Linear Algebra and Geometry[1] 3

Inner product spaces, quadratic forms, and more advanced problem solving

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A short table of contents

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An extremely detailed table of contents; the videos (titles in green) are numbered

In blue: problems solved on an iPad (the solving process presented for the students; active problem solving)

In red: solved problems demonstrated during a presentation (a walk-through; passive problem solving)

In magenta: additional problems solved in written articles (added as resources).

C1 Eigendecomposition, spectral decomposition

S1 Introduction to the course

1 Introduction to the course. Extra material: this list with all the movies and problems.

S2 Geometrical operators in the plane and in the 3-space

You will learn: using eigenvalues and eigenvectors of geometrical operators such as symmetries, projections, and rotations in order to get their standard matrices; you will also strengthen your understanding of geometrical transformations.

2 Eigendecomposition, recap.

3 Eigendecomposition and operators.

4 Problem 1: Line symmetry in the plane.

Problem 1: Determine the standard matrix for the linear transformation T which for each point $(x, y) \in \mathbb{R}^2$ assigns its reflection about the line $y = 2x$.

Extra material: notes with solved Problem 1.

5 Problem 2: Projection in the plane.

Problem 2: Determine the standard matrix for the linear transformation T which for each point $(x, y) \in \mathbb{R}^2$ assigns its projection on the line $y = 2x$ along vector $(1, 1)$.

Extra material: notes with solved Problem 2.

6 Problem 3: Symmetry in the 3-space.

Problem 3: The linear operator $T : \mathbb{R}^3 \to \mathbb{R}^3$ is the symmetry along the plane $P:\ x + y + 2z = 0$.

a) Find an ON-basis $\{\mathbf{b}_1,\ \mathbf{b}_2,\ \mathbf{b}_3\}$ in $\mathbb{R}^3$ consisting of eigenvectors for T,

b) Determine the standard matrix of T (in standard basis).

Extra material: notes with solved Problem 3.

7 Problem 4: Projection in the 3-space.

Problem 4: The linear operator $T : \mathbb{R}^3 \to \mathbb{R}^3$ is the projection on the plane $P:\ x + 2y + 3z = 0$ in the direction of vector $\mathbf{v} = (1, 1, 1)$.

a) Find a basis $\{\mathbf{b}_1,\ \mathbf{b}_2,\ \mathbf{b}_3\}$ in $\mathbb{R}^3$ consisting of eigenvectors for T,

b) Determine the standard matrix of T (in standard basis).

Extra material: notes with solved Problem 4.

8 Problem 5: Projection in the 3-space.

Problem 5: The linear operator $T : \mathbb{R}^3 \to \mathbb{R}^3$ is defined as

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 8x_1 \\ -x_1 + 6x_2 - 3x_3 \\ -2x_1 - 4x_2 + 2x_3 \end{pmatrix}.$$

a) Show that T is linear,

b) Show that T describes a projection on a plane through the origin. Determine an equation for this plane and the directional vector of the projection.

Extra material: notes with solved Problem 5.

9 Another formulation of eigendecomposition: Spectral decomposition.

10 Powers of matrices: Two methods.

11 Spectral decomposition, Problem 6.

Problem 6: The linear operator $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ is defined as

$$T_A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_1 + 2x_2 - 6x_3 \\ 2x_1 + 3x_2 - 6x_3 \\ x_1 + 2x_2 - 4x_3 \end{pmatrix}.$$

a) Show that T_A is diagonalizable; determine a basis for each eigenspace, an invertible matrix B, and a diagonal matrix D such that $A = BDB^{-1}$.

b) Show that A can be represented as $A = \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3$ where

$$\lambda_1,\, \lambda_2,\, \lambda_3 \in \mathbb{R}, \quad P_1 + P_2 + P_3 = I, \quad P_1^2 = P_1, \quad P_2^2 = P_2, \quad P_3^2 = P_3$$

and

$$P_1P_2 = P_2P_1 = P_1P_3 = P_3P_1 = P_2P_3 = P_3P_2 = \mathbf{O}.$$

12 Spectral decomposition, Problem 7.

Problem 7: The linear operator $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ is defined as

$$T_A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 5x_1 + 7x_2 + 2x_3 \\ 7x_1 + 5x_2 - 2x_3 \\ 2x_1 - 2x_2 - 4x_3 \end{pmatrix}.$$

a) Show that T_A is diagonalizable; determine a basis for each eigenspace, an invertible matrix B, and a diagonal matrix D such that $A = BDB^{-1}$.

b) Show that A can be represented as $A = \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3$ where

$$\lambda_1,\, \lambda_2,\, \lambda_3 \in \mathbb{R}, \quad P_1 + P_2 + P_3 = I, \quad P_1^2 = P_1, \quad P_2^2 = P_2, \quad P_3^2 = P_3$$

and

$$P_1P_2 = P_2P_1 = P_1P_3 = P_3P_1 = P_2P_3 = P_3P_2 = \mathbf{O}.$$

13 Spectral decomposition, Geometrical illustration, Problem 8.

Problem 8: The linear operator $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ is defined with help of my favorite matrix:

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}.$$

a) We know that T_A is diagonalizable (Video 197 in Part 2), and we know an invertible matrix B, and a diagonal matrix D such that $A = BDB^{-1}$.

b) Show that A can be represented as $A = \lambda_1 P_1 + \lambda_2 P_2$ where

$$\lambda_1,\, \lambda_2 \in \mathbb{R}, \quad P_1 + P_2 = I, \quad P_1^2 = P_1, \quad P_2^2 = P_2, \quad P_1P_2 = P_2P_1 = \mathbf{O}.$$

c) Show a geometrical illustration of the spectral decomposition above.

Extra material: notes with solved Problem 8.

S3 More problem solving; spaces different from $\mathbb{R}^n$

You will learn: work with eigendecomposition of matrices for linear operators on various vector spaces.

14 Eigendecomposition, Problem 1.

Problem 1: The linear operator $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by the following matrix:

$$A = \begin{pmatrix} 2 & 1 & a \\ 1 & 2 & -a \\ a & 0 & 1 \end{pmatrix}.$$

For which values of $a \in \mathbb{R}$ is T_A diagonalizable? For each such a determine a basis consisting entirely of eigenvectors. (We solve this problem only for $a \neq 0$; the solution for $a = 0$ comes in Video 149.)

Extra material: notes with solved Problem 1.

15 Eigendecomposition, Problem 2.

Problem 2: The linear operator $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by the following matrix:

$$A = \begin{pmatrix} a+1 & 1 & a+1 \\ 1 & 2 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

For which values of $a \in \mathbb{R}$ is T_A diagonalizable? For each such a determine a basis consisting entirely of eigenvectors.

Extra material: notes with solved Problem 2.

16 Powers and roots, Problem 3.

Problem 3: Let

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix}.$$

Find a solution to the equation $X^5 = A$.

Extra material: notes with solved Problem 3.

17 Powers and roots, Problem 4.

Problem 4: Let

$$A = \begin{pmatrix} -1 & -1 & -1 \\ 0 & 2 & -2 \\ 0 & -1 & 3 \end{pmatrix}.$$

Find a solution to the equation $X^n = A$ for each positive odd exponent n.

Extra material: notes with solved Problem 4.

18 In the space of polynomials, Problem 5.

Problem 5: $\mathcal{P}_2 = \mathbb{P}_2[t]$ is the space of all the polynomials $p(t)$ of degree $\leqslant 2$. The linear operator $F : \mathcal{P}_2 \to \mathcal{P}_2$ is defined by

$$F(p) = p(t-1) - tp'(t) \qquad \text{for} \qquad p(t) = x_1 + x_2 t + x_3 t^2 \in \mathcal{P}_2.$$

Determine $\mathrm{Im}(F)$, $\mathrm{Ker}(F)$, all the eigenvalues and eigenspaces of F.

Extra material: notes with solved Problem 5.

19 In the space of polynomials, Problem 6.

Problem 6: $\mathcal{P}_3 = \mathbb{P}_3[t]$ is the space of all the polynomials $p(t)$ of degree $\leqslant 3$. The linear operator $F : \mathcal{P}_3 \to \mathcal{P}_3$ is defined by

$$F(p) = p''(t) + (t+1)p'(t) + 2p(t) \qquad \text{for} \qquad p(t) = x_1 + x_2 t + x_3 t^2 + x_4 t^3 \in \mathcal{P}_3.$$

Determine whether F is diagonalizable and, if the answer is yes, determine a basis of $\mathcal{P}_3$ containing only eigenvectors of F.

Extra material: notes with solved Problem 6.

20 In the space of matrices, Problem 7.

Problem 7: $\mathcal{S}_2$ is the space of all the symmetrical 2×2 matrices with real entries. The linear operator $F : \mathcal{S}_2 \to \mathcal{S}_2$ is defined by

$$F(\mathbf{x}) = \begin{pmatrix} -4x_1 + 9x_2 - 6x_3 & -5x_1 + 10x_2 - 6x_3 \\ -5x_1 + 10x_2 - 6x_3 & -5x_1 + 11x_2 - 7x_3 \end{pmatrix} \qquad \text{for} \qquad \mathbf{x} = \begin{pmatrix} x_1 & x_3 \\ x_3 & x_2 \end{pmatrix} \in \mathcal{S}_2.$$

Determine the standard matrix for F and show that F is not diagonalizable.

Extra material: notes with solved Problem 7.

S4 *Intermezzo*: isomorphic vector spaces

You will learn: about certain similarities between different spaces and how to measure them.

21 You wouldn't see the difference...

22 Different spaces with the same structure.

An observation: the neutral element of addition (*additive identity*) maps onto the neutral element of addition under each isomorphism.

Example 1: The following three vector spaces over $\mathbb{R}$ are isomorphic: $\mathcal{P}_2 \cong \mathbb{R}^3 \cong \mathcal{S}_2$.

23 More examples of isomorphic vector spaces.

Example 2: $\mathcal{P}_{n-1} \cong \mathbb{R}^n$. The following three vector spaces over $\mathbb{R}$ are isomorphic: $\mathcal{P}_3 \cong \mathbb{R}^4 \cong \mathcal{M}_{2\times 2}$.

24 A necessary condition for isomorphic vector spaces.

25 A necessary and sufficient condition for isomorphic vector spaces.

26 Why you don't see the difference.

27 Isomorphic spaces: Problem 1.

Problem 1: Compute the dimension of $\mathcal{S}_n$ (symmetric $n \times n$ matrices with real entries) for each natural number $n \geqslant 1$.

28 Isomorphic spaces: Problem 2.

Problem 2: Which of the following spaces are isomorphic?:

a) $\mathbb{R}^2$ and $\mathbb{R}^4$,

b) $\mathcal{P}_5$ and $\mathbb{R}^5$,

c) $\mathcal{M}_{2\times 3}$ and $\mathbb{R}^6$

d) $\mathcal{P}_5$ and $\mathcal{M}_{2\times 3}$,

e) $\mathcal{M}_{2\times k}$ and $\mathbb{C}^k$,

f) $\mathbb{R}^2$ and the xy-plane in $\mathbb{R}^3$.

Extra material: notes with solution to Problem 2.

29 Isomorphic spaces: Problem 3.

Problem 3: Which of the following maps are isomorphisms?:

a) $f : \mathcal{M}_{2\times 2} \to \mathbb{R}$ given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto ad - bc$,

b) $f : \mathcal{M}_{2\times 2} \to \mathbb{R}^4$ given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a+b+c+d \\ a+b+c \\ a+b \\ a \end{pmatrix}$,

c) $f : \mathcal{M}_{2\times 2} \to \mathcal{P}_3$ given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto c + (d+c)x + (b+a)x^2 + ax^3$,

d) $f : \mathcal{M}_{2\times 2} \to \mathcal{P}_3$ given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto c + (d+c)x + (b+a+1)x^2 + ax^3$.

Extra material: notes with solution to Problem 3.

30 Vector spaces, fields, rings; ring homomorphisms and isomorphisms.

An observation: ring homomorphism $f : \mathbb{Z} \to \mathcal{M}_{2\times 2}$ defined by $x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$ does not map the multiplicative identity of $\mathbb{Z}$ to the multiplicative identity of $\mathcal{M}_{2\times 2}$.

31 Vector spaces, fields, rings, Problem 4.

Problem 4: Some examples of rings:

a) each field, for example $\mathbb{R}$ and $\mathbb{C}$,

b) $\mathbb{R}[x]$ and $\mathcal{M}_{2\times 2}$ (**not** fields); **not** $\mathcal{P}_3$,

c) diagonal $n \times n$ matrices,

d) upper/lower-triangular $n \times n$ matrices.

Extra material: notes with solution to Problem 4.

32 Vector spaces, fields, rings, Problem 5.

Problem 5: Motivate the following statements:

a) Each field is a vector space *over itself*,

b) $\mathbb{R}$ is a vector space over $\mathbb{R}$, dimension 1,

c) $\mathbb{C}$ is a vector space over $\mathbb{C}$, dimension 1,

d) Fields $\mathbb{R}$ and $\mathbb{C}$ are not isomorphic,

e) $\mathbb{C}$ is a vector space over $\mathbb{R}$, dimension 2,

f) $\mathbb{R}$ is not a vector space over $\mathbb{C}$,

g) $\mathbb{R}^2$ and $\mathbb{C}$ are isomorphic vector spaces over $\mathbb{R}$, but only $\mathbb{C}$ has a field structure.

Extra material: notes with solution to Problem 5.

S5 Recurrence relations, dynamical systems, Markov matrices

You will learn: more exciting applications of eigenvalues and diagonalization.

33 Continuous versus discrete.

34 Two famous examples of recurrence.

Examples: Factorial; Geometric sequence.

Extra material: notes with an induction proof showing that the recurrence $x_0 = 1,\ x_{n+1} = 2x_n$ describes the geometric sequence $x_n = 2^n$.

35 Linear discrete dynamical systems.

36 Systems of difference equations, Problem 1.

Problem 1: Let X_n be a sequence of vectors satisfying the difference equation

$$X_{n+1} = AX_n \text{ for } n = 0, 1, 2, \ldots, \text{ where } A = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \text{ and } X_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Determine X_n and examine $\lim_{n\to\infty} X_n$. Show two different methods of solving this problem.

Extra material: notes with solved Problem 1.

37 Systems of difference equations, Problem 2.

Problem 2: Let $u_n,\ v_n$ be two sequences satisfying following recursive relation:

$$\begin{cases} u_{n+1} = u_n + 3v_n \\ v_{n+1} = 3v_n + u_n \end{cases}, \quad \text{for } n = 0, 1, 2, \ldots \quad \text{and } u_0 = 2,\ v_0 = 1.$$

Determine both sequences. Show two different methods of solving this problem.

Extra material: notes with solved Problem 2.

38 Systems of difference equations, Problem 3.

Problem 3: Let $u_n,\ v_n,\ w_n$ be three sequences satisfying following recursive relation:

$$\begin{cases} u_{n+1} = 2u_n - 2v_n + w_n \\ v_{n+1} = 2u_n + 4v_n - 2w_n \\ w_{n+1} = 4u_n + 2v_n - w_n \end{cases}, \quad \text{for } n = 0, 1, 2, \ldots \quad \text{and } u_0 = 1,\ v_0 = 0,\ w_0 = 1.$$

Determine these sequences.

Extra material: notes with solved Problem 3.

39 Higher order difference equations, Problem 4.

Problem 4: Rewrite the following (third order) difference equation as a first-order system:

$$u_{n+3} - 2u_{n+2} - 5u_{n+1} + 6u_n = 0.$$

Extra material: notes with solved Problem 4.

40 Higher order difference equations, Problem 5.

Problem 5: Rewrite the following difference equations as first-order systems:

a) $u_{n+2} = 5u_{n+1} - 6u_n$,

b) $u_{n+2} = -4u_{n+1} + 3u_n$,

c) $u_{n+3} = 5u_{n+2} + 6u_{n+1} + 7u_n$,

d) $u_{n+3} = 3u_{n+2} - 12u_n$.

Write corresponding matrix equations.

Extra material: notes with solved Problem 5.

41 Higher order difference equations, Problem 6.

Problem 6: Solve the following difference equation with initial conditions:

$$u_{n+2} = 8u_{n+1} - 12u_n, \ \ u_0 = 5,\ u_1 = 4.$$

Extra material: notes with solved Problem 6.

42 Markov matrices.

Examples: Eight matrices: Markov or not?

43 Each Markov matrix has eigenvalue 1.

44 Steady-state vector (equilibrium vector), Problem 7.

Problem 7: Find a steady-state vector for Markov matrix $\begin{pmatrix} .6 & .3 \\ .4 & .7 \end{pmatrix}$. Illustrate graphically long-time behaviour of the system in three different processes: one with the initial state $X_0 = (50, 20)$, one with the initial state $X_0 = (10, 30)$, and one with the initial state $X_0 = (100, 10)$.

Extra material: notes with solved Problem 7.

45 Markov matrices, Problem 8, Restaurant.

Problem 8: A college with 3000 students has two restaurants: U and V. All the students have lunch there every day. They change restaurant from time to time, according to the following pattern:

- approximately 20% of the students having lunch in restaurant U one day change to V the next day,
- approximately 10% of the students having lunch in restaurant V one day change to U the next day.

We know that today (day 0) there are 1500 students in restaurant U and 1500 students in restaurant V. Denote by u_n and v_n the number of students having lunch in the restaurants U and V (respectively) on day n.

a) Determine u_n and v_n for each natural number n.

b) What is the long-term prediction for the lunch situation? (Determine the equilibrium vector / steady-state vector.)

Extra material: notes with solved Problem 8.

46 Markov matrices, Problem 9, Migration.

Problem 9: Each year about 5% of the population of Uppsala moves to Vattholma (and 95% stays in Uppsala), while 3% of the population of Vattholma moves to Uppsala (and 97% remains in Vattholma). Denote by $X_n = (u_n, v_n)$ the population distribution vector (in percent) under year n. What is the long-term prediction for the population distribution if under year 0 (now) 60% of the entire population of both cities lives in Uppsala and 40% in Vattholma?

Extra material: notes with solved Problem 9.

47 Markov matrices, Problem 10, Election.

Problem 10: Congressional election is held every second year, and there are three ways to vote: D (Democratic), R (Republican) and I (Independent). Assume that the election outcomes $X_n = (d_n, r_n, i_n)^T$ in some district with constant number of voters form a Markov chain X_n with transition matrix P, i.e. $X_{n+1} = PX_n$:

$$\begin{bmatrix} d_{n+1} \\ r_{n+1} \\ i_{n+1} \end{bmatrix} = \begin{bmatrix} .70 & .10 & .30 \\ .20 & .80 & .30 \\ .10 & .10 & .40 \end{bmatrix} \begin{bmatrix} d_n \\ r_n \\ i_n \end{bmatrix}$$

The outcome vector $X_n = (d_n, r_n, i_n)^T$ contains percentage of voting R, D or I during the election n, X_{n+1} describes the percentages during next elections (two years later).

a) Draw a diagram illustrating the situation described by this Markov process.

b) Suppose that outcome of one election is given by $X_0 = (.55, .40, .05)^T$. Determine the likely outcome of the next election and the likely outcome of the election after that.

c) Determine the steady-state vector for this process.

d) What percentage of the voters are likely to vote R in some election many years from now, assuming that the election outcomes form a Markov chain?

Extra material: notes with solved Problem 10.

48 Dynamical systems, Problem 11.

Problem 11: The matrix A has eigenvalues 1, $\frac{2}{3}$, and $\frac{1}{3}$, with corresponding eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$:

$$A = \frac{1}{9}\begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 5 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}.$$

Find the general solution of the equation $X_{n+1} = AX_n$ if $X_0 = (1, 11, -2)^T$. Compute $\lim_{n\to\infty} X_n$.

Extra material: notes with solved Problem 11.

49 Where to read more on this topic?

S6 Solving systems of linear ODE, and solving higher order ODE

You will learn: solve systems of linear ODE and linear ODE of higher order with help of diagonalization.

50 What is an ODE and what kinds of ODE we are going to deal with.

51 Solutions to first order linear ODE with constant coefficients.

52 Systems of first order linear ODE with constant coefficients.

53 A very simple example.

Example 1: Determine all the solutions to the system of ODE:

$$\begin{cases} x_1' = 2x_1 \\ x_2' = 2x_2 \end{cases}$$

54 The method.

Example 2: Determine all the solutions to the system of ODE:

$$\begin{cases} y_1'(t) = 2y_1(t), & y_1(0) = 1 \\ y_2'(t) = -3y_2(t), & y_2(0) = -2 \\ y_3'(t) = 0, & y_3(0) = 3. \end{cases}$$

Example 3: Determine all the solutions to the system of ODE:

$$\begin{cases} y_1' = y_2 \\ y_2' = y_1 \end{cases}$$

Extra material: notes with solved Example 3.

55 System of ODE, Problem 1.

Problem 1: Solve the system of ODE $\mathbf{y}' = A\mathbf{y}$ satisfying the initial condition $\mathbf{y}(0) = (1, -2, 3)^T$, where:

$$A = \begin{bmatrix} 2 & 5 & -5 \\ 2 & 8 & -8 \\ 2 & 11 & -11 \end{bmatrix}.$$

Hint: We diagonalized this matrix in Video 203 in Part 2; eigenvalues in Video 186.

56 System of ODE, Problem 2.

Problem 2: Solve the system of ODE:

$$\begin{cases} y_1'(t) = 3y_1(t) + 2y_2(t) - 6y_3(t) \\ y_2'(t) = 2y_1(t) + 3y_2(t) - 6y_3(t) \\ y_3'(t) = y_1(t) + 2y_2(t) - 4y_3(t) \end{cases}$$

with initial conditions $y_1(0) = 1,\ y_2(0) = 2,\ y_3(0) = 3$.

57 System of ODE, Problem 3.

Problem 3: Determine the solution to the system of ODE:

$$\begin{cases} y_1' = y_1 + y_2 \\ y_2' = 4y_1 - 2y_2 \end{cases}$$

which satisfies the initial condition $y_1(0) = 1,\ y_2(0) = 6$.

Extra material: notes with solved Problem 3.

58 How to deal with higher order linear ODE?

Problem 4: Find general solution to the ODE $y''' + 2y'' - y' - 2y = 0$.

Extra material: notes with solved Problem 4.

Extra material: an article with a supplement to Video 81 in Part 2; inhomogenous second order linear ODE with constant coefficients.

- ⋆ **Example 0**: Determine the general solution to $y'' + 3y' + 2y = 0$.
- ⋆ **Example 1**: Determine the general solution to $y'' + 3y' + 2y = t^2$.
- ⋆ **Example 2**: Determine the general solution to $y'' + 3y' + 2y = e^{2t}$.

59 Another way of looking at the same problem.

Problem 5: Determine the solution to the system of ODE:

$$\begin{cases} y_1' = y_1 + 2y_2 \\ y_2' = 3y_1 + 2y_2 \end{cases}$$

satisfying the initial conditions $y_1(0) = 0,\ y_2(0) = -4$.

Extra material: notes with solved Problem 5.

1. Extra material: an article with more theory and one example.
   - ⋆ **Section 1**: Solution spaces and their generators.
   - ⋆ **Section 2**: Matrix exponential and how to use it.
   - ⋆ **Example**: Determine all the solutions to the system of ODE:

$$\begin{cases} x_1' = -x_2 \\ x_2' = x_1 \end{cases}$$

2. Extra material: an article with more solved problems on eigenvalues and ODE.
   - ⋆ **Extra problem 1**: Solve the following system of differential equations: $\begin{cases} y_1' = y_1 + 4y_2 \\ y_2' = y_1 + y_2 \end{cases}$ where $y_1(0) = 1$ and $y_2(0) = 2$.
   - ⋆ **Extra problem 2**: Solve the initial value problem $\begin{cases} y''' + 4y'' + y' - 6y = 0 \\ y(0) = y'(0) = y''(0) = 1. \end{cases}$
   - ⋆ **Extra problem 3**: Solve the system of linear ODE

$$\begin{cases} x_1' = x_1 + 2x_2 \\ x_2' = 4x_1 + 3x_2 \end{cases}$$

   - ⋆ **Extra problem 4**: Solve the initial value problem $\begin{cases} y''' - 3y'' + 2y' = 0 \\ y(0) = 1,\ y'(0) = 3,\ y''(0) = 5. \end{cases}$

C2 Inner product spaces

S7 Inner product as a generalization of dot product

You will learn: about other products with similar properties as dot product, and how they can look in different vector spaces.

60 Between concrete and abstract.

61 Dot product in Part 1.

62 Dot product and orthogonality in Part 2.

63 From $\mathbb{R}^2$ to inner product spaces.

64 Inner product spaces.

Example: Inner products are examples of bi-linear forms.

Extra material: notes with the explanation of bi-linearity of inner products.

65 Euclidean n-space.

Example: $\mathbb{R}^n$ with dot product is an inner product space.

Extra material: proof that dot product satisfies IP1–IP4.

66 A very important remark about notation.

67 Inner and outer products.

68 Weighted Euclidean inner product, Problem 1.

Problem 1: Let $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ be vectors in $\mathbb{R}^2$. Show that the weighted Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2$ satisfies the four inner product axioms.

Extra material: notes with the solution of Problem 1.

69 Remember transposed matrices?

70 Positive definite matrices.

Example: If A is a square invertible matrix, AA^T is positive definite.

Extra material: notes with the proof.

71 Quadratic forms and how to read them.

Example: Compute $\varphi(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ where:

$$\text{a)} \quad A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \qquad \text{b)} \quad A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

Extra material: notes with the computations.

72 Matrix inner products on $\mathbb{R}^n$, Problem 2.

Problem 2: Show that matrices defined in the previous video define an inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T A\mathbf{y}$.

Extra material: notes with the solution of Problem 2.

73 Gram matrix, Problem 3.

Problem 3: Compute Gram matrix G for the matrix inner product defined by $C = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$.

Extra material: notes about Gram matrix from Video 72; Gram matrix for Euclidean inner product; Gram matrix for weighted Euclidean inner product; solution to Example above.

74 Gram matrix, Problem 4.

Problem 4: We introduce the inner product in $\mathbb{R}^2$ by $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T G\mathbf{y}$ where $G = \frac{1}{5}\begin{pmatrix} 17 & -6 \\ -6 & 8 \end{pmatrix}$. Find a matrix C which defines the same inner product as a matrix inner product. (Hint: find a **symmetric** C.)

75 Inner product in the space of continuous functions.

Extra material: notes with the proof that it really is inner product space.

76 Gram matrix for an inner product in the space $\mathcal{P}_n$ of polynomials.

77 Two inner products on the space of polynomials $\mathcal{P}_n$.

Extra material: notes with the proof of IP4 for the evaluation inner product.

78 The evaluation inner products on $\mathcal{P}_2$, Problem 5.

Problem 5: Let $\mathcal{P}_2$ have the evaluation inner product at the points

$$x_0 = -2, \quad x_1 = 0, \quad x_2 = 2.$$

Compute $\langle \mathbf{p}, \mathbf{q} \rangle$ for the polynomials $\mathbf{p} = p(x) = x^2$ and $\mathbf{q} = q(x) = 1 + x$. What is the value of the standard inner product for these polynomials? Compute also the inner product from Video 76 for these polynomials, with two methods: by computing by Gram matrix, and by computing the integral.

Extra material: notes with the solution of Problem 5.

79 Inner product in the space of $m \times n$ matrices.

80 Inner product in the space of square matrices.

Example: Compute the inner product for matrices $A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

81 Inner product in the space of matrices, Problem 6.

Problem 6: Compute two different inner products (defined in Video 79 and in Video 80) for matrices

$$A = \begin{pmatrix} 2 & 1 \\ -3 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 5 & -1 \\ 0 & 4 \end{pmatrix}.$$

Extra material: notes with the solution of Problem 6.

82 Frobenius inner product; Hadamard product of matrices.

Completing information to videos 79–81: Both inner products (from Videos 79 and 80) are the same, and Frobenius inner product is defined for non-square matrices, too.

S8 Norm, distance, angles, and orthogonality in inner product spaces

You will learn: how to define geometric concepts in non-geometric setups.

83 Norm in inner product spaces.

Extra material: notes with the proof of properties of norm.

84 Weird geometry in the Euclidean space with weighted inner product.

Example: Draw the circle with radius 1 and centred at the origin in the metric induced by the weighted inner product $\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{9}u_1v_1 + \frac{1}{4}u_2v_2$.

85 Frobenius norm of matrices, Problem 1.

Problem 1: Compute Frobenius norm of the matrices

$$A = \begin{pmatrix} 2 & 1 \\ -3 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 5 & -1 \\ 0 & 4 \end{pmatrix}.$$

Extra material: notes with solved Problem 1.

86 Norm in the space of functions, Problem 2.

Problem 2: Compute the norm of $\mathbf{p} = p(x) = x$ and of $\mathbf{q} = q(x) = 2x^2$ on interval $[0, 1]$.

Extra material: notes with solved Problem 2.

87 Distance in inner product spaces.

Extra material: notes with the proof of properties of distance.

88 Frobenius distance between matrices, Problem 3.

Problem 1: Compute Frobenius distance between the matrices

$$A = \begin{pmatrix} 2 & 1 \\ -3 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 5 & -1 \\ 0 & 4 \end{pmatrix}.$$

Extra material: notes with solved Problem 3.

89 Distance in the space of functions, Problem 4.

Problem 4: Compute the distance between $\mathbf{p} = p(x) = x$ and $\mathbf{q} = q(x) = x^2$, and between $\mathbf{q} = q(x) = x^2$ and $\mathbf{r} = r(x) = x^3$ on interval $[0, 1]$.

90 First step to defining abstract angles.

91 Cauchy–Schwarz inequality, proof 1.

Extra material: notes with the proof of Cauchy–Schwarz inequality.

92 Cauchy–Schwarz inequality, proof 2.

Extra material: notes with another proof of Cauchy–Schwarz inequality.

93 Cauchy–Schwarz inequality in the space of continuous functions.

94 Angles in inner product spaces.

95 More weird geometry: Angles in inner product spaces, Problem 5.

Problem 5: Compute the angle between the polynomials $p(x) = x$ and $q(x) = 1$ in the inner product space of all the polynomials on the interval $[0, 1]$, with inner product $\langle \mathbf{p}, \mathbf{q} \rangle = \int_0^1 p(x)q(x)\, dx$.

96 Angles in inner product spaces, Problem 6.

Problem 6: Compute the angle between the polynomials $p(x) = x$ and $q(x) = 2x^2$ in the inner product space of all the polynomials on the interval $[0, 1]$, with inner product $\langle \mathbf{p}, \mathbf{q} \rangle = \int_0^1 p(x)q(x)\, dx$. (The norms were computed in Video 86.)

97 Orthogonality in inner product spaces.

98 Orthogonality in inner product spaces depends on inner product.

Example: Polynomials $\mathbf{p} = p(x) = x^2$ and $\mathbf{q} = q(x) = 1 + x$ form different angles in different IP spaces:

a) $\mathcal{P}_2$ with the evaluation inner product at the points $x_0 = -2,\ x_1 = 0,\ x_2 = 2$,

b) $\mathcal{P}_2$ with standard inner product,

c) $\mathcal{P}_2$ with the integral inner product defined in V. 75 & 76 (with the factor $\frac{1}{2}$ and on the interval $[-1, 1]$).

All the necessary computations were made in Video 78.

99 Orthogonality in inner product spaces, Problem 7.

Problem 7: Determine whether the angle between matrices A and B in the IP space with Frobenius IP is acute, right, or obtuse:

$$A = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 3 \\ 5 & -5 \end{pmatrix}.$$

Extra material: notes with solved Problem 7.

100 What is triangle inequality?

101 Triangle inequality in inner product spaces.

Extra material: notes with the proof of triangle inequality in inner product spaces.

Example: Illustrate the triangle inequality in the IP space of matrices with Frobenius IP on the example from Video 88:

$$A = \begin{pmatrix} 2 & 1 \\ -3 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 5 & -1 \\ 0 & 4 \end{pmatrix}.$$

102 Generalized Theorem of Pythagoras.

Extra material: notes with the proof of Theorem of Pythagoras in inner product spaces.

103 Generalized Theorem of Pythagoras, Problem 8.

Problem 8: Illustrate Pythagorean theorem in the IP space of matrices with Frobenius IP on A and B:

$$A = \begin{pmatrix} 3 & 2 & 5 \\ 0 & 1 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & -5 & 2 \\ 7 & 0 & 3 \end{pmatrix}.$$

.

Extra material: notes with solved Problem 8.

104 Generalized Theorem of Pythagoras, Problem 9.

Problem 9: Illustrate Pythagorean theorem in $\mathcal{P}_2$ with standard IP, using (orthogonal, by computations from Video 98) polynomials $\mathbf{p} = p(x) = x^2$ and $\mathbf{q} = q(x) = 1 + x$.

Extra material: notes with solved Problem 9.

105 Generalized Theorem of Pythagoras, Problem 10.

Problem 10: Show that the same polynomials as in V104 ($\mathbf{p} = p(x) = x^2$ and $\mathbf{q} = q(x) = 1 + x$) are **not** orthogonal with respect to the integral IP in $\mathcal{P}_2$ (defined in V76). Show that $\|\mathbf{p} + \mathbf{q}\|^2 \neq \|\mathbf{p}\|^2 + \|\mathbf{q}\|^2$.

Extra material: notes with solved Problem 10.

S9 Projections and Gram–Schmidt process in various inner product spaces

You will learn: apply Gram–Schmidt process in inner product spaces different from $\mathbb{R}^n$ (which were already covered in Part 2); work with projections on subspaces.

106 Different but still awesome!

107 ON bases in IP spaces.

108 Why does normalizing work in the same way in all IP spaces?

109 Orthonormal sets of continuous functions, Problem 1.

Problem 1: Show that the set of functions $\{f, g, h\}$, where $f(x) = 1$, $g(x) = \sin x$, $h(x) = \cos x$, is orthogonal in the inner product space of all continuous functions on the interval $[-\pi, \pi]$, i.e. $C[-\pi, \pi]$, with inner product $\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx$. Normalize the functions.

Extra material: notes with solved Problem 1.

Extra material: an article about computing some trigonometric integrals.

110 Orthogonal complements, Problem 2.

Problem 2: Find a first degree polynomial $q(x)$ that is orthogonal to the polynomial $p(x) = 1 - x$ in the IP space of all the polynomials on the interval $[0, 1]$ with inner product $\langle \mathbf{p}, \mathbf{q} \rangle = \int_0^1 p(x)q(x)\,dx$.

Extra material: notes with solved Problem 2.

111 Orthogonal sets are linearly independent, Problem 3.

Problem 3: Show that the polynomials $p(x) = 1 - x$ and $q(x) = 1 - 3x$ from V110 are linearly independent.

Extra material: notes with solved Problem 3.

112 Coordinates in orthogonal bases in IP spaces.

113 Projections and orthogonal decomposition in IP spaces.

Extra material: notes with the proof of the decomposition formula in inner product spaces.

114 Orthogonal projections on subspaces of an IP space, Problem 4.

Problem 4: The IP space $\mathcal{P}_3$ of all the polynomials of degree $\leqslant 3$ with inner product $\langle \mathbf{p}, \mathbf{q} \rangle = \int_0^1 p(x)q(x)\,dx$ has $\mathcal{P}_2$ as subspace. The following polynomials form an ON-basis in $\mathcal{P}_2$:

$$p_1(x) = 1, \quad p_2(x) = \sqrt{3}(2x - 1), \quad p_3(x) = \sqrt{5}(6x^2 - 6x + 1).$$

a) Compute the orthogonal projection of the polynomial x^3 on the subspace $\mathcal{P}_2$.

b) Find a polynomial $\mathbf{q}$ of degree 3, s.t. $\int_0^1 p(x)q(x)\,dx = 0$ for all $\mathbf{p} \in \mathcal{P}_2$ (and, later, the orthogonal complement for $\mathcal{P}_2$ in $\mathcal{P}_3$).

Extra material: notes with solved Problem 4.

115 Orthogonal projections on subspaces of an IP space, Problem 5.

Problem 5: Consider the IP space $\mathcal{P}$ of all the polynomials, with inner product $\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^1 p(x)q(x)\,dx$. Polynomials $\mathbf{p} = p(x) = x$ and $\mathbf{q} = q(x) = 3x^2 - 1$ belong to this space.

a) Compute the norms $\|\mathbf{p}\|$ and $\|\mathbf{q}\|$.

b) Show that the polynomials $\mathbf{p}$ and $\mathbf{q}$ are orthogonal.

c) Compute the orthogonal projection of $\mathbf{f} = f(x) = x + 1$ on the subspace of $\mathcal{P}$ generated by $\mathbf{p}$ and $\mathbf{q}$.

Extra material: notes with solved Problem 5.

116 Gram–Schmidt in IP spaces.

117 Gram–Schmidt in IP spaces, Problem 6: Legendre polynomials.

Problem 6: Consider the vector space $\mathcal{P}_2$ of all the polynomials with degree $\leqslant 2$, with inner product $\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^1 p(x)q(x)\,dx$. Apply the Gram–Schmidt process to transform the standard basis $\{1, x, x^2\}$ for $\mathcal{P}_2$ into an orthogonal basis $\{\phi_1(x), \phi_2(x), \phi_3(x)\}$. The polynomials in this new basis are called **Legendre polynomials**.

Extra material: notes with solved Problem 6.

118 Gram–Schmidt in IP spaces, Problem 7.

Problem 7: Consider the vector space $\mathcal{P}_2$ of all the polynomials with degree $\leqslant 2$, with inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle = p(0)q(0) + p(1)q(1) + p(2)q(2).$$

Apply the Gram–Schmidt process to transform the standard basis $\{1, x, x^2\}$ for $\mathcal{P}_2$ into an ON-basis.

Extra material: notes with solved Problem 7.

119 Easy computations of IP in ON bases, Problem 8.

Problem 8: Consider the IP space $\mathcal{P}$ of all the polynomials, with inner product $\langle \mathbf{p}, \mathbf{q} \rangle = \int_0^1 p(x)q(x)\,dx$.

$$\text{Polynomials} \quad q_1(x) = 1, \quad q_2(x) = \sqrt{3}(2x-1), \quad \text{and} \quad q_3(x) = \sqrt{5}(6x^2 - 6x + 1)$$

are orthonormal. Determine the angle β between the polynomials $\mathbf{v} = v(x)$ and $\mathbf{w} = w(x)$ defined as

$$v(x) = (1 - \sqrt{3}) + 2\sqrt{3}x, \qquad w(x) = (-\sqrt{3} + \sqrt{5}) + (2\sqrt{3} - 6\sqrt{5})x + 6\sqrt{5}x^2.$$

Extra material: notes with solved Problem 8.

Extra material: one more solved problem.

⋆ **Extra problem**: Let $M_{2\times 2}(\mathbb{R})$ be equipped with the inner product

$$\left\langle \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}, \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} \right\rangle = x_1 y_1 + \frac{x_2 y_2}{2} + \frac{x_3 y_3}{3} + \frac{x_4 y_4}{4}.$$

a) Find an ON-basis of $M_{2\times 2}(\mathbb{R})$ (you have to prove that your basis is ON).

b) Let $U \subset M_{2\times 2}(\mathbb{R})$ be the subspace of all 2×2 matrices such that their diagonal elements are zero. In other words

$$U = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix};\ a = 0,\ d = 0 \right\}.$$

Find $\operatorname{proj}_U \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$.

c) Find a non-zero vector in $U^\perp$.

S10 Min-max problems, best approximations, and least squares

You will learn: solve some simple min-max problems with help of Cauchy–Schwarz inequality, find the shortest distance to subspaces in IP spaces, handle inconsistent systems of linear equations.

120 In this section.

121 Min-max, Problem 1.

Problem 1: Given $x^2 + y^2 \leqslant 16$, what is the maximum value for $3x + 4y$? Hint: Cauchy–Schwarz inequality.

Extra material: notes with solved Problem 1.

122 Min-max, Problem 2.

Problem 2: Given $x^2 + y^2 + z^2 = 1$, what is the maximum value for $x + 4y + 8z$? Hint: Cauchy–Schwarz inequality.

Extra material: notes with solved Problem 2.

123 Min-max, Problem 3.

Problem 3: Given $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$, what is the maximum value for $x_1 - x_2 - 3x_3 + 5x_4$? Hint: Cauchy–Schwarz inequality.

Extra material: notes with solved Problem 3.

124 Min-max, Problem 4.

Problem 4: Given $x^2 + y^2 + 4z^2 = 4$, what is the minimum and maximum value for $x - y + z$? Hint: Cauchy–Schwarz inequality.

Extra material: notes with solved Problem 4.

125 Min-max, Problem 5.

Problem 5: Determine the max and the min values of $2x_1 + 3x_2 - 2x_3$ with constraint $x_1^2 + x_2^2 + (x_2 - x_3)^2 = 1$ (i.e., on an ellipsoid), using Cauchy–Schwarz inequality for the following inner product on $\mathbb{R}^3$:
$\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 + x_2y_2 + (x_2 - x_3)(y_2 - y_3)$.

Extra material: notes with solved Problem 5.

126 Another look at orthogonal projections as matrix transformations.

127 Orthogonal projections, Problem 6.

Problem 6: Let $M = \{(x, y, z) \in \mathbb{E}^3;\ x - y - z = 0,\ x + y + 2z = 0\}$. Operator $T : \mathbb{E}^3 \to \mathbb{E}^3$ is the orthogonal projection on M. Compute the matrix P of T, and the matrix Q of the orthogonal projection on $M^\perp$.

128 Orthogonal projections, Problem 7.

Problem 7: Given the following subspace of $\mathbb{E}^4$: $M = \text{span}\{(1, 1, 1, 1)^t,\ (1, 1, 0, -2)^t\}$. Determine the standard matrix of the orthogonal projection on M.

129 Shortest distance from a subspace.

Extra material: notes with a proof of **Best Approximation Theorem**.

130 Shortest distance, Problem 8.

Problem 8: Continuation of Problem 7: Given the following subspace of $\mathbb{E}^4$: $M = \text{span}\{(1, 1, 1, 1)^t,\ (1, 1, 0, -2)^t\}$. Determine the standard matrix Q of the orthogonal projection on $M^\perp$. Decompose $\mathbf{w} = (1, 2, 3, 0)^t$ as $\mathbf{w} = \mathbf{u} + \mathbf{v}$ where $\mathbf{u} \in M$ and $\mathbf{v} \in M^\perp$. Determine the shortest distance from $\mathbf{w}$ to M.

131 Shortest distance, Problem 9.

Problem 9: Given a subspace M of $\mathbb{E}^4$: $M = \{(x_1, x_2, x_3, x_4) \in \mathbb{E}^4;\ 2x_1 + x_2 - x_3 = 0,\ x_1 + x_2 + x_4 = 0\}$. Determine ON-bases in M and $M^\perp$. Decompose $\mathbf{w} = (1, 0, 0, 0)^t$ as $\mathbf{w} = \mathbf{u} + \mathbf{v}$ where $\mathbf{u} \in M$ and $\mathbf{v} \in M^\perp$. Determine the shortest distance from $\mathbf{w}$ to M and from $\mathbf{w}$ to $M^\perp$.

132 Shortest distance, Problem 10.

Continuation from Video 118. Compute the shortest distance from the polynomial x^2 to the subspace $\mathcal{P}_1 \subset \mathcal{P}_2$ containing polynomials of degree 1 or 0.

133 Solvability of systems of equations in terms of the column space.

134 Least squares solution and residual vector.

135 Four fundamental matrix spaces and the normal equation. Derivation of two methods for finding least squares solutions.

136 Least squares, Problem 11, by normal equation.

Problem 11: Find a least squares solution to the inconsistent system $A\mathbf{x} = \mathbf{b}$ for

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$$

using the method by the normal equation.

Extra material: notes with solved Problem 11.

137 Least squares, Problem 11, by projection.

Problem 11: Find a least squares solution to the inconsistent system $A\mathbf{x} = \mathbf{b}$ for

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$$

using the method by projection on the column space.

Extra material: notes with solved Problem 11.

138 Least squares straight line fit, Problem 12.

Problem 12: Find the least square straight line fit to the four points: $(0, 2)$, $(1, 1)$, $(2, 1)$, and $(3, 4)$.

Extra material: notes with solved Problem 12.

139 Least squares, fitting a quadratic curve to data, Problem 13.

Problem 13: Fit a quadratic curve to the four points: $(0, 2)$, $(1, 1)$, $(2, 1)$, and $(3, 4)$.

Extra material: notes with solved Problem 13.

C3 Symmetric matrices and quadratic forms

S11 Diagonalization of symmetric matrices

You will learn: about various nice properties of symmetric matrices, and about orthogonal diagonalization.

140 The link between symmetric matrices and quadratic forms, Problem 1.

Problem 1: Write down the symmetric matrices corresponding to the following quadratic forms:

a) $\varphi(x, y) = x^2 - 2xy + y^2$

b) $\varphi(x, y, z) = x^2 + 2y^2 - 3z^2 - 2xy + 6xz$

c) $\varphi(x_1, x_2, x_3, x_4) = 5x_1^2 - 7x_3^2 + 9x_4^2 - 5x_1x_2 + 6x_1x_4 - 7x_2x_3 + 8x_3x_4$.

Extra material: notes with solved Problem 1.

141 Some properties of symmetric matrices.

142 Eigenvectors corresponding to distinct eigenvalues for a symmetric matrix are orthogonal.

143 Complex numbers: a brief repetition.

144 Eigenvalues for a (real) symmetric matrix are real.

Example: Matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ with real entries has two complex eigenvalues and two eigenvectors in $\mathbb{C}^2$.

Extra material: notes with the proof that all (real) symmetric $n \times n$ matrices have n real eigenvalues.

145 Orthogonal diagonalization.

146 If a matrix is orthogonally diagonalizable, it is symmetric.

147 The Spectral Theorem: Each symmetric matrix is orthogonally diagonalizable.

Extra material: notes with a proof of Lemma 2.

148 Orthogonal diagonalization: how to do it.

149 Orthogonal diagonalization, Problem 2.

Problem 2: The linear operator $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by the following matrix:

$$A = \begin{pmatrix} 2 & 1 & a \\ 1 & 2 & -a \\ a & 0 & 1 \end{pmatrix}.$$

For which values of $a \in \mathbb{R}$ is T_A orthogonally diagonalizable? Perform the diagonalization for these a. (We solved this problem for diagonalizable—but not orthogonally diagonalizable—matrices in Video 14.)

150 Spectral decomposition for symmetric matrices, Problem 3. Three ways of looking at linear transformations defined by symmetric matrices.

Problem 3: Perform and illustrate geometrically spectral decomposition of the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$.

Make a geometrical illustration (in three different ways) for $A\mathbf{x}$ with $\mathbf{x} = (1,1)^T$.

151 Orthogonal diagonalization, Problem 4.

Problem 4: Is it possible to construct a symmetric 2×2 matrix which has:

a) eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 2$, and eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$;

b) eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 2$, and eigenvectors $\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$;

c) one eigenvalue $\lambda_1 = 5$ with algebraic multiplicity 2, and eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$;

d) one eigenvalue $\lambda_1 = 5$ with algebraic multiplicity 2, and eigenspace $E_{\lambda_1} = \text{span}\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$.

Extra material: notes with solved Problem 4.

152 Orthogonal diagonalization, Problem 5.

Problem 5: Determine a symmetric 2×2 matrix which has one eigenvalue $\lambda_1 = 1$ with corresponding eigenvector $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and the second eigenvalue $\lambda_2 = 2$.

Extra material: notes with solved Problem 5.

153 Orthogonal diagonalization, Problem 6.

Problem 6: Is it possible to construct a 3×3 symmetric matrix with one eigenvalue $\lambda_1 = 3$, and the second one, double $\lambda_{2,3} = 2$ (i.e., with algebraic multiplicity 2), and the corresponding eigenspaces

$$E_{\lambda_1=3} = \text{span}\{\begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}\} \qquad \text{and} \qquad E_{\lambda=2} = \text{span}\{\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}\}.$$

Extra material: notes with solved Problem 6.

154 Orthogonal diagonalization, Problem 7.

Problem 7: Is it possible to construct a 3×3 symmetric matrix with one eigenvalue $\lambda_1 = 3$, and the second one, double $\lambda_{2,3} = 2$ (i.e., with algebraic multiplicity 2), and the corresponding eigenspaces

$$E_{\lambda_1=3} = \text{span}\{\begin{bmatrix}-2\\0\\2\end{bmatrix}\} \qquad \text{and} \qquad E_{\lambda=2} = \text{span}\{\begin{bmatrix}-1\\2\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\}.$$

Extra material: notes with solved Problem 7.

155 Spectral decomposition, Problem 8.

Problem 8: Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be defined as $F(\mathbf{x}) = (4x - 4y + 2z,\ -4x + 4y + 2z,\ 2x + 2y + 7z)^t$. Show that F is orthogonally diagonalizable (which is the same as to show that the standard matrix A of F is orthogonally diagonalizable). Determine an ON-basis in each eigenspace of F, an orthogonal matrix B and a diagonal matrix D s.t. $A = BDB^{-1} = BDB^t$. Show that A can be represented as

$$A = \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3$$

where

$$\lambda_1,\, \lambda_2,\, \lambda_3 \in \mathbb{R}, \quad P_1 + P_2 + P_3 = I, \quad P_1^2 = P_1 = P_1^t, \quad P_2^2 = P_2 = P_2^t, \quad P_3^2 = P_3 = P_3^t$$

and

$$P_1P_2 = P_2P_1 = P_1P_3 = P_3P_1 = P_2P_3 = P_3P_2 = \mathbf{O}.$$

156 Positive and negative definite matrices, semidefinite and indefinite matrices, Problem 9.

Problem 9: **A warning**: all plus signs in a matrix don't guarantee its positive definiteness. Show it on the following examples:

$$A = \begin{pmatrix}1 & \frac{1}{2}\\ \frac{1}{2} & 1\end{pmatrix}, \qquad A = \begin{pmatrix}1 & 1\\ 1 & \frac{1}{8}\end{pmatrix}, \qquad A = \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}.$$

Extra material: notes with solved Problem 9.

Extra material: an article with more solved problems on definiteness of symmetric matrices, determined by completing the square.

- ⋆ **Extra problem 1**: Determine whether $A = \begin{pmatrix}-6 & 4\\ 4 & -4\end{pmatrix}$ is positive/negative definite, semidefinite, or indefinite. Use the method of completing the square.
- ⋆ **Extra problem 2**: Determine whether $A = \begin{pmatrix}-2 & 4\\ 4 & -4\end{pmatrix}$ is positive/negative definite, semidefinite, or indefinite. Use the method of completing the square.
- ⋆ **Extra problem 3**: Determine whether $A = \begin{pmatrix}12 & -4\\ -4 & 4\end{pmatrix}$ is positive/negative definite, semidefinite, or indefinite. Use the method of completing the square.
- ⋆ **Extra problem 4**: Determine whether $A = \begin{pmatrix}6 & 3\\ 3 & 2\end{pmatrix}$ is positive/negative definite, semidefinite, or indefinite. Use the method of completing the square.
- ⋆ **Extra problem 5**: Determine whether $A = \begin{pmatrix}6 & 3\\ 3 & 1\end{pmatrix}$ is positive/negative definite, semidefinite, or indefinite. Use the method of completing the square.
- ⋆ **Extra problem 6**: Determine whether $A = \begin{pmatrix}6 & 2\\ 2 & 2\end{pmatrix}$ is positive/negative definite, semidefinite, or indefinite. Use the method of completing the square.
- ⋆ **Extra problem 7**: Determine whether $A = \begin{pmatrix}-2 & 2\\ 2 & 2\end{pmatrix}$ is positive/negative definite, semidefinite, or indefinite. Use the method of completing the square.

⋆ **Extra problem 8**: Determine whether $A = \begin{pmatrix} 6 & -6 & -6 \\ -6 & 8 & 8 \\ -6 & 8 & 12 \end{pmatrix}$ is positive/negative definite, semidefinite, or indefinite. Use the method of completing the square.

157 The wonderful strength of an orthogonally diagonalized matrix.

158 Three tests for definiteness of symmetric matrices, Problem 10.

Problem 10: Show that $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is positive definite. Use three methods:

1. completing the square,
2. eigenvalue test,
3. determinant test.

Extra material: notes with solved Problem 10.

159 Symmetric square roots of symmetric positive definite matrices; singular values; Problem 11.

Problem 11: (See Video 74): Let $G = \frac{1}{5}\begin{pmatrix} 37 & -16 \\ -16 & 13 \end{pmatrix}$. Find a symmetric matrix C such that $C^2 = G$.

S12 Quadratic forms and their classification

You will learn: how to describe (geometrically) and recognise (from their equation) quadratic curves and surfaces.

160 The correspondence between quadratic forms and symmetric matrices is 1-to-1.

Extra material: notes with a proof of the theorem about 1-to-1 correspondence, and an example showing that representation by non-symmetric matrices is **not** unique.

161 Completing the square is not unique.

Example: Use the form $q(x, y) = 2x^2 - 4xy + y^2$ to show that completing the square is not unique.

Extra material: notes with solved example.

162 What kind of questions we want to answer.

163 Quadratic forms in two variables, Problem 1.

Problem 1: Let $c > 0$. What kind of set is described by $q(\mathbf{x}) = h$, where the quadratic form $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ is defined by $A = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$?

164 Quadratic forms in two variables, Problem 2.

Problem 2: Let $a, b, \lambda_1, \lambda_2$ be some positive constants, and

$$A = \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix}, \qquad B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix},$$

where C is any symmetric matrix with eigenvalues λ_1 and λ_2. What kind of set is described by $q(\mathbf{x}) = 1$, where the quadratic form $q(\mathbf{x}) = \mathbf{x}^T M\mathbf{x}$ is defined by $M = A$, $M = B$, and $M = C$?

165 Quadratic curves, generally.

167 Quadratic curves as conic sections.

166 Quadratic curves by distances; shortest distance from the origin.

168 Principal axes; The shortest distance from the origin, Problem 3.

Problem 3: Plot the curve described by $\mathbf{x}^T A\mathbf{x} = 36$, where $A = \begin{pmatrix} 8 & 2 \\ 2 & 5 \end{pmatrix}$.

169 Classification of quadratic forms in two variables.

Example: The equation $13x_1^2 + 13x_2^2 - 10x_1x_2 = 72$ describes a curve in the plane. Draw the curve and find the shortest and the longest distance from a point on the curve to the origin.

170 Classification of curves, Problem 4.

Problem 4: The equation $19x_1^2 + 11x_2^2 - 6x_1x_2 = 10$ describes a curve in the plane. Draw the curve and find the shortest and the longest distance from a point on the curve to the origin, and the coordinates (in the standard basis) of the points where these values occur.

Extra material: notes with solved Problem 4.

171 Classification of curves, Problem 5.

Problem 5: Find principal axes for the quadratic form $q(x_1, x_2) = x_1^2 - 4x_1x_2 + x_2^2$. Draw the curve $q(x_1, x_2) = 1$ and its counterpart in the standard position. Can you find the shortest and the longest distance from the curve to the origin?

Extra material: notes with solved Problem 5.

172 Different roles of symmetric matrices (back to Videos 150 and 168), Problem 6.

Problem 6: Let $A = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$.

a) We have seen in Video 150 that $T_A(\mathbf{x}) = A\mathbf{x}$ maps the unit circle on an ellipse with principal axes in the direction of the eigenvectors of A. Find an equation of this ellipse.

b) Consider the symmetric matrix B generating the ellipse from the previous question. Draw the curve.

c) Find the curve defined by $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = 1$ for the original matrix. What kind of curve do you get?

d) What do the curves from b) and c) have in common?

Extra material: notes with solved Problem 6.

173 Classification of curves, Problem 7.

Problem 7: Find principal axes for the quadratic form $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = 6x_1^2 + 4x_1x_2 + 3x_2^2$. Draw the curve $q(x_1, x_2) = 1$ and its counterpart in the standard position. Compare the curve to the image of the unit circle under $T_A\mathbf{x} = A\mathbf{x}$.

Extra material: notes with solved Problem 7.

174 Generally about quadratic surfaces.

175 Some nice visuals on quadratic surfaces.

176 Quadratic surfaces, shortest distance, Problem 8.

Example: Determine the shortest distance from the origin:

a) for the ellipsoid $4x^2 + 9y^2 + z^2 = 36$,

b) for the hyperboloid of one sheet $4x^2 - 9y^2 + z^2 = 36$,

c) for the hyperboloid of two sheets $-4x^2 - 9y^2 + z^2 = 36$.

Problem 8: Surface Y is described as the set of all the points in $\mathbb{E}^3$ satisfying the equation

$$11x^2 + 11y^2 + 8z^2 + 2xy + 8xz + 8yz = 4.$$

Determine what kind of surface it is, the shortest distance from the surface to the origin, and the coordinates of the closest points (in the standard basis).

Extra material: notes with solved Example, and with solved Problem 8.

177 Quadratic surfaces, Problem 9.

Problem 9: Surface Y is described as the set of all the points in $\mathbb{E}^3$ satisfying the equation

$$4x^2 + 4y^2 + 7z^2 - 8xy + 4xz + 4yz = 2.$$

Determine what kind of surface it is, the shortest distance from the surface to the origin, and the coordinates of the closest points (in the standard basis). Show that the surface is a rotational surface (a surface of revolution) and determine the axis of this rotation.

Extra material: notes with solved Problem 9.

178 Quadratic surfaces, Problem 10.

Problem 10: Determine, for each value of $a \in \mathbb{R}$, the type of the surface

$$-2x^2 + 4xy + 2xz + y^2 + 4yz - 2z^2 = a.$$

Determine the shortest distance from the surface to the origin. For which values of a is it a surface of revolution? Determine the directional vector of the rotation axis for all such a.

Extra material: notes with solved Problem 10, and (in the beginning of the document) with a supplement to Video 177, clarifying the part about the coordinates of the points with shortest distance to the origin.

179 Law of inertia for quadratic forms; Signature of a form, Problem 11.

Problem 11: Perform the symmetric matrix operations in order to determine the signature of the quadratic form defined by the matrix $\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$.

Extra material: notes with solved Problem 11.

180 Four methods of determining definiteness; Problem 12.

Problem 12: Consider the quadratic form

$$h(x_1, x_2, x_3) = -x_1^2 + 2x_1x_2 + x_2^2 - 4x_2x_3.$$

Use Lagrange's method of systematic completing the square in order to establish the type of the surface this form generates, and the signature of the form.

Extra material: notes with solved Problem 12.

Extra material: an article with two solved problems on quadratic forms.

- ⋆ **Problem 1**: Let $q = x_1^2 + 3x_2^2 + 19x_4^2 - 2x_1x_2 + 4x_1x_3 + 2x_1x_4 - 6x_2x_4 - 12x_3x_4$. Find an equivalent form with no mixed products by using the method of symmetric matrix operations.
- ⋆ **Problem 2**: Let $q = -4x_1^2 - 8x_2^2 - 4x_3^2 + 4x_1x_2 - 12x_1x_3 + 4x_2x_3$. Describe the surface geometrically, find the shortest distance from the surface to the origin, and the coordinates of the closest points.

S13 Constrained optimization

You will learn: how to determine the range of quadratic forms on (generalized) unit spheres in $\mathbb{R}^n$.

181 The theory for this section.

182 Constrained optimization, Problem 1.

Problem 1: (See Video 173.) Find the largest and the smallest value of the quadratic form $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = 6x_1^2 + 4x_1x_2 + 3x_2^2$ for arguments on the unit circle $\|\mathbf{x}\| = 1$. Make an illustration.

183 Constrained optimization, Problem 2.

Problem 2: (See Video 172.) Find the largest and the smallest value of the quadratic form $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = x_1^2 + 4x_1x_2 - 2x_2^2$ for arguments on the unit circle $\|\mathbf{x}\| = 1$. Make an illustration.

184 Constrained optimization, Problem 3.

Problem 3: Find the largest and the smallest value of the quadratic form $q(x, y, z) = x^2 + y^2 - 4xy + z^2$ for arguments on the unit sphere $x^2 + y^2 + z^2 = 1$.

185 Constrained optimization, Problem 4.

Problem 4: Find the largest and the smallest value of the quadratic form $q(x_1, x_2, x_3)$ for arguments on the unit sphere $x_1^2 + x_2^2 + x_3^2 = 1$. Here $q(x_1, x_2, x_3) = 5x_1^2 + 14x_1x_2 + 4x_1x_3 + 5x_2^2 - 4x_2x_3 - 4x_3^2$.

C4 The Grand Finale

S14 Singular value decomposition

You will learn: about singular value decomposition: how it works and why it works; about pseudoinverses.

186 All our roads led us to SVD.

187 Why do we need SVD?

188 We know really a lot about $A^T A$ for any rectangular matrix A.

189 New facts about $A^T A$: eigenvalues and eigenvectors. Singular values of A.

Extra material: an article with proof that $A^T A$ and AA^T have the same non-zero eigenvalues.

Extra material: notes with proof that $A^T A$ and AA^T are positive semi-definite; singular values of A are norms of $A\mathbf{v}_i$ where $\mathbf{v}_i$ are eigenvectors for $A^T A$.

190 ON-bases containing only eigenvectors of certain matrix products.

Extra material: notes with proof of the theorem above.

191 Singular value decomposition with proof and geometric interpretation.

Extra material: notes with proof of the theorem above.

Fun fact: Given SVD of a matrix A, find the SVD of $A^T A$ and AA^T. (Note: as these matrices are symmetric and positive semi-definite, their SVD is also their eigendecomposition.)

192 SVD, reduced singular value decomposition, Problem 1.

Problem 1: Find the singular value decomposition and reduced singular value decomposition of $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Extra material: notes with solved Problem 1.

193 SVD, Problem 2.

Problem 2: Find the singular value decomposition of $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$.

Extra material: notes with solved Problem 2.

194 More new facts about $A^T A$: six equivalent statements.

Extra material: notes with proof of the theorem above.

195 Least squares, SVD, and pseudoinverse (Moore–Penrose inverse).

196 Pseudoinverse, Problem 3.

Problem 3: Compute the pseudoinverse for the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ from Video 192, using two methods of computation (for one of the methods, use the results from Video 192). Verify that the nullspace of A^T is mapped by $A^\dagger$ onto the zero vector of $\mathbb{R}^2$.

Extra material: notes with solved Problem 3.

197 SVD and Fundamental Theorem of Linear Algebra.

S15 Wrap-up Linear Algebra and Geometry

198 Linear Algebra and Geometry, Wrap-up.
199 So, what's next?
200 Final words.