that:

a) $f(0) = 0$,
b) the product fg is an odd function,
c) the sum $f + g$ does not need to be odd.

Extra material: notes with solved Problem 16.

S7 Logic

You will learn: the meaning of the symbols used in logic; conjunction, disjunction, implication, equivalence, negation; basic rules of logic (tautologies) and how to prove them; two kinds of quantifiers: existential and universal; necessary and sufficient conditions.

107 What is a statement and logical value; open and closed statements.

Example 1: Difference between statements and algebraic expressions; true and false statements.

108 To be or not to be in the delta neighbourhood of some point.

109 Unary logical connective: Negation NOT.

Example 2: Write the negations $\neg p$ for all the statements from Video 107. Determine whether they are false or true. Statements: 1. Number 7 is prime, 2. $98 = 26$, 3. $(-2)^2 > 0$, 4. $x^2 = 6$, 5. Function f is continuous, 6. $x = 26$, 7. $\sqrt{x} + \sin x > 0$, 8. $|x - a| < \delta$.

Extra material: notes with solved Example 2.

110 Binary logical connective 1: Conjunction AND.

Example 3: Determine whether the statement $p \wedge q$ is true or false:

a) p: $2 = 4$, q: $x^2 = 0$
b) p: $x^2 = 4$, q: $x < 0$
c) p: Number 7 is prime, q: $4 < 3$
d) p: $x - 5 < 0$, q: $x > -3$
e) p: Stockholm is the capital of Sweden, q: $6^2 = 36$

f) p: $x^2 - y^2 = (x-y)(x+y)$, q: $x = y$.

In case of open statements, determine for what values of the variables $p \wedge q$ is true.

Extra material: notes with solved Example 3.

111 Binary logical connective 2: Disjunction OR.

Example 4: Determine whether the statement $p \vee q$ is true or false:

a) p: $2 = 4$, q: $x^2 = 0$
b) p: $x^2 = 4$, q: $x < 0$
c) p: Number 7 is prime, q: $4 < 3$
d) p: $x - 5 < 0$, q: $x > -3$
e) p: Stockholm is the capital of Sweden, q: $6^2 = 36$
f) p: $x^2 - y^2 = (x-y)(x+y)$, q: $x = y$.

In case of open statements, determine for what values of the variables $p \vee q$ is true.

Extra material: notes with solved Example 4.

112 Connectives, Example 5.

a) Suppose p is false, q is false, s is true. Find the logical value of $(s \vee p) \wedge (q \wedge \neg s)$.
b) Suppose p is true, q is true, r is false, s is false. Find the logical value of $(s \vee p) \wedge (\neg r \vee \neg s)$.
c) Suppose p is true, q is true, s is false. Find the logical value of $(\neg s \vee p) \vee (q \wedge \neg s)$.
d) Suppose p is false, s is false, r is true. Find the logical value of $\neg[(s \wedge p) \vee \neg r]$.
e) Suppose p is false, q is true, s is true. Find the logical value of $(p \wedge \neg q) \vee \neg s$.
f) Suppose p is false, q is true, r is false. Find the logical value of $(p \vee \neg q) \vee r$.
g) Suppose p is true, q is true, r is true, s is false. Find the logical value of $(\neg p \vee s) \vee (s \wedge r)$.

Extra material: notes with solved Example 5.

113 Binary logical connective 3: Implication IF THEN. A necessary condition.

Example 6: Necessary conditions:

a) $4|n \quad \Rightarrow \quad 2|n$.
b) If a quadrilateral is a square then it has four right angles.
c) $x = 2 \quad \Rightarrow \quad x^2 = 4$.
d) $x = 0 \quad \Rightarrow \quad \sin x = 0$.
e) $2x + 4 < 0 \quad \Rightarrow \quad x < -2$.

114 Converse, inverse, and contrapositive statements.

Example 7: Consider the implication $x = 2 \Rightarrow x^2 = 4$. Formulate its *converse*, its *inverse*, and its *contrapositive*. Which of them are true?

Extra material: notes with solved Example 7.

115 Binary logical connective 4: Equivalence IFF. A necessary and sufficient condition.

Example 8: Correct the necessary conditions in Video 113 so that they become sufficient conditions.

116 Equivalence versus equality.

Example 9: Equal or equivalent?:

a) $x^2 - y^2$ $(x-y)(x+y)$
b) $x + y > 0$ $x > -y$
c) $x^2 + y^2 = 1$ $y = \pm\sqrt{1 - x^2}$
d) $6 - 5 - 1$ $\ln 1$
e) $x^2 - 2xy + y^2$ $(x-y)^2$
f) $xy < 0$ $\frac{x}{y} < 0$.

117 Tautology as a logical formula.

118 How to verify tautologies.

Example 10: Prove both de Morgan's Laws: *The negation of a disjunction is the conjunction of the negations* and *The negation of a conjunction is the disjunction of the negations.*

119 Tautologies, Problem 1.

Problem 1: Prove the following tautology: $(p \Rightarrow q) \Leftrightarrow \neg p \vee q$. Use this tautology to prove that the negation of implication is done according to the rule: $\neg(p \Rightarrow q) \Leftrightarrow (p \wedge \neg q)$. Show that the following tautology follows as a consequence of the previous ones: $(p \Rightarrow q) \;\Leftrightarrow\; [(p \wedge \neg q) \;\Rightarrow\; \perp]$.

Extra material: notes with solved Problem 1.

120 Tautologies, Problem 2.

Problem 2: Prove that the contrapositive to a given implication is equivalent to this implication, but the converse and inverse are not. The two last cases are examples of formal fallacies, i.e. incorrect reasoning; the first one is an example of a valid, correct reasoning, and is used for proofs by contraposition which will be discussed in Section 11.

Extra material: notes with solved Problem 2.

121 Tautologies, Problem 3.

Problem 3: Prove the distributivity of conjunction over disjunction: $p \wedge (q \vee r) \;\Leftrightarrow\; (p \wedge q) \vee (p \wedge r)$.

Extra material: notes with solved Problem 3.

122 Tautologies, Problem 4.

Problem 4: Make a truth table for the following expression: $(p \vee q) \wedge \neg(\neg q \wedge r)$.

Extra material: notes with solved Problem 4.

123 Tautologies, Problem 5.

Problem 5: Make a truth table for the following expression: $(p \Rightarrow q) \Rightarrow [(q \Rightarrow r) \Rightarrow (p \Rightarrow r)]$.

Extra material: notes with solved Problem 5.

124 **Advanced**: Tautologies, Problem 6.

Problem 6: Draw the following set in the plane:

$$\{(x, y) \in \mathbb{R}^2;\ |x| + |y| < 2 \;\Rightarrow\; x^2 + y^2 < 1\}.$$

Extra material: notes with solved Problem 6.

125 Existential quantifier THERE EXISTS.

Example 11: Determine whether the statements are true or false:

a) $\exists x \in \mathbb{R}\;\; x^2 < 0$,
b) $\exists! x \in \mathbb{R}\;\; x^2 \leqslant 0$,
c) $\exists! x \in \mathbb{R}\;\; x^2 > 0$,
d) $\exists x \in \mathbb{N}\;\; x + 5 = 3$,
e) $\exists x \in \mathbb{R}\;\; \exists y \in \mathbb{R}\;\; x + y = 0$,
f) $\exists x \in \mathbb{R}\;\; (x - 5 < 0 \;\wedge\; x > -3)$
g) $\exists x \in \mathbb{R}\;\; \exists y \in \mathbb{R}\;\; (x + y = 0 \;\vee\; x^2 < 0)$,
h) $\exists x \in \mathbb{R}\;\; (2x + 3 < 0 \;\wedge\; 3x - 6 > 4)$.

Extra material: notes with solved Example 11.

126 Universal quantifier FOR ALL. Order matters. Precedence rules.

Example 12: Let $p(x, y) : x + y = 0$ and $q(x, y) : xy = 1$. Determine whether the following statements are true or false. Parts b) and c) show that the order of quantifiers matters (if they are different):

a) $\forall x \in \mathbb{R}\ \ \forall y \in \mathbb{R}\ \ p(x,y)$; $\quad \forall x \neq 0\ \ \forall y \neq 0\ \ q(x,y)$,
b) $\forall x \in \mathbb{R}\ \ \exists y \in \mathbb{R}\ \ p(x,y)$; $\quad \forall x \neq 0\ \ \exists y \neq 0\ \ q(x,y)$,
c) $\exists y \in \mathbb{R}\ \ \forall x \in \mathbb{R}\ \ p(x,y)$; $\quad \exists y \neq 0\ \ \forall x \neq 0\ \ q(x,y)$,
d) $\exists x \in \mathbb{R}\ \ \exists y \in \mathbb{R}\ \ p(x,y)$; $\quad \exists x \neq 0\ \ \exists y \neq 0\ \ q(x,y)$.

Example 13: Illustrate that the following two statements are **not** equivalent

$$\forall x\ (p(x) \vee q(x)) \quad \text{and} \quad \forall x\ p(x) \vee q(x).$$

Use statements $p(x) : x > -3$ and $q(x) : x < 5$.

Extra material: notes with solved Examples 12 and 13.

127 Negations: De Morgan's Laws for quantifiers, Problem 7.

Problem 7: Use logical symbols to write the definitions of monotone functions (as in Video 84). Show that $f(x) = \frac{1}{x}$ is not decreasing on $\mathbb{R} \setminus \{0\}$.

128 Logic, Problem 8.

Problem 8:

a) Use logical symbols to write the definition of a bounded function (as in Video 85).
b) Use logical symbols to write the definitions of a local (global) maximum, and local (global) minimum of a function (as in Video 85).
c) Use logical symbols to write the definitions of an injection and surjection (as in Video 82). Write with mathematical symbols that some function is not an injection / surjection.

Extra material: notes with solved Problem 8.

129 Logic, Problem 9.

Problem 9: Write the following sentences with mathematical symbols:

a) Natural number n is even.
b) Natural number n is a sum of squares of two integers.
c) Natural number n is prime.
d) Natural number n is not prime.
e) There is no largest element in the set of natural numbers.
f) There exists no real number whose square is negative.

Extra material: notes with solved Problem 9.

130 Logic, Problem 10.

Problem 10: Translate *The sum of two positive integers is always positive* into a logical expression.

Extra material: notes with solved Problem 10.

131 **Future, advanced**: Sequences, Problem 11.

Problem 11: *Almost all* is defined as *all except for some finite number*. Let $a_n = \frac{1}{n}$. Write with help of logical symbols that the elements of a_n come arbitrarily close to zero with n tending to infinity, by saying that for each $\varepsilon > 0$ almost all elements of a_n belong to the ε neighbourhood of zero. Make a picture.

Extra material: notes with solved Problem 11.

132 Epsilon-delta, here we come!

Example 14: Epsilon-delta definition of a limit: now we can read and understand its meaning.

133 **Advanced**: Negation: L is not the limit.

Example 15: How to write with help of logical symbols that L is not the limit.

S8 Sets

You will learn: the basic terms and formulas from the Set Theory and the link to Logic; union, intersection, set difference, subset, complement; cardinality of a set; Inclusion–exclusion principle.

134 Primitive notions: set, belonging to a set, the empty set.

135 The universe and its subsets.

136 If you know logic, you know the set theory.

137 Intersection defined with help of conjunction.

Example 1.1: Determine the domain of $f+g$ where $f(x)=\frac{\sqrt{9-x^2}}{x-3}$ and $g(x)=\frac{\sqrt{5-x}}{\sqrt{x-1}}$. $\bigcap_{n=1}^{\infty}(-\frac{1}{n},\frac{1}{n})=\{0\}$.

Extra material: notes with an explanation about the big intersection symbol; old Polish universal quantifier.

138 Union defined with help of disjunction.

Example 2: Back to Problem 13 from Video 103: Determine the domain D_f of

$$f(x)=\frac{1}{\ln(\sin x)}.$$

Extra material: notes with solved Example 2 and with an explanation about the big union symbol; old Polish existential quantifier.

139 Equality of sets defined with help of equivalence: Axiom of Extensionality.

140 Subset defined with help of implication.

Example 3: The set $A=\{0,1,2,3\}$ is not a subset of $B=\{1,2,3,4,5\}$.

141 The empty set gives a false statement.

Example 4: The empty set is a subset of each set A.

142 The universe gives a true statement.

143 Set difference.

Example 5: Show that the set difference is **not** commutative using $A=\{0,1,2,3\}$ and $B=\{1,2,3,4,5\}$.

144 Symmetric difference.

Example 6: Compute symmetric difference of $A=\{1,2,3,4,6,8\}$ and $B=\{1,2,4,5,7,8,9,10\}$.

145 Complement defined with help of negation.

Example 7: Describe A^c with respect to the universe U, where:

a) A: set of all the positive even integers, $U=\mathbb{N}$.
b) A: set of all the positive even integers, $U=\mathbb{R}$.
c) $A=(-1,2]$, $U=\mathbb{R}$.
d) $A=\{-4,-1,2,3,5\}$, $U=\{x\in\mathbb{Z};\ |x|\leqslant 5\}$.
e) $A=[-4,3]\cap\mathbb{N}$, $U=\{x\in\mathbb{Z};\ |x|\leqslant 5\}$.
f) $A=[-4,3]\cap\mathbb{N}$, $U=\{x\in\mathbb{R};\ |x|\leqslant 5\}$.

Extra material: notes with solved Example 7.

146 The laws of set theory and the laws of logic.

147 De Morgan's Laws, an illustration and proof.

Extra material: notes from the iPad.

148 The distributive law, an illustration and proof.

Extra material: notes from the iPad.

149 Set difference and symmetric difference, Problem 1.

Problem 1: Show that the symmetric difference of two sets can be expressed as $A \triangle B = (A \cup B) \setminus (A \cap B)$.

Extra material: notes with solved Problem 1.

150 Cartesian product of sets.

Example 8: Describe $A \times B$ if $A = \{0, 1, 2, 3\}$ and $B = \{1, 4, 5\}$.

Example 9: Describe $A \times B$ if $A = (2, 5]$ and $B = [1, 6)$.

151 Power set of a given set; an example of Boolean algebra.

Example 10: Describe the power set 2^X, where:

a) $X = \{a, b\}$.
b) $X = \{1, 2, 3\}$.
c) $X = \{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$.

152 Cardinality of sets.

153 Cardinality of sets, Problem 2.

Problem 2: Let $|X| = n$. Show that cardinality of the power set 2^X is 2^n.

Extra material: notes with solved Problem 2.

154 **Optional**: Equinumerous (equipotent) sets; cardinal numbers.

155 **Optional**: Finite, countable and uncountable sets.

156 **Optional**: Comparing cardinalities of sets, Problem 3.

Problem 3: Construct surjections showing that the following sets are countable:

a) $\{4, 5, 6, 7, 8, \dots\}$,
b) the set of all even natural numbers,
c) the set of all integers,
d) the set of all rational numbers.

Extra material: notes with solved Problem 3.

157 **Optional**: Comparing cardinalities of sets, Problem 4.

Problem 4: Find a bijection showing that the interval $(-\pi/2, \pi/2)$ has the same cardinality as $\mathbb{R}$. Show that each pair of open intervals have the same cardinality. This shows that each open interval has the cardinality $\mathfrak{c}$.

Extra material: notes with solved Problem 4.

158 Inclusion–exclusion principle.

Example 11: In a group of 30 students: 18 students like tea and 25 students like coffee.

a) How many students like both coffee and tea?
b) How many students like only coffee?
c) How many students like only tea? (We assume that each student likes *at least* one of them: coffee or tea.)

159 Inclusion–exclusion principle, Problem 5.

Problem 5: In a group of 57 students: 42 students can speak English, 38 students can speak Swedish, 35 students can speak Polish. In the same group: 23 students can speak both Swedish and English, 20 students can speak both English and Polish, and 27 students can speak both Swedish and Polish. How many students in this group can speak all the three languages? How many students can speak just one language (if each speaks *at least* 1)?

Extra material: notes with solved Problem 5.

160 Transposition law and subsets, Problem 6.

Problem 6: Illustrate the Transposition law for implications (formulated in Video 117, proven in Video 120) $(p \Rightarrow q) \ \Leftrightarrow \ (\neg q \Rightarrow \neg p)$ with help of Venn diagrams.

S9 Relations

You will learn: about binary relations generally, and specifically about RST (Reflexive–Symmetric–Transitive) relations, equivalence classes, and about order (partial order) relations.

161 Relations in a regular sense.

162 Relations in a mathematical sense.

Example 1: Examples of relations.

a) parallelity relation, similarity relation: $l \parallel k$, $\triangle ABC \sim \triangle PQR$,
b) being divisor of: $2|8$, $5|35$, $\dots$,
c) less than / greater than / $\dots$: $x < y$, $x > y$, $x \leqslant y$, $x \geqslant y$,
d) equality relation: $x = y$, $A = B$, $(x_1, y_1) = (x_2, y_2), \dots$,
e) being a subset of: $\mathbb{N} \subset \mathbb{Z}$, $\mathbb{N} \subset \mathbb{R}$, $\{1, 2, \pi, \sqrt{3}\} \subset \mathbb{R}$, $\dots$.

163 Relation: a formal definition and notation.

164 How to plot relations.

Example 2: Let $X = \mathbb{R}$. Illustrate the following relations on X:

a) $xRy \quad \Leftrightarrow \quad x = y$,
b) $xRy \quad \Leftrightarrow \quad x < y$,
c) $xRy \quad \Leftrightarrow \quad x \leqslant y$,
d) $xRy \quad \Leftrightarrow \quad x > y$.

165 How to plot relations, Problem 1.

Problem 1: Let $X = \{1, 2, 3, 4, 5\}$. Illustrate the following relations on X:

a) $xRy \quad \Leftrightarrow \quad x < y$,
b) $xRy \quad \Leftrightarrow \quad x|y$,
c) $xRy \quad \Leftrightarrow \quad y = x^2$,
d) $xRy \quad \Leftrightarrow \quad x \equiv y \pmod 2$.

166 Relations on finite sets, and their graphs, Problem 2.

Problem 2: Draw the graphs depicting the relations from Problem 1.

Extra material: notes with solved Problem 2.

167 RST relations / equivalence relations.

Problem 3: Which relations described in Video 162 are RST?

Extra material: notes with solved Problem 3.

168 How to recognize an RST relation from its plot?

Problem 4: Which relations described in Videos 164 and 165 are RST?

169 How to recognize an RST relation on a finite set from the graph?

Problem 5: Use the relations described in Video 165 and their graphs from Video 166 to verify the graph method of testing relations for being RST.

Extra material: notes with solved Problem 5.

170 Relation congruence modulo n.

Extra material: notes with the proof that the relation modulo is an equivalence relation.

171 Equivalence classes and partitions of a set.

Example 3: Analyse the relation modulo 5 and its equivalence classes.

Example 4: Describe equivalence classes in relations of parallelity of lines, similarity of triangles; vectors.

Extra material: notes with solved Example 4.

172 More properties of relations: irreflexive, antisymmetric, strongly connected.

Example 5: Test the new properties for $<$, $>$, $\leqslant$, $\geqslant$ on $X = \mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}, \mathbb{R}$, and for $x|y$ on $\mathbb{N}^+$.

Extra material: notes with solved Example 5.

173 Order relations.

174 Partial orders and Hasse diagrams.

Example 6: Draw Hasse diagram for Example 10 c) from Video 151 (subset relation) and for relation of being divisor of described in Video 162 (use number 60).

175 Relations, Problem 6.

Problem 6: Let $X = \{a, b, c, d\}$. Examine the properties of $R \subset X \times X$ if:

a) $R = \{(a, a), (b, b), (a, b)\}$,
b) $R = \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, a)\}$,
c) $R = \{(a, a), (b, b), (c, c), (a, b), (a, c), (b, c)\}$.

Extra material: notes with solved Problem 6.

176 Relations, Problem 7.

Problem 7: The relation $R \subset \mathbb{N} \times \mathbb{N}$ is defined as:

$$\forall \ x, y \in \mathbb{N} \quad [xRy \quad \Leftrightarrow \quad (x \leqslant 5 \ \wedge \ y \leqslant 5 \ \wedge \ x = y) \ \vee \ (x > 5 \ \wedge \ y > 5 \ \wedge \ 2|x + y)].$$

Examine properties of this relation.

Extra material: notes with solved Problem 7.

177 Relations, Problem 8.

Problem 8: The relation $R \subset \mathbb{R} \times \mathbb{R}$ is defined as:

$$\forall \ x, y \in \mathbb{R} \quad (xRy \quad \Leftrightarrow \quad |x - 2| = |y + 2|).$$

Examine properties of this relation.

Extra material: notes with solved Problem 8.

178 Relations, Problem 9.

Problem 9: Let $\mathcal{P}(\mathbb{N})$ be the set of all the subsets of $\mathbb{N}$ and let $\mathbb{E}$ denote the set of all the even natural numbers. We define the relation $R \subset \mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N})$ in the following way:

$$\forall \ X, Y \in \mathcal{P}(\mathbb{N}) \quad (XRY \quad \Leftrightarrow \quad X \cap \mathbb{E} = Y \cap \mathbb{E}).$$

Examine properties of this relation.

Extra material: notes with solved Problem 9.

S10 Functions as relations

You will learn: definition of a function as relation between sets: domain and co-domain; injections, surjections, bijections, inverse functions.

179 Some early signs.

180 Each function is a relation, not every relation is a function.

181 Each relation is invertible!

Example: Find the inverses to the relations b) and c) from Video 165. Here $X = \{1, 2, 3, 4, 5\}$. Make a picture and write explicitly both R and R^{-1}: b) $\ xRy \quad \Leftrightarrow \quad x|y$, c) $\ xRy \quad \Leftrightarrow \quad y = x^2$.

182 Relations and functions, Problem 1.

Problem 1: Relation R from $X = \{a, b, c, d\}$ to $Y = \{m, n, p\}$ is defined as

$$R = \{(a, m), (a, n), (b, m), (b, n)\}.$$

Determine the following for R:

a) its departure set and its target set,
b) its domain and its range,
c) its inverse R^{-1}.

Extra material: notes with solved Problem 1.

183 Relations and functions, Problem 2.

Problem 2: Which of the following relations between $X = \{1, 2, 3, 4, 5\}$ and $Y = \{a, b, c, d\}$ are functions?:

a) $R_1 = \{(1, a), (2, b), (3, c), (3, d)\}$,
b) $R_2 = \{(1, a), (2, b), (2, c), (3, d)\}$,
c) $R_3 = \{(1, a), (1, b), (1, c), (3, d)\}$,
d) $R_4 = \{(1, a), (2, a), (4, d), (5, d)\}$.

Extra material: notes with solved Problem 2.

184 Relations and functions, Problem 3.

Problem 3: Let $X = \{1, 2, 3, 4\}$ and $Y = \{5, 6, 7\}$.

a) Write down (explicitly) the Cartesian product $X \times Y$.
b) Give an example of a function $f : X \to Y$ that is surjective but not injective. Are there any injective functions $f : X \to Y$ (such that $D_f = X$)?
c) Give an example of a function $f : Y \to X$ that is injective but not surjective. Are there any surjective functions $f : Y \to X$ (such that $D_f = Y$)?
d) Determine the inverse of the bijection $f : X \to X$ defined by: $f(1) = 3$, $f(2) = 4$, $f(3) = 1$, $f(4) = 2$. Verify that $f \circ f = \mathrm{Id}_X$ where $\mathrm{Id}_X(x) = x$ for all $x \in X$.

Extra material: notes with solved Problem 3.

185 Equality of functions, Problem 4.

Problem 4: Given $f : A \to B$ and $g : C \to D$. Verify if they are equal to each other if:

a) $A = \{1, 2, 3\},\ B = \{m, n, p, q\},\ C = \{1, 2, 3, 4\},\ D = \{m, n, p, q\},\ f = \{(1, m), (2, p)\},\ g = \{(1, m), (2, p)\}$,
b) $A = \{1, 2, 3\},\ B = \{m, n, p, q\},\ C = \{1, 2, 3\},\ D = \{m, n, p\},\ f = \{(1, m), (2, p)\},\ g = \{(1, m), (2, p)\}$,
c) $A = \{1, 2, 3\},\ B = \{m, n, p, q\},\ C = \{1, 2, 3\},\ D = \{m, n, p, q\},\ f = \{(1, m), (2, p)\},\ g = \{(1, m), (2, p), (3, q)\}$,
d) $A = \{1, 2, 3\},\ B = \{m, n, p, q\},\ C = \{1, 2, 3\},\ D = \{m, n, p, q\},\ f = \{(1, m), (2, p)\},\ g = \{(1, m), (2, p)\}$.

186 Surjections and injections, Problem 5.

Problem 5: Let $f : A \to B$ and $g : B \to C$ be two functions, and $h = g \circ f$ be their composition, i.e. $h(x) = g(f(x))$ for all $x \in A$. Answer the following questions and motivate your answers:

a)
- Assume that f and g are both surjective. Show that then h is surjective, too.
- Assume that f and g are both injective. Show that then h is injective, too.

b)
- Assume that h and f are both surjective. Is then g necessarily surjective, too?
- Assume that h and f are both injective. Is then g necessarily injective, too?

c)
- Assume that h and g are both surjective. Is then f necessarily surjective, too?
- Assume that h and g are both injective. Is then f necessarily injective, too?

Extra material: notes with solved Problem 5.

187 **Future**: Cancellation law for injective functions and why it is important.

Theorem: If $f : Y \to Z$ is injective, then it is **left cancellable**, i.e.

$$\forall X \ \ \forall g_1, g_2 : X \to Y \quad (f \circ g_1 = f \circ g_2 \quad \Rightarrow \quad g_1 = g_2).$$

Example: Solve two equations: $\ln(7x - 9) = \ln(x^2 - x - 29), \quad e^{7x-9} = e^{x^2-x-29}$.

Extra material: notes with the proof of Theorem and with a solution to Example.

188 **Advanced**: A theorem about inverse function.

Theorem: Let $f : X \to Y$. If there are two functions $g : Y \to X$ and $h : Y \to X$ such that $g(f(x)) = x$ for every $x \in X$ and $f(h(y)) = y$ for every $y \in Y$, then f is bijective and $g = h = f^{-1}$.

Extra material: notes with the proof of Theorem.

189 **Advanced, future**: A really important theorem about compositions of functions.

Theorem

a) If $f : V \to Y$, $g : Z \to V$, $h : X \to Z$, then $(f \circ g) \circ h = f \circ (g \circ h)$,
b) If $f : V \to X$ and $g : Y \to V$ are invertible then also $f \circ g$ is invertible and $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$,
c) If $f : X \to Y$ is invertible then also f^{-1} is invertible and $(f^{-1})^{-1} = f$.

Example: Let $R = \{(n, n+1);\ n \in \mathbb{N}\}$. Draw a graph of this relation, and of the composition $R \circ R$.

Extra material: notes with the proof of Theorem.

S11 Axioms, definitions, theorems, and proofs

You will learn: the meaning of words *axiom*, *definition*, *theorem*, *lemma*, *proposition*, *corollary*, *proof*; Various types of proofs with some examples: direct proof, proof by induction, indirect proof, proof by contradiction.

Read along with this section: Section 9.3 (Mathematical induction) in the Precalculus book.

190 How we build mathematical theories.

191 Examples of primitive notions and definitions.

192 Axioms.

193 Theorems, propositions, lemmas, corollaries.

194 Types of proofs.

195 Direct proof (deduction and reduction).

Prove by direct proof (deduction): If an integer number n is even, then n^2 is also even.

Prove by direct proof (deduction): If n is an integer such that $4|(n+1)$, then $8|(n^2-1)$.

Prove by direct proof (reduction): The arithmetic mean of two non-negative numbers is greater than or equal to their geometric mean.

Extra material: notes with the proof of the theorem above.

196 Proof by cases.

Prove by cases: If ABC is any triangle, then:

- $|\sphericalangle BAC| < 90^\circ \quad \Rightarrow \quad |AB|^2 + |AC|^2 > |BC|^2$,
- $|\sphericalangle BAC| = 90^\circ \quad \Rightarrow \quad |AB|^2 + |AC|^2 = |BC|^2$,
- $|\sphericalangle BAC| > 90^\circ \quad \Rightarrow \quad |AB|^2 + |AC|^2 < |BC|^2$.

Prove by cases: If an integer number n is not divisible by 3, then $n^2 - 1$ is divisible by 3.

Extra material: notes with the proof of the second theorem above.

197 Proof by contrapositive.

Prove by contraposition: If the product of two integers is an even number, then at least one of these numbers must be even. (A special case of this one will be treated as lemma for the proof that the square root of 2 is an irrational number, in Video 215.)

Prove by contraposition: Given n positive natural numbers: $k_1, k_2, \ldots, k_n$. Show that if

$$(*) \quad \frac{1}{k_1} + \cdots + \frac{1}{k_n} > \frac{n}{2},$$

then $k_i = 1$ for some i.

Extra material: notes with the proof of the first theorem above.

198 Proof of the extended version of the Cancellation law from Video 187.

Theorem: Function $f : Y \to Z$ is injective if and only if

$$\forall X \ \ \forall g_1, g_2 : X \to Y \quad (f \circ g_1 = f \circ g_2 \quad \Rightarrow \quad g_1 = g_2).$$

Extra material: notes with the proof of Theorem.

199 Series of equivalences.

200 Proof by contradiction.

Prove by contradiction: If $x + x = x$ then $x = 0$.

Prove by contradiction: Let a, b, c be three integers such that $a \neq 0$, $a|b$, and $a \not| c$. Then $a \not| (b + c)$.

Extra material: notes with the proof of the theorem above.

201 Existence, uniqueness, counter-examples.

Prove that there exists a prime number p such that $p + 2$ and $p + 6$ are also prime numbers.

Disprove: For each prime number p, both $p + 2$ and $p + 6$ are also prime numbers.

Prove that in each group (see the definition in Video 192): the neutral element is unique, and the opposite (inverse) to each element is also unique.

Extra material: notes with the proof of the theorem above.

202 Proof of the partition theorem from Video 171.

Prove by contradiction: *Theorem about partitions*: Let R be an equivalence relation on a set X. Then every element of X belongs to exactly one equivalence class.

Extra material: notes with the proof of the theorem above.

203 Construction of the natural numbers; Peano and induction.

Prove by induction: For each positive natural number n we have

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Extra material: notes with the proof of the theorem above.

204 Proof by induction, Problem 1.

Prove by induction: For each positive natural number n we have $9|4^n + 15n - 1$.

Extra material: notes with the proof of the theorem above.

205 Proof by induction, Problem 2.

Prove by induction: For each natural number n we have

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{(n+1)(n+2)} = \frac{n+1}{n+2}.$$

Extra material: notes with the proof of the theorem above.

206 Proof by strong induction, Problems 3 and 4.

Prove by strong induction:

P3. Let a_n be the sequence defined by: $a_1 = 1, a_2 = 8, a_n = a_{n-1} + 2a_{n-2}$ for all $n \geqslant 3$.
Show that $a_n = 3 \cdot 2^{n-1} + 2 \cdot (-1)^n$ for all $n \in \mathbb{N}^+$.

P4. **Fundamental Theorem of Arithmetic**: Every integer $n \geqslant 2$ can be factored into a product of primes.

Extra material: notes with the proof of the statements above.

207 Proof of three statements about natural numbers.

Prove by any method:

1. There is no largest integer;
2. The statement from Video 137: $\bigcap_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n}) = \{0\}$;
3. There exist infinitely many prime numbers.

Extra material: notes with the proof of the statements above.

208 Prove or disprove.

Prove or disprove the following formulas. For all sets A, B, C holds:

a) $A \times (B \cup C) = (A \times B) \cup (A \times C)$,
b) $A \setminus (B \times C) = (A \setminus B) \times (A \setminus C)$,
c) $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$.

Extra material: notes with solved Problem 2.

209 How to define the set of integers using an equivalence relation and the set of natural numbers.

Extra material: notes from the iPad (showing that our relation is RST, and that the addition is well-defined).

210 How to define the set of rational numbers using an equivalence relation and the set of integers.

Extra material: notes from the iPad (showing that our relation is RST, and that the multiplication is well-defined).

211 Some algebraic structures and where to find them.

212 Construction of the real numbers.

The following three properties are equivalent, and they follow from the Axiom of (Dedekind) Completeness:

1. The set $\mathbb{N}$ is not bounded above;
2. The Archimedean Principle;
3. The density of $\mathbb{Q}$ in $\mathbb{R}$.

Extra material: notes with the proof of some parts of the statement above.

213 Some examples of proofs directly from the axioms.

Prove directly from the axioms that for all $x, y, z \in \mathbb{R}$:

1. if $x + y = x + z$ then $y = z$ (cancellation);
2. $-(-x) = x$;
3. if $x \neq 0$ and $xy = xz$ then $y = z$ (cancellation);
4. if $x \neq 0$ then $(x^{-1})^{-1} = x$;
5. $0x = 0 = x0$;
6. $-x = (-1)x$;
7. $x(-y) = (-x)y = -(xy)$;
8. if $x > 0$ then $-x < 0$; if $x < 0$ then $-x > 0$;
9. if $x < y$ and $z < 0$ then $xz > yz$; Thus: $0 < 1$.

Extra material: notes with the proof of some parts of the statement above.

214 The glorious and majestic Zero-Product Property.

Prove by any method: If $ab = 0$ then $a = 0$ or $b = 0$.

Extra material: notes with the proof of the theorem above.

215 The square root of 2 is irrational.

Prove by contradiction: The square root of 2 is an irrational number.

Extra material: notes with the proof of the theorem above.

216 Density of $\mathbb{Q}$ and $\mathbb{R} \smallsetminus \mathbb{Q}$ in $\mathbb{R}$.

Contemplate the following function (the characteristic function of $[0,1] \cap \mathbb{Q}$): $\chi(x) = \begin{cases} 1\,, & x \in [0,1] \cap \mathbb{Q} \\ 0\,, & x \in [0,1] \smallsetminus \mathbb{Q} \end{cases}$

217 A real number is irrational iff its decimal expansion is infinite and aperiodic.

218 **Optional**: The set of real numbers and the set of irrational numbers are not countable.

219 The last piece of the epsilon–delta definition.

220 **Future**: Proof that the limit of sum is equal to the sum of limits.

Theorem: If $\lim\limits_{x \to a} f(x) = L_1$ and $\lim\limits_{x \to a} g(x) = L_2$ then $\lim\limits_{x \to a} (f(x) + g(x)) = L_1 + L_2$.

Extra material: notes with the proof of the theorem above.

S12 Sequences and series; AP, GP, HP

You will learn: how to use the symbols Sigma and Pi; you will also get an introduction to sequences and series, with some examples; arithmetic, geometric, and harmonic progressions.

Read along with this section: Chapter 9 in the Precalculus book (sections 9.1 and 9.2).

221 The Sigma symbol, Problem 1.

Problem 1: Write the sums without (1–4) or with (5–8) the Sigma:

1. $\sum\limits_{k=1}^{7} k^3$;
2. $\sum\limits_{k=3}^{9} (2k-1)^2$;
3. $\sum\limits_{k=1}^{10} (-1)^k$;
4. $\sum\limits_{k=5}^{30} (\frac{1}{k} - b_k)$;
5. $1 + 2 + 4 + 8 + 16 + 32$;
6. $x^2 - x^3 + \cdots - x^{19} + x^{20}$;
7. $e + e^2 + e^3 + \cdots + e^{10}$;
8. $1 - 3 + 9 - 27 + 81 - 243$.

Extra material: notes with the solution to Problem 1.

222 The Pi symbol, Problem 2.

Problem 2: Write the products without (1–4) or with (5–8) the Pi:

1. $\prod\limits_{k=1}^{5} k$;
2. $\prod\limits_{k=0}^{9} (2k-1)$;

3. $\prod_{k=1}^{10} a_k$;
4. $\prod_{k=5}^{30} (1-b_k)^{k-1}$;
5. $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7$;
6. $1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13$;
7. $\left(1-\frac{4}{1}\right)\left(1-\frac{4}{9}\right)\left(1-\frac{4}{25}\right)\left(1-\frac{4}{49}\right)\left(1-\frac{4}{81}\right)\left(1-\frac{4}{121}\right)$;
8. $\sqrt{a_1 a_2 a_3 \cdot \ldots \cdot a_{20}}$.

Extra material: notes with the solution to Problem 2.

223 What is a sequence: two ways of illustrating sequences.

224 Explicit and recursive ways of defining sequences.

Example: Write the first elements of the following sequences:

1. with explicit definition: $a_n = \frac{1}{n}$, $b_n = \frac{n+1}{n}$, $c_n = (-1)^n$, $d_n = n^2$
2. with a recursive definition: $a_1 = 1,\ a_2 = 1,\ a_{n+2} = a_n + a_{n+1}$ for $n \geqslant 1$, $b_1 = 1,\ b_{n+1} = b_n^2 - 1$ for $n \geqslant 1$.

Extra material: notes with solved Example.

225 Some examples of famous sequences.

226 Recursive to explicit: two examples.

Examples: Factorial; Geometric sequence.

Extra material: notes with an induction proof showing that the recurrence $x_0 = 1,\ x_{n+1} = 2x_n$ describes the geometric sequence $x_n = 2^n$.

227 Arithmetic progressions and arithmetic mean, Problem 3.

Problem 3: Which of the following sequences are arithmetic progressions?:

a) $\frac{1}{3},\ 1,\ \frac{5}{3},\ \frac{7}{3},\ \ldots,$
b) $4,\ -1,\ -6,\ -11,\ \ldots,$
c) $9,\ 12,\ 16,\ \ldots.$

For these sequences, write the formula for a_{10}.

Extra material: notes with solved Problem 3.

228 Geometric progressions and geometric mean, Problem 4.

Problem 4: Which of the following sequences are geometric progressions?:

a) $-\frac{2}{5},\ \frac{4}{5},\ -\frac{8}{5},\ \ldots,$
b) $1,\ 3,\ 6,\ 9,\ \ldots,$
c) $-32,\ 16,\ ,-8,\ 4,\ \ldots.$

For these sequences, write the formula for a_{10}.

Extra material: notes with solved Problem 4.

229 Harmonic progressions and harmonic mean, Problem 5.

Derive the formula for the harmonic mean, and establish the relationship between the three means for two positive numbers a and b.

Problem 5: Find term k for the following harmonic progressions:

a) $\frac{2}{3},\ \frac{1}{2},\ \frac{2}{5},\ \ldots,$ term number $k = 8$,
b) $5,\ \frac{30}{7},\ \frac{15}{4},\ \ldots,$ term number $k = 10$,
c) $\frac{10}{3},\ 2,\ \frac{10}{7},\ \ldots,$ term number k (general).

Extra material: notes with solved Problem 5.

230 Series as sums of the elements in a sequence.

Example: Can we assign any meaningful sum to the following sequences?:

1. $a_n = 0$ for all natural $n > 0$;
2. $a_n = 1$ for all natural $n > 0$;
3. $a_n = (-1)^n$ for all natural $n > 0$;
4. $a_n = \left(\frac{1}{2}\right)^n$ for all natural $n > 0$.

231 Convergent and divergent series.

232 Arithmetic series, Problem 6.

1. Derive the formula for the partial sum S_n for the sequence $a_n = n$;
2. Derive the formula for the partial sum S_n for the sequence $a_n = a + (n-1)d$.

Problem 6: Show that the sum of n consecutive odd integers beginning with 1 is n^2.

Extra material: notes with solved Problem 6.

233 Geometric series, Problem 7.

1. Derive the formula for the partial sum S_{n+1} for the sequence $a_n = aq^n$ from $n = 0$;
2. Four different ways of writing the series $3\sum_{k=0}^{\infty}\left(-\frac{2}{5}\right)^k$.

Problem 7: Compute the following sums:

$$-3\cdot\left(\frac{2}{5}\right)^3 + 3\cdot\left(\frac{2}{5}\right)^4 - 3\cdot\left(\frac{2}{5}\right)^5 + \dots, \quad 2+1+\frac{1}{2}+\dots+\frac{1}{128}, \quad e+e^2+\dots+e^{10}, \quad 1-x+x^2-x^3+\dots-x^9.$$

Extra material: notes with solved Problem 7.

234 Harmonic series, Problem 8.

Problem 8: Show with a computation that the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges to infinity. I will show you a geometrical interpretation of this fact, with a method from Calculus 2.

235 Limit of a sequence; Cauchy sequences.

236 **Future**: Proof that the limit of a sum is equal to the sum of limits.

Theorem: If $\lim_{n\to\infty} a_n = L_1$ and $\lim_{n\to\infty} b_n = L_2$ then $\lim_{n\to\infty}(a_n + b_n) = L_1 + L_2$.

Extra material: notes with the proof of the theorem above.

237 So, what's next?

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S13 Bonus section

You will learn: about all the courses we offer, and where to find discount coupons. You will also get a glimpse into our plans for future courses, with approximate (very hypothetical!) release dates.

B Bonus lecture.

Extra material 1: a pdf with all the links and coupon codes.

Extra material 2: a pdf with an advice about optimal order of studying our courses.

Extra material 3: a pdf with information about course books, and how to get more practice.

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