Linear Algebra and Geometry 2[1]

Much more about matrices; abstract vector spaces and their bases

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A short table of contents

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An extremely detailed table of contents; the videos (titles in green) are numbered

In blue: problems solved on an iPad (the solving process presented for the students; active problem solving)
In red: solved problems demonstrated during a presentation (a walk-through; passive problem solving)
In magenta: additional problems solved in written articles (added as resources).

C1 Abstract vector spaces and related stuff

S1 Introduction to the course

1 Introduction to the course. Extra material: this list with all the movies and problems.

S2 Real vector spaces and their subspaces

You will learn: the definition of vector spaces and the way of reasoning around the axioms; determine whether a subset of a vector space is a subspace or not.

2 From abstract to concrete.

3 From concrete to abstract.

4 Our prototype.

5 Formal definition of vector spaces. Example 1: $\mathbb{R}^n$.
Extra material: some notes from the iPad.

6 Vector spaces, Example 2: $m \times n$ matrices with real entries.
Example 2: The set $\mathbb{R}^{m\times n}$ (another notation: $\mathcal{M}_{m\times n}(\mathbb{R})$) of all the m-by-n matrices with real entries.

7 Vector spaces, Example 3: real-valued functions on some interval.
Example 3: The set $F[a, b]$ of all the functions $f : [a, b] \to \mathbb{R}$ for given constants a and b s.t. $a < b$.
Extra material: some notes from the iPad.

8 Vector spaces, Example 4: complex numbers.
Example 4: The set $\mathbb{C}$ of all complex numbers is a vector space over $\mathbb{R}$.
Extra material: some notes from the iPad.

9 Cancellation property.
Let V be a vector space. Show that the following property (*Cancellation*) holds: if $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are any elements of V such that $\mathbf{v} + \mathbf{u} = \mathbf{v} + \mathbf{w}$ then $\mathbf{u} = \mathbf{w}$.
Extra material: some notes from the iPad.

10 Two properties of vector spaces; Definition of difference.
Let V be a vector space. Show that its neutral element is unique. Show also that the additive inverse to each element of V is unique.
Extra material: some notes from the iPad.

11 Some properties of vector spaces.
Let V be a vector space with neutral element $\mathbf{0}$. Let $\mathbf{v}$ be any element in V and $\alpha \in \mathbb{R}$. Show that:

a) $0 \cdot \mathbf{v} = \mathbf{0}$

b) $\alpha \cdot \mathbf{0} = \mathbf{0}$

c) $(-1) \cdot \mathbf{v} = -\mathbf{v}$

d) if $\alpha \cdot \mathbf{v} = \mathbf{0}$ then either $\alpha = 0$ or $\mathbf{v} = \mathbf{0}$ (or both).

The last law is called *Zero-product property.*

Extra material: some notes from the iPad.

12 What is a subspace.

Observation 1: Intersection of subspaces of V is a subspace of V.

13 All the subspaces in $\mathbb{R}^2$.

Observation 2: Union of subspaces of V is usually not a subspace of V.

14 All the subspaces in $\mathbb{R}^3$.

15 Subspaces, Problem 1.

Problem 1: Which of the following sets are subspaces of $\mathbb{R}^3$?:

a) $\{(a,\, 0,\, 0);\ \ a \in \mathbb{R}\}$

b) $\{(a,\, 1,\, 0);\ \ a \in \mathbb{R}\}$

c) $\{(a,\, b,\, c);\ \ b = a + c\}$.

16 Subspaces, Problem 2.

Problem 2: Which of the following sets are subspaces of $F[-\infty, \infty]$?:

a) the set of all continuous functions $C^0[-\infty, \infty]$

b) the set of all discontinuous functions

c) the set of all polynomials $\mathbb{R}[x]$

d) the set of all polynomials of degree 2

e) the set of all polynomials of degree less than or equal to 2

f) the set of all polynomials $p \in \mathbb{R}[x]$ s.t. $p(5) = 0$

g) the set of all even (or: all odd) functions.

Extra material: notes with solution to Problem 2.

17 Subspaces, Problem 3.

Problem 3: Which of the following sets are subspaces of $\mathbb{R}^{n\times n}$?:

a) all diagonal matrices

b) all symmetrical matrices

c) all matrices with determinant equal to 1

d) all matrices with determinant 0 (all singular/non-invertible matrices)

e) all matrices with trace equal to 0

f) all matrices with trace equal to 1.

Extra material: notes with solution to Problem 3.

18 Subspaces, Problem 4.

Problem 4: If $\mathbf{u}_0 \in \mathbb{R}^n$ then the set $\{\mathbf{v} \in \mathbb{R}^n;\ \mathbf{u}_0 \cdot \mathbf{v} = 0\}$ is a subspace.

Extra material: notes with solution to Problem 4.

S3 Linear combinations and linear independence

You will learn: the concept of linear combination and span, linearly dependent and independent sets; apply Gaussian elimination for determining whether a set is linearly independent; geometrical interpretation of linear dependence and linear independence.

19 Our unifying example.

20 Linear combinations in Part 1.

21 Linear combinations, new stuff. Example 1.

Example 1: Linear combinations in $\mathbb{R}^n$.

22 Linear combinations. Example 2.

Example 2: Linear combinations in the space of 2-by-2 matrices (all of them, and symmetric ones).

23 Linear combinations, Problem 1.

Problem 1: Are the following vectors linear combinations of $\mathbf{u} = (1, -3, 2)$ and $\mathbf{v} = (1, 0, -4)$?:

a) $(0, -3, 6)$

b) $(3, -9, -2)$.

Give a geometrical interpretation of your answers.

Extra material: notes with solution to Problem 1.

24 Linear combinations, Problem 2.

Problem 2: Express the following vectors as linear combinations of vectors $\mathbf{u} = (2, 1, 4)$, $\mathbf{v} = (1, -1, 3)$ and $\mathbf{w} = (3, 2, 5)$:

a) $(-9, -7, -15)$

b) $(6, 11, 6)$

c) $(0, 0, 0)$

d) $(7, 8, 9)$.

Extra material: notes with solution to Problem 2.

25 What is a span, definition and some examples.

Some examples of span in $\mathbb{R}^2$ and $\mathbb{R}^3$.

26 Span, Problem 3.

Problem 3: Let M denote the subspace of $\mathbb{R}^4$ spanned by
$\mathbf{u}_1 = (2, 1, 1, 1)$, $\mathbf{u}_2 = (1, 1, 0, 0)$, $\mathbf{u}_3 = (3, 1, 2, 2)$, $\mathbf{u}_4 = (1, 1, 1, 0)$ and $\mathbf{u}_5 = (1, 0, -2, 1)$.
For which values of the parameter a does vector $\mathbf{v} = (3 + a, 3, 3 + a, 2 + 2a)$ belong to M?

Extra material: notes with solution to Problem 3.

27 Span, Problem 4.

Problem 4: Let M denote the subspace of $\mathbb{R}^4$ spanned by
$\mathbf{u}_1 = (1, 2, 1, -2)$, $\mathbf{u}_2 = (-2, -1, 3, 1)$, $\mathbf{u}_3 = (-1, 1, 4, -1)$, $\mathbf{u}_4 = (3, 3, -2, -3)$ and $\mathbf{u}_5 = (1, 0, -1, 0)$.
For which values of the parameter a does vector $\mathbf{v} = (a, -1, 5 - a, 1)$ belong to M?

Extra material: notes with solution to Problem 4.

28 Span, Problem 5.

Problem 5: Let M denote the subspace of $\mathbb{R}^4$ spanned by
$\mathbf{u}_1 = (1, 2, 1, 0)$, $\mathbf{u}_2 = (-2, -1, 3, 0)$, $\mathbf{u}_3 = (-1, 1, 4, 0)$, $\mathbf{u}_4 = (3, 3, -2, 0)$ and $\mathbf{u}_5 = (1, 0, -1, 0)$.
For which values of the parameter a does vector $\mathbf{v} = (a, -1, 5 - a, 1)$ belong to M?

29 What do we mean by *trivial*?

30 Linear independence and linear dependence.

A set of vectors is linearly dependent iff some of the vectors in the set is a linear combination of the other vectors.

31 Geometry of linear independence and linear dependence.

32 An important remark on linear independence in $\mathbb{R}^n$.

33 Linearly independent generators, Problem 6.

Problem 6: Find a linearly independent set generating the row vectors of matrix A:

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}.$$

Extra material: notes with solved Problem 6.

34 Linear independence in the set of matrices, Problem 7.

Problem 7: Find a linearly independent set generating:

a) the set of all symmetrical matrices in $\mathbb{R}^{3\times 3}$

b) the following subset of $\mathbb{R}^{3\times 3}$: the set of all matrices $\begin{bmatrix} x & 0 & y \\ 0 & x-y & x+z \\ y & x-z & x \end{bmatrix}$, where $x, y, z \in \mathbb{R}$.

Extra material: notes with solved Problem 7.

35 Linear independence in $C^0[-\infty, \infty]$, Problem 8.

Problem 8: Show that:

a) the polynomials $\mathbf{p}_1(t) = 1,\ \mathbf{p}_2(t) = t,\ \mathbf{p}_3(t) = t^2$ are linearly independent in $C^0[-\infty, \infty]$ and that they generate the set of all the polynomials of degree less than or equal to 2,

b) the polynomials $\mathbf{q}_1(t) = 1,\ \mathbf{q}_2(t) = t,\ \mathbf{q}_3(t) = 4 - t$ are linearly dependent in $C^0[-\infty, \infty]$.

Extra material: notes with solved Problem 8.

36 Vandermonde determinant and polynomials.

Example: Show that the set $S = \{1,\ t,\ t^2,\ \ldots,\ t^{n-1}\}$ is linearly independent in $C^0[-\infty, \infty]$ and that it generates the set of all the polynomials of degree less than or equal to $n-1$.

37 Linear independence in $C^\infty(\mathbb{R})$, Problem 9.

Problem 9: Show that the set $\{e^t,\ e^{2t},\ e^{3t}\}$ is linearly independent in $C^\infty(\mathbb{R})$.

Extra material: notes with solved Problem 9.

38 Wronskian and linear independence in $C^\infty(\mathbb{R})$.

Back to Problem 8a: Use Wronskian to show that the polynomials $\mathbf{p}_1(t) = 1,\ \mathbf{p}_2(t) = t,\ \mathbf{p}_3(t) = t^2$ and $\mathbf{p}_4(t) = t^3$ are linearly independent in $C^\infty(\mathbb{R})$.

39 Linear independence in $C^\infty(\mathbb{R})$, Problem 10.

Problem 10: Show that the set $\{1,\ \cos t,\ \sin t\}$ is linearly independent in $C^\infty(\mathbb{R})$.

Extra material: notes with solved Problem 10.

40 Linear independence in $C^\infty(\mathbb{R})$, Problem 11.

Problem 11: Show that the set $\{e^{\lambda_1 t},\ e^{\lambda_2 t}, \ldots, e^{\lambda_n t}\}$ (where λ_i for $i = 1, 2, \ldots, n$ are distinct real numbers) is linearly independent in $C^\infty(\mathbb{R})$.

Extra material: notes with solved Problem 11.

S4 Coordinates, basis, and dimension

You will learn: about the concept of basis for a vector space, the coordinates w.r.t. a given basis, and the dimension of a vector space; you will learn how to apply the determinant test for determining whether a set of n vectors is a basis of $\mathbb{R}^n$.

41 What is a basis and dimension?

42 Bases in the 3-space, Problem 1.

Problem 1: Which of the following sets form a basis in $\mathbb{R}^3$?

a) $\mathbf{v}_1 = (1,2,3),\ \mathbf{v}_2 = (0,2,0)$

b) $\mathbf{v}_1 = (1,2,3),\ \mathbf{v}_2 = (0,2,0),\ \mathbf{v}_3 = (1,0,3)$

c) $\mathbf{v}_1 = (1,2,3),\ \mathbf{v}_2 = (0,2,0),\ \mathbf{v}_3 = (1,0,0)$

d) $\mathbf{v}_1 = (1,2,3),\ \mathbf{v}_2 = (0,2,0),\ \mathbf{v}_3 = (1,0,0), \mathbf{v}_4 = (5,1,-3)$

e) $\mathbf{v}_1 = (\lambda,1,2),\ \mathbf{v}_2 = (1,\lambda,2),\ \mathbf{v}_3 = (1,2,\lambda)$.

Extra material: notes with solved Problem 1.

43 Bases in the plane and in the 3-space.

44 Bases in the 3-space, Problem 2.

Problem 2: Let $\mathbf{v} = (1,2,3)$, $\mathbf{u} = (0,7,1)$. Determine all the vectors $\mathbf{w}$ such that $\mathbb{R}^3 = \text{span}\{\mathbf{v},\ \mathbf{u},\ \mathbf{w}\}$.

Extra material: notes with solved Problem 2.

45 Bases in the 4-space, Problem 3.

Problem 3: Let $\mathbf{v} = (2,3,-6,7)$, $\mathbf{u} = (0,1,-1,1)$ and $\mathbf{w} = (1,2,-5,5)$. Determine a basis for the intersection $M \cap N$ where $M = \text{span}\{\mathbf{v},\ \mathbf{u},\ \mathbf{w}\}$ and $N = \{\mathbf{x} = (x_1,x_2,x_3,x_4);\ x_1 - 2x_2 - x_3 = 0\}$.

Extra material: notes with solved Problem 3.

46 Bases in the 4-space, Problem 4.

Problem 4: Let $\mathbf{a}_1 = (1,2,-3,4)$, $\mathbf{a}_2 = (2,4,-6,7)$, $\mathbf{a}_3 = (0,1,-2,2)$, $\mathbf{a}_4 = (0,1,-1,1)$, $\mathbf{a}_5 = (2,4,-7,8)$, $\mathbf{a}_6 = (1,2,-2,3)$. Determine a basis for the intersection $M \cap N$ where $M = \text{span}\{\mathbf{a}_1,\ \mathbf{a}_2,\ \mathbf{a}_3\}$ and $N = \text{span}\{\mathbf{a}_4,\ \mathbf{a}_5,\ \mathbf{a}_6\}$.

Extra material: notes with solved Problem 4.

47 Bases in the space of polynomials, Problem 5.

Problem 5: Determine a basis for the subspace $U \subset \mathbb{R}[x]_3$ spanned by

$$f_1(x) = x^2+1, \quad f_2(x) = x+2, \quad f_3(x) = 2x^2 - x, \quad f_4(x) = 2x^3 - 1.$$

Complete it to a basis of the entire $\mathbb{R}[x]_3$.

Extra material: notes with solved Problem 5.

48 Coordinates with respect to a basis.

49 Coordinates with respect to a basis are unique.

Extra material: notes from the iPad, with the proof of uniqueness of coordinates.

50 Coordinates in our unifying example.

51 Dimension of a subspace, Problem 6.

Problem 6: Find the dimension of the subspace

$$\left\{ \begin{bmatrix} a-3b+6c \\ 5a+4d \\ b-2c-d \\ 5d \end{bmatrix},\quad a,b,c,d \in \mathbb{R} \right\}.$$

Extra material: notes with solved Problem 6.

52 Bases in a space of functions, Problem 7.

Problem 7: Let $\mathbb{R}^X$ denote the vector space of all the functions from the set $X = \{1,2,3,4\}$ to $\mathbb{R}$. Determine a basis for this space and determine the coordinates w.r.t. this basis for function $g : X \to \mathbb{R}$ given by:

$$g(1) = -11, \quad g(2) = 7, \quad g(3) = -\frac{3}{2}, \quad g(4) = 5.$$

Extra material: notes with solved Problem 7.

S5 Change of basis

You will learn: how to recalculate coordinates between bases by solving systems of linear equations, by using transition matrices, and by using Gaussian elimination; the geometry behind different coordinate systems.

53 Coordinates in different bases.

54 It is easy to recalculate from the standard basis.

Example 1: Show that the vectors $(1,1,3)$, $(2,2,3)$ and $(3,1,3)$ form a basis in $\mathbb{R}^3$. Determine the coordinates of the vector $(2,1,2)$ with respect to this basis.

55 Transition matrix, a derivation.

56 Previous example with transition matrix.

Example 2: Determine the coordinates of the vector $(2,1,2)$ with respect to the basis formed by $(1,1,3)$, $(2,2,3)$ and $(3,1,3)$. Use the transition matrix.

57 Our unifying example.

Example 3: $B = \{\mathbf{f}_1, \mathbf{f}_2\}$ with $\mathbf{f}_1 = (2,4)$ and $\mathbf{f}_2 = (3,1)$ is a basis in $\mathbb{R}^2$. Given four vectors with following coordinates in basis B: $\mathbf{v}_1 = (1,-1)$, $\mathbf{v}_2 = (1,1)$, $\mathbf{v}_3 = (-1,2)$ and $\mathbf{v}_4 = (-\frac{1}{2},-1)$. Compute their coordinates in the standard basis in $\mathbb{R}^2$ with help of the transition matrix. Make a picture.

58 One more simple example and bases.

Example 4: Compute the coordinates for $\mathbf{v} = (17,-1)$ in basis $\mathbf{w}_1 = (8,1)$, $\mathbf{w}_2 = (7,4)$.

59 Two non-standard bases, Method 1.

Example 5: Determine the transition matrix $P_{B'\to B}$ between B' och B where $B = \{\mathbf{u}_1 = (1,1),\ \mathbf{u}_2 = (-1,1)\}$ and $B' = \{\mathbf{u}'_1 = (2,1),\ \mathbf{u}'_2 = (1,2)\}$.

60 Two non-standard bases, Method 2.

The same example as above, with the Method 2.

61 How to recalculate coordinates between two non-standard bases? An algorithm.

The same example as above, with the Method 3.

62 Change of basis, Problem 1.

Problem 1: Let $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B' = \{\mathbf{v}_1, \mathbf{v}_2\}$ be two bases in $\mathbb{R}^2$:

$$\mathbf{u}_1 = (2,2), \quad \mathbf{u}_2 = (4,-1), \qquad \mathbf{v}_1 = (1,3), \quad \mathbf{v}_2 = (-1,-1).$$

a) Use the following algorithm for computing $P_{B'\to B}$:

$$[\ P_{B\to S} \mid P_{B'\to S}\ ] \xrightarrow{\text{elementary row operations}} [\ I \mid P_{B\to S}^{-1} P_{B'\to S} = P_{B'\to B}\ ].$$

b) Use the following algorithm for computing $P_{B\to B'}$:

$$[\ P_{B'\to S} \mid P_{B\to S}\ ] \xrightarrow{\text{elementary row operations}} [\ I \mid P_{B'\to S}^{-1} P_{B\to S} = P_{B\to B'}\ ].$$

c) Confirm that $P_{B'\to B}$ and $P_{B\to B'}$ are inverse to each other.

d) Let $\mathbf{w} = (5,-3)$. Determine $[\mathbf{w}]_B$ and use the transition matrix $P_{B\to B'}$ for computing $[\mathbf{w}]_{B'}$ from $[\mathbf{w}]_B$ using the formula:

$$[\mathbf{w}]_{B'} = P_{B\to B'}[\mathbf{w}]_B.$$

e) Let $\mathbf{w} = (3,-5)$. Determine $[\mathbf{w}]_{B'}$ and use the transition matrix $P_{B'\to B}$ for computing $[\mathbf{w}]_B$ from $[\mathbf{w}]_{B'}$ using the formula:

$$[\mathbf{w}]_B = P_{B'\to B}[\mathbf{w}]_{B'}.$$

Extra material: notes with solved Problem 1.

63 Change of basis, Problem 2.

Problem 2: Find the coordinates for vector $\mathbf{w}$ in the basis $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ for $\mathbb{R}^2$:

a) $\mathbf{u}_1 = (1,0),\ \mathbf{u}_2 = (0,1),\ \mathbf{w} = (-4,3)$

b) $\mathbf{u}_1 = (1,-1),\ \mathbf{u}_2 = (2,5),\ \mathbf{w} = (3,7)$

c) $\mathbf{u}_1 = (1,2),\ \mathbf{u}_2 = (-2,1),\ \mathbf{w} = (a,b)$.

Extra material: notes with solved Problem 2.

64 Change of basis, Problem 3.

Problem 3: Let
$\mathbf{b}_1 = (1,1,-1),\ \mathbf{b}_2 = (1,-2,1),\ \mathbf{b}_3 = (1,3,-2)$
$\mathbf{c}_1 = (2,1,-1),\ \mathbf{c}_2 = (1,2,2),\ \mathbf{c}_3 = (-1,0,1)$.
Show that $B = \{\mathbf{b}_1,\ \mathbf{b}_2,\ \mathbf{b}_3\}$ and $C = \{\mathbf{c}_1,\ \mathbf{c}_2,\ \mathbf{c}_3\}$ are bases in $\mathbb{R}^3$.
For vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ we have
$\mathbf{u} = (-1,1,1),\ [\mathbf{v}]_B = (-1,1,1),\ [\mathbf{w}]_C = (-1,1,1)$.
Determine the coordinates of $\mathbf{r} = \mathbf{u} - 2\mathbf{v} + \mathbf{w}$
with respect to bases S, B and C.

Extra material: notes with solved Problem 3.

65 Change of basis, Problem 4.

Problem 4: If B_1, B_2, and B_3 are bases for $\mathbb{R}^2$, and if

$$P_{B_1 \to B_2} = \begin{bmatrix} 4 & 2 \\ 3 & 2 \end{bmatrix} \quad \text{and} \quad P_{B_2 \to B_3} = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix},$$

what is $P_{B_3 \to B_1}$?

Extra material: notes with solved Problem 4.

66 Change of basis, Problem 5.

Problem 5: The set $B = \{1+t,\ 1+t^2,\ t+t^2\}$ is a basis for $\mathbb{P}_2[x]$ (the subspace of all the polynomials with grade less than or equal to 2). Find the coordinates of $p(t) = 6 + 3t - t^2$ w.r.t. this basis.

Extra material: notes with solved Problem 5.

67 Change to an orthonormal basis in $\mathbb{R}^2$.

S6 Row space, column space, and nullspace of a matrix

You will learn: concepts of row and column space, and the nullspace for a matrix; find bases for span of several vectors in $\mathbb{R}^n$ with different conditions for the basis.

68 What you are going to learn in this section.

69 Row space and column space for a matrix.

70 What are the elementary row operations doing to the row spaces?

Problem 1: Determine a basis for the row space of matrix A:

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}$$

Extra material: notes with solved Problem 1.

71 What are the elementary row operations doing to the column spaces?

72 Column space, Problem 2.

Problem 2: Determine a basis for the column space of matrix A:

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}$$

Extra material: notes with solved Problem 2.

73 Determining a basis for a span, Problem 3.

Problem 3: Find a basis for the subspace of $\mathbb{R}^5$ spanned by the vectors

$$\mathbf{v}_1 = (1, -2, 0, 0, 3), \quad \mathbf{v}_2 = (2, -5, -3, -2, 6), \quad \mathbf{v}_3 = (0, 5, 15, 10, 0), \quad \mathbf{v}_4 = (2, 6, 18, 8, 6).$$

Extra material: notes with solved Problem 3.

74 Determining a basis for a span consisting of a subset of given vectors, Problem 4.

Problem 4: Given five vectors in $\mathbb{R}^4$: $\mathbf{u}_1 = (2, 1, 1, 1)$, $\mathbf{u}_2 = (1, 1, 0, 0)$, $\mathbf{u}_3 = (3, 1, 2, 2)$, $\mathbf{u}_4 = (1, 1, 1, 0)$ and $\mathbf{u}_5 = (1, 0, -2, 1)$. Which of these vectors span the entire subspace generated by these vectors?
In Video 26 we showed that vector $\mathbf{v} = (1, 3, 1, -2)$ belongs to the span; express this vector as a linear combination of the basis vectors you have identified.

Extra material: notes with solution to Problem 4.

75 Determining a basis for a span consisting of a subset of given vectors, Problem 5.

Problem 5: Given five vectors in $\mathbb{R}^4$: $\mathbf{u}_1 = (1, 2, 1, -2)$, $\mathbf{u}_2 = (-2, -1, 3, 1)$, $\mathbf{u}_3 = (-1, 1, 4, -1)$, $\mathbf{u}_4 = (3, 3, -2, -3)$ and $\mathbf{u}_5 = (1, 0, -1, 0)$. Which of these vectors span the entire subspace generated by these vectors? Determine the coordinates of all the given vectors w.r.t. the basis you have just established.

Extra material: notes with solution to Problem 5.

76 A tricky one: Let rows become columns, Problem 6.

Problem 6: Determine a basis for the row space of matrix A, consisting entirely of row vectors from A:

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}$$

Extra material: notes with solved Problem 6.

77 A basis in the space of polynomials, Problem 7.

Problem 7: Polynomials $p_1(t) = -1 + t + 2t^2 - t^3$, $p_2(t) = -3 + 2t + 2t^2 - t^3$, $p_3(t) = -7 + 4t + 2t^2 - t^3$, $p_4(t) = 10 - 7t - 10t^2 + 5t^3$, and $p_5(t) = -26 + 19t + 28t^2 - 14t^3$ span a subspace of $\mathbb{P}_3[t]$. Determine a basis B of this subspace consisting only of vectors named above. What are the coordinates of all the given polynomials in basis B?

Extra material: notes with solved Problem 7.

78 Nullspace for a matrix.

Extra material: proof that the nullspace of an $m \times n$ matrix is a subspace of $\mathbb{R}^n$.

79 How to find the nullspace, Problem 8.

Problem 8: Find a basis for the nullspace of the matrix A:

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}.$$

80 Nullspace, Problem 9.

Problem 9: Find a basis for and the dimension of the solution space of the homogenous system:

$$\begin{cases} x_1 + 3x_2 - 2x_3 \qquad\qquad + 2x_5 \qquad\quad = 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = 0 \\ \qquad\qquad 5x_3 + 10x_4 \qquad\quad + 15x_6 = 0 \\ 2x_1 + 6x_2 \qquad\quad + 8x_4 + 4x_5 + 18x_6 = 0 \end{cases}$$

Extra material: notes with solved Problem 9.

81 Nullspace, Problem 10.

Problem 10: Find a basis for the nullspace of the matrix A and solve the (non-homogenous) system of equations:

$$A = \begin{bmatrix} 1 & -2 & 10 \\ 2 & -3 & 18 \\ 0 & -7 & 14 \end{bmatrix}, \qquad \begin{cases} x - 2y + 10z = 9 \\ 2x - 3y + 18z = 17 \\ \quad - 7y + 14z = 7 \end{cases}$$

Give a geometrical interpretation of your solution.

Extra material: notes with solved Problem 10.

S7 Rank, nullity, and four fundamental matrix spaces

You will learn: determine the rank and the nullity for a matrix; find orthogonal complement to a given subspace; four fundamental matrix spaces and the relationship between them.

82 Rank of a matrix.

83 Nullity.

Example: Find a basis for the row space and a basis for the nullspace of the matrix A. What are the values of rank(A) and nullity(A)?

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}.$$

84 Relationship between rank and nullity.

85 Relationship between rank and nullity, Problem 1.

Problem 1: In each part, find the largest possible value for the rank of A and the smallest possible value of nullity of A:

(a) A is 4×6, (b) A is 5×5, (c) A is 6×4.

Extra material: notes with solved Problem 1.

86 Relationship between rank and nullity, Problem 2.

Problem 2: In each part, use the information in the table to determine whether the linear system $A\mathbf{x} = \mathbf{b}$ is consistent. If so, state the number of parameters in its general solution:

	(a)	(b)	(c)	(d)	(e)	(f)	(g)
Size of A	3×3	3×3	3×3	5×9	5×9	4×4	6×2
Rank(A)	3	2	1	2	2	0	2
Rank$[\,A\,|\,\mathbf{b}\,]$	3	3	1	2	3	0	2
Consistent?							
No of parameters							

87 Relationship between rank and nullity, Problem 3.

Problem 3: Are there values of r and s for which

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r-2 & 2 \\ 0 & s-1 & r+2 \\ 0 & 0 & 3 \end{bmatrix}$$

has rank 1? Has rank 2? If so, find those values.

88 Orthogonal complements, Problem 4.

Problem 4: Find a basis for the orthogonal complement $M^\perp$ of $M \subset \mathbb{R}^4$, where

$M = \{\, (a, b, c, d) \in \mathbb{R}^4; \;\; a + b + c = 0,\; a + b + d = 0,\; 2a + 2b + c + d = 0 \,\}$.

89 Four fundamental matrix spaces.

90 The Fundamental Theorem of Linear Algebra and Gilbert Strang.

C2 Linear transformations

S8 Matrix transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$

You will learn: about matrix transformations: understand the way of identifying linear transformations with matrices (produce the standard matrix for a given transformation, and produce the transformation for a given matrix); concepts: kernel, image and inverse operators; understand the link between them and nullspace, column space and inverse matrix.

91 What do we mean by linear?

92 Some terminology.

93 How to think about functions from $\mathbb{R}^n$ to $\mathbb{R}^m$?

Example 1: Determine the set of departure and the codomain
for the following functions:

a) $f(x, y, z, w) = (2x - \sin y,\; x^2 - yz,\; w - x)$

b) $f(x, y, z, w, t) = xyz + wt - \sin y$

c) $f(x, y) = (3x - y,\; 5x - 8y,\; -x + 6y,\; 7x)$

d) $f(x_1, x_2, x_3, x_4, x_5) = (6x_1 - 8x_4,\; 2x_3 + 4x_4 - x_5,\; x_2 - 9x_3)$

e) $f(x, y, z, w) = (2 - x - w,\; 7x + 3z - 3w)$

f) $f(x, y, z) = (-x - z,\; 7y + 3z,\; x + z)$.

94 When is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$ linear? Approach 1.

Example 2: Which of the functions from Example 1 are linear? Motivate.

95 When is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$ linear? Approach 2.

Example 3: Write standard matrices for all linear transformations in Example 1.

Extra material: notes with solved Example 3.

96 When is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$ linear? Approach 3.

Example 4: Linearity in Linear Algebra and in Calculus:

a) $f(x) = x + 2$ is linear in Calculus but not in Linear Algebra.

b) $f(x) = x$ is linear in both Calculus and Linear Algebra.

c) $f(x) = 3x$ (or, generally, $f(x) = cx$) is linear in both Calculus and Linear Algebra.

d) $f(x, y, z) = ax + by + cz$ is linear in both Calculus and Linear Algebra.

e) $f(x,y,z) = ax + by + cz + d$ (where $d \neq 0$) is linear in Calculus but not in Linear Algebra.

97 Approaches 2 and 3 are equivalent.

98 Matrix transformations, Problem 1.

Problem 1: Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $w_1 = 4x_1 - 3x_2 + x_3$, $w_2 = 2x_1 - x_2 + 5x_3$, $w_3 = x_1 + 2x_2 - 2x_3$. Compute $T(1,0,2)$ in two ways: by plugging in the coordinates in the formulas, and with help of the standard matrix.

Extra material: notes with solved Problem 1.

99 Image, kernel, and inverse operators, Problem 2.

Problem 2: Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $w_1 = 4x_1 - 3x_2 + x_3$, $w_2 = 2x_1 - x_2 + 5x_3$, $w_3 = x_1 + 2x_2 - 2x_3$. Is $(-6, 16, -5)$ a value under T of some vector? Determine $\ker(T)$ and $\mathrm{Im}(T)$.

Extra material: notes with solved Problem 2.

100 Basis for the image, Problem 3.

Problem 3: Find a basis for the image of $L : \mathbb{R}^4 \to \mathbb{R}^3$ where L defines as

$$L(x,y,z,w) = (x + 9y + 7z + 3w,\ \ 2x + 8y - z + w,\ \ -x + 7y + 17z + 5w).$$

Extra material: notes with solved Problem 3.

101 Kernel, Problem 4.

Problem 4: Let $L : \mathbb{R}^4 \to \mathbb{R}^3$ be defined as

$$L(x,y,z,w) = (x + y + z + w,\ \ x + 2y + 2z + 3w,\ \ 2x + 3y + 3z + 4w).$$

a) Determine the nullspace $\ker(L)$.

b) Determine a basis for the nullspace.

c) Determine the dimension of the nullspace.

Extra material: notes with solved Problem 4.

102 Image and kernel, Problem 5.

Problem 5: Let $L : \mathbb{R}^4 \to \mathbb{R}^2$ be defined as

$$L(x,y,z,w) = (x + 2y + z + 2w,\ \ 2x + 4y + 2z + 4w).$$

a) Determine the nullspace $\ker(L)$.

b) Determine a basis for the nullspace.

c) Determine the dimension of the nullspace.

d) Determine the image $\mathrm{Im}(L)$.

e) Determine a basis for the image.

f) Determine the dimension of the image.

Extra material: notes with solved Problem 5.

103 Inverse operators, Problem 6.

Problem 6: Determine whether the following operators $T : \mathbb{R}^2 \to \mathbb{R}^2$ are one-to-one. If it is the case, determine the inverse operator T^{-1}.

a) $w_1 = 2x_2,\ \ w_2 = -x_1$.

b) $w_1 = 9x_1 + 5x_2,\ \ w_2 = 2x_1 - 7x_2$.

c) $w_1 = -x_2,\ \ w_2 = -x_1$.

d) $w_1 = 3x_1,\ \ w_2 = -5x_1$.

Extra material: notes with solved Problem 6.

104 Linear transformations, Problem 7.

Problem 7: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Determine its standard matrix knowing that $T(\mathbf{v}_1) = \mathbf{w}_1$ and $T(\mathbf{v}_2) = \mathbf{w}_2$ where

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{w}_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Extra material: notes with solved Problem 7.

105 Kernel and geometry, Problem 8.

Problem 8: Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $T(x, y, z) = (x + y + z, \;\; x + 2y + 2z)$.

a) Determine whether the vectors $\mathbf{a} = (0, -2, 2)$, $\mathbf{b} = (1, 2, 1)$, $\mathbf{c} = (0, 0, 0)$ belong to $\ker(T)$.

b) Determine $\ker(T)$.

c) If all the vectors in kernel start in the origin, then their endpoints form a subspace of $\mathbb{R}^3$. Give a geometrical interpretation of this subspace (a point, a straight line, a plane through the origin).

Extra material: notes with solved Problem 8.

106 Linear transformations, Problem 9.

Problem 9: Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation such that

$$T(\mathbf{u}_1) = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}, \quad T(\mathbf{u}_2) = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}, \quad \text{where} \quad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 5 \\ 5 \\ 10 \\ 10 \end{bmatrix}.$$

Is it possible to, with this information only, determine $T(\mathbf{v})$ where

$$a) \quad \mathbf{v} = \begin{bmatrix} 11 \\ 11 \\ 21 \\ 21 \end{bmatrix}, \quad b) \quad \mathbf{v} = \begin{bmatrix} 6 \\ 6 \\ 11 \\ 15 \end{bmatrix}?$$

Extra material: notes with solved Problem 9.

S9 Geometry of matrix transformations on $\mathbb{R}^2$ and $\mathbb{R}^3$

You will learn: about transformations such as rotations, symmetries, projections and their matrices; you will learn how to illustrate the actions of linear transformations in the plane.

107 Our unifying example: linear transformations and change of basis.

Example 1: Picture the following matrix transformation:

$$T_A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + 3y \\ 4x + y \end{bmatrix}$$

(An animation in MANIM included. More animations in videos 108, 111, 112, and 114.)

108 An example with nontrivial kernel.

Example 2: (Example (d) from Problem 6 in Video 103.) Consider the linear operator defined by $T(x_1, x_2) = (w_1, w_2)$ where $w_1 = 3x_1, \;\; w_2 = -5x_1$. Compute its kernel and image, and explain the geometry of the situation.

Extra material: notes from the iPad.

109 Line symmetries in the plane.

Example 3: Line symmetry (reflection):

a) about x-axis,

b) about y-axis,

c) about the diagonal $y = x$.

Extra material: notes from the iPad.

110 Projection on a given vector, Problem 1.

Problem 1: Determine the standard matrix for the linear transformation T which for each vector $\mathbf{v} \in \mathbb{R}^3$ assigns its projection on vector $\mathbf{u} = (1, 2, 3)$, i.e. $T(\mathbf{v}) = \text{proj}_{\mathbf{u}}\mathbf{v}$.

Extra material: notes with solved Problem 1.

111 Symmetry about the line $y = kx$, Problem 2.

Problem 2: Determine the standard matrix for the linear transformation T which for each point $(x, y) \in \mathbb{R}^2$ assigns its reflection about the line $y = 2x$.

Extra material: notes with solved Problem 2.

112 Rotation by 90 degrees about the origin.

Extra material: notes from the iPad.

113 Rotation by the angle α about the origin.

114 Expansion, compression, scaling, and shear.

115 Plane symmetry in the 3-space, Problem 3.

Problem 3: Plane symmetry (reflection) about:

a) the xy-plane,

b) the yz-plane,

c) the xz-plane.

Use matrix multiplication to find the image of the vector $(3, 4, 5)$ under these operators.

Extra material: notes with solved Problem 3.

116 Projections on planes in the 3-space, Problem 4.

Problem 4: Orthogonal projection on:

a) the xy-plane,

b) the yz-plane,

c) the xz-plane.

Use matrix multiplication to find the image of the vector $(3, 4, 5)$ under these operators.

Extra material: notes with solved Problem 4.

117 Symmetry about a given plane, Problem 5.

Problem 5: Determine the standard matrix of the symmetry about the plane $3x + 2y + z = 0$.

Extra material: notes with solved Problem 5.

118 Projection on a given plane, Problem 6.

Problem 6: Determine the standard matrix of the orthogonal projection on the plane $3x + 2y + z = 0$.

Extra material: notes with solved Problem 6.

119 Rotations in the 3-space, Problem 7.

Problem 7: Use matrix multiplication to find the image of the vector $(-2, 1, 2)$ if it is rotated:

a) $30°$ about the x-axis,

b) $45°$ about the y-axis,

c) $90°$ about the z-axis.

Extra material: notes with solved Problem 7.

S10 Properties of matrix transformations

You will learn: what happens with subspaces and affine spaces (points, lines and planes) under linear transformations; what happens with the area and volume; composition of linear transformations as matrix multiplication.

120 What kind of properties we will discuss.

121 What happens with vector subspaces and affine subspaces under linear transformations.

122 Parallel lines transform into parallel lines, Problem 1.

Problem 1: Take our favorite matrix transformation:

$$T_A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x+3y \\ 4x+y \end{bmatrix}$$

Show that the parallel lines $y = -\frac{1}{2}x$ and $y = -\frac{1}{2}x + \frac{3}{2}$ transform onto two parallel lines.

123 Transformations of straight lines, Problem 2.

Problem 2: The invertible matrix:

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

maps the line $y = 2x + 1$ into another line. Find its equation.

Extra material: notes with solved Problem 2.

124 Change of area (volume) under linear operators in the plane (space).

Compare these two transformations:

$$T_A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \qquad T_B \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

The determinant of the first matrix is negative, the determinant of the second one is positive.

125 Change of area under linear transformations, Problem 3.

Problem 3: Let a and b be positive numbers. Find the area of the region E bounded by the ellipse whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

126 Compositions of linear transformations.

127 How to obtain the standard matrix of a composition of linear transformations?

Example: Show that the order of performing transformations matter:

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Use these matrices as standard matrices for operators. (The same matrices were used in Video 74 in Part 1, to show that matrix multiplication is not commutative.)

128 Why does it work?

Extra material: proof that composition of linear transformations is linear transformation.

129 Compositions of linear transformations, Problem 4.

Problem 4: The linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^4$ is such that

$$T(1,0) = (2,-1,0,3), \quad T(0,1) = (0,1,-4,2).$$

The transformation $R : \mathbb{R}^2 \to \mathbb{R}^2$ is the counterclockwise rotation through the angle $\pi/6$ around the origin. Only one of the two compositions $R \circ T$ and $T \circ R$ is possible to perform. Find the standard matrix for this composition, and explain why the other one is impossible to perform.

Extra material: notes with solved Problem 4.

130 Compositions of linear transformations, Problem 5.

Problem 5: Find the standard matrix for the stated composition in $\mathbb{R}^2$:

a) A reflection about the line $y = x$, followed by an orthogonal projection on the y-axis, followed by a rotation by $60°$,

b) A reflection about the x-axis, followed by a rotation by $45°$, followed by dilation (*scaling with a scale factor larger than* 1) with factor $k = 3$,

c) A rotation by $-60°$, followed by a rotation by $105°$, followed by a rotation by $45°$.

Extra material: notes with solved Problem 5.

Extra material: an article with more solved problems on linear transformations.

- **Extra problem 1**: The matrix transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$T(x_1, x_2, x_3) = (x_1 + x_2,\ x_2,\ x_3).$$

Define the matrix transformation $S : \mathbb{R}^3 \to \mathbb{R}^3$ as orthogonal projection on the vector $\mathbf{v} = (1, 2, 0)$, that is $S(\mathbf{x}) = \text{proj}_{\mathbf{v}}\mathbf{x}$. Determine the standard matrix of $S \circ T$.

- **Extra problem 2**: The matrix transformation $F : \mathbb{R}^2 \to \mathbb{R}^2$ rotates vectors counterclockwise about the origin by the angle $\pi/4$. The matrix transformation $G : \mathbb{R}^2 \to \mathbb{R}^2$ is the orthogonal projection on the vector $(2, 1)$. Find the standard matrix for the composition $G \circ F$.

- **Extra problem 3**: The matrix transformation $F : \mathbb{R}^2 \to \mathbb{R}^2$ rotates vectors counterclockwise about the origin by the angle $2\pi/3$. The matrix transformation $G : \mathbb{R}^2 \to \mathbb{R}^2$ rotates vectors clockwise about the origin by the angle $\pi/6$. Find the standard matrix for the composition $G \circ F$.

S11 General linear transformations in different bases

You will learn: solving problems involving linear transformations between two vector spaces; work with linear transformations in different bases.

131 Linear transformations between two linear spaces.

132 Linear transformations, Problem 1.

Problem 1: Vector space $\mathcal{P}_3 = \mathbb{P}_3[t]$ is the space of all the polynomials $p(t)$ of degree $\leqslant 3$:

a) Show that the transformation $F : \mathcal{P}_3 \to \mathcal{P}_3$ defined by $F(p(t)) = (1 + t)p'(t) - 2p(t)$ is linear.

b) Determine the standard matrix of F in the basis $\{1, t, t^2, t^3\}$.

c) Determine a basis for $\text{Im}(F)$ and $\ker(F)$. (The elements in the bases must be polynomials.)

d) Is F a bijection? Give a motivation for your answer.

Extra material: notes with solved Problem 1.

133 Linear transformations, Problem 2.

Problem 2: $\mathcal{P}_2 = \mathbb{P}_2[t]$ is the space of all the polynomials $p(t)$ of degree $\leqslant 2$ and $\mathcal{S}_2$ is the space of all the symmetrical 2×2 matrices with real entries. Show that $F : \mathcal{P}_2 \to \mathcal{S}_2$ defined by

$$F(p) = \begin{pmatrix} p(-1) & p(0) \\ p(0) & p(1) \end{pmatrix} \qquad \text{for} \qquad p(t) = x_0 + x_1 t + x_2 t^2 \in \mathcal{P}_2$$

is linear and invertible. Determine the standard matrix of F^{-1} in standard bases in $\mathcal{P}_2$ and $\mathcal{S}_2$.

Extra material: notes with solved Problem 2.

134 Linear transformations, Problem 3.

Problem 3: $\mathcal{P}_3 = \mathbb{P}_3[t]$ is the space of all the polynomials $p(t)$ of degree $\leqslant 3$ and $\mathcal{S}_3$ is the space of all the symmetrical 3×3 matrices with real entries. The transformation $F : \mathcal{P}_3 \to \mathcal{S}_3$ defined by

$$F(p) = \begin{pmatrix} p(0) & p'(0) & p''(0) \\ p'(0) & p'(1) & p''(1) \\ p''(0) & p''(1) & p''(-1) \end{pmatrix} \qquad \text{for} \qquad p(t) = x_1 + x_2 t + x_3 t^2 + x_4 t^3 \in \mathcal{P}_3$$

is linear. Determine the standard matrix of F in standard bases in $\mathcal{P}_3$ and $\mathcal{S}_3$. Determine also the image and kernel of F.

Extra material: notes with solved Problem 3.

135 Linear transformations, Problem 4.

Problem 4: Let $\mathcal{S}_2$ be the space of all the symmetrical 2×2 matrices with real entries. A linear transformation $F : \mathcal{S}_2 \to \mathcal{S}_2$ has the following matrix in the standard basis of $\mathcal{S}_2$:

$$[F] = \begin{pmatrix} 1+2a & 1 & 1 \\ 1 & 2 & -1 \\ 4a & 3 & -2a \end{pmatrix}$$

For which values of the constant a does the intersection of $\ker(F)$ and $\operatorname{Im}(F)$ contain more elements than only the zero vector of the space? Point out such vectors in these cases. (Don't forget that our *vectors* are symmetrical 2×2 matrices.)

Extra material: notes with solved Problem 4.

136 Linear transformations, Problem 5.

Problem 5: $\mathcal{P}_2$ is the space of all the polynomials $p(t)$ of degree $\leqslant 2$. Show that $F : \mathcal{P}_2 \to \mathbb{R}^3$ defined by

$$F(p) = \begin{pmatrix} p(0) \\ p(0) + p(1) \\ p'(-1) \end{pmatrix} \qquad \text{for} \qquad p(t) = x_1 + x_2 t + x_3 t^2 \in \mathcal{P}_2$$

is linear and invertible. Determine the standard matrix of F and F^{-1} in standard bases in $\mathcal{P}_2$ and $\mathbb{R}^3$.

Extra material: notes with solved Problem 5.

137 Linear transformations in different bases, Problem 6.

Problem 6: Let $\mathcal{S}_2$ be the space of all the symmetrical 2×2 matrices with real entries. A linear transformation $F : \mathcal{S}_2 \to \mathbb{R}^3$ is defined as $F(X) = BX\mathbf{a}$ where:

$$X = \begin{pmatrix} x_1 & x_3 \\ x_3 & x_2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Determine the image and the kernel of this transformation.

Extra material: notes with solved Problem 6.

138 Linear transformations in different bases.

139 Linear transformations in different bases, Problem 7.

Problem 7: Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined geometrically as a projection of all vectors in the space onto the plane $x + y + 2z = 0$ along the line $(x, y, z) = (t, -t, t),\ t \in \mathbb{R}$. Determine the matrix of F in the standard basis of $\mathbb{R}^3$.

Extra material: notes with solved Problem 7.

140 Linear transformations in different bases, Problem 8.

Problem 8: Transformation $F : \mathbb{R}^2 \to \mathbb{R}^2$ is linear and it satisfies $F(\mathbf{u}_1) = 2\mathbf{u}_1$ and $F(\mathbf{u}_2) = -\mathbf{u}_2$ where $B = \{\mathbf{u}_1,\ \mathbf{u}_2\}$ is the basis of $\mathbb{R}^2$ with $\mathbf{u}_1 = (-1, -1)$ and $\mathbf{u}_2 = (1, 2)$. Determine the standard matrix for

F in the standard basis of $\mathbb{R}^2$. Present two method: one with help of transition matrices, one by solving systems of equations.

Extra material: notes with solved Problem 8.

141 Linear transformations in different bases, Problem 9.

Problem 9: Transformation $F : \mathcal{P}_2 \to \mathcal{P}_2$ is defined as $F(\mathbf{p}) = F(p(t)) = p(t-1)$ where $p(t) = x_1 + x_2 t + x_3 t^2$. Determine the standard matrix of F in the basis $B = \{\mathbf{b}_1,\ \mathbf{b}_2,\ \mathbf{b}_3\} = \{t,\ t+1,\ t^2-1\}$ and show that F is a bijection.

Extra material: notes with solved Problem 9.

142 Linear transformations, Problem 10.

Problem 10: Show that the matrix transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^3$ with the standard matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ is injective but not surjective.

Extra material: notes with solved Problem 10.

143 Linear transformations, Problem 11.

Problem 11: Show that the matrix transformation $T_A : \mathbb{R}^3 \to \mathbb{R}^2$ with the standard matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ is surjective but not injective.

Extra material: notes with solved Problem 11.

C3 Orthogonality

S12 Gram–Schmidt Process

You will learn: about orthonormal bases and their superiority above the other bases; about orthogonal projections on subspaces to $\mathbb{R}^n$; produce orthonormal bases for given subspaces of $\mathbb{R}^n$ with help of Gram–Schmidt process.

144 Dot product and orthogonality until now.

145 Orthonormal bases are awesome.

146 Orthonormal bases are awesome, Reason 1: distance.

147 Orthonormal bases are awesome, Reason 2: dot product.

148 Orthonormal bases are awesome, Reason 3: transition matrix.

149 Orthonormal bases are awesome, Reason 4: coordinates.

150 Coordinates in ON bases, Problem 1.

Problem 1: Verify that the vectors

$$\mathbf{v}_1 = (-\tfrac{3}{5},\ \tfrac{4}{5},\ 0), \quad \mathbf{v}_2 = (\tfrac{4}{5},\ \tfrac{3}{5},\ 0), \quad \mathbf{v}_3 = (0, 0, 1)$$

form an ON basis for $\mathbb{R}^3$. Express the following vectors as linear combinations of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$:

$$a)\ \ (2, 1, -1), \quad b)\ \ (1, 3, 4), \quad c)\ \ (\tfrac{1}{7},\ -\tfrac{3}{7},\ \tfrac{5}{7}).$$

Extra material: notes with solved Problem 1.

151 Coordinates in orthogonal bases, Theorem and proof.

152 Each orthogonal set is linearly independent, Proof.

153 Coordinates in orthogonal bases, Problem 2.

Problem 2: Verify that the vectors

$$\mathbf{v}_1 = (1,\ -2,\ 3,\ -4),\quad \mathbf{v}_2 = (2,\ 1,\ -4,\ -3),\quad \mathbf{v}_3 = (-3,\ 4,\ 1,\ -2),\quad \mathbf{v}_4 = (4,\ 3,\ 2,\ 1)$$

form an orthogonal basis for $\mathbb{R}^4$. Express vector $\mathbf{u} = (-1,\ 2,\ 3,\ 7)$ as linear combinations of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$.

Extra material: notes with solved Problem 2.

154 Orthonormal bases, Problem 3.

Problem 3: Verify that the vectors

$$\mathbf{v}_1 = (\tfrac{2}{3},\ \tfrac{1}{3},\ -\tfrac{2}{3}),\quad \mathbf{v}_2 = (\tfrac{1}{3},\ \tfrac{2}{3},\ \tfrac{2}{3}),\quad \mathbf{v}_3 = (\tfrac{2}{3},\ -\tfrac{2}{3},\ \tfrac{1}{3})$$

form an ON basis in $\mathbb{R}^3$.

Extra material: notes with solved Problem 3.

155 Projection Theorem 1.

Extra material: notes from the iPad.

156 Projection Theorem 2.

Extra material: notes from the iPad.

157 Projection Formula, an illustration in the 3-space.

158 Calculating projections, Problem 4.

Problem 4: Let $W = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ where $\mathbf{v}_1 = (0,\ 1,\ 0)$ and $\mathbf{v}_2 = (-\frac{4}{5},\ 0,\ \frac{3}{5})$. Decompose $\mathbf{u} = (1,\ 1,\ 1)$ as the sum $\mathbf{w}_1 + \mathbf{w}_2$ where $\mathbf{w}_1 \in W$ and $\mathbf{w}_2 \in W^\perp$. Verify also that $\mathbf{w}_2$ is orthogonal to the generators of W.

Extra material: notes with solved Problem 4.

159 Calculating projections, Problem 5.

Problem 5: (*The same as Problem 4, but solved by different method*)
Let $W = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ where $\mathbf{v}_1 = (0,\ 1,\ 0)$ and $\mathbf{v}_2 = (-\frac{4}{5},\ 0,\ \frac{3}{5})$. Decompose $\mathbf{u} = (1,\ 1,\ 1)$ as the sum $\mathbf{w}_1 + \mathbf{w}_2$ where $\mathbf{w}_1 \in W$ and $\mathbf{w}_2 \in W^\perp$. Use normal vector to W instead of the new formula.

Extra material: notes with solved Problem 5.

160 Gram–Schmidt Process.

161 Gram–Schmidt Process, Our unifying example.

Example: Perform Gram–Schmidt Process for $B = \{\mathbf{f}_1, \mathbf{f}_2\}$ with $\mathbf{f}_1 = (2, 4)$ and $\mathbf{f}_2 = (3, 1)$. Make a picture.

162 Gram–Schmidt Process, Problem 6.

Problem 6: The vectors $\mathbf{u}_1 = (1, 1, 1, 1)$, $\mathbf{u}_2 = (4, 2, 4, 2)$ and $\mathbf{u}_3 = (1, 0, 0, 0)$ span a subspace of $\mathbb{R}^4$. Determine an ON-basis for this subspace.

Extra material: notes with solved Problem 6.

163 Gram–Schmidt Process, Problem 7.

Problem 7: Let $\mathbb{R}^4 \supset W = \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2\}$ where $\mathbf{u}_1 = (-1,\ 0,\ 1,\ 2)$ and $\mathbf{u}_2 = (0,\ 1,\ 0,\ 1)$. Decompose $\mathbf{u} = (3,\ -4,\ 2,\ 5)$ as the sum $\mathbf{w}_1 + \mathbf{w}_2$ where $\mathbf{w}_1 \in W$ and $\mathbf{w}_2 \in W^\perp$. Verify also that $\mathbf{w}_2$ is orthogonal to the generators of W.

Extra material: notes with solved Problem 7.

Extra material: an article with more solved problems on Gram–Schmidt process.

- ⋆ **Extra problem 1**: Vectors $\mathbf{u}_1 = (4, 3, 2, 1)$ and $\mathbf{u}_2 = (3, 2, 1, 4)$ span a subspace of $\mathbb{R}^4$. Determine an ON-basis for this subspace using Gram–Schmidt Process.
- ⋆ **Extra problem 2**: The vectors $\mathbf{u}_1 = (3, 2, 1, 2)$, $\mathbf{u}_2 = (0, -5, -1, 1)$ and $\mathbf{u}_3 = (7, 3, 2, 5)$ span a subspace of $\mathbb{R}^4$. Determine an ON-basis for this subspace.
- ⋆ **Extra problem 3**: The vectors $(1, 1, 1, 1)$, $(2, 0, 1, 1)$, $(1, -1, 0, 2)$ span a subspace of the vector space $\mathbb{R}^4$. Find an ON-basis for this subspace.

S13 Orthogonal matrices

You will learn: definition and properties of orthonormal matrices; their geometrical interpretation.

164 Product of a matrix and its transposed is symmetric.

165 Definition and examples of orthogonal matrices.

Example 1: The following matrices are orthogonal:

$$A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}, \qquad B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \quad \text{and} \quad R_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

for a and b s.t. $a^2 + b^2 = 1$ (the first two groups of matrices), and for each $\alpha \in \mathbb{R}$ (the third group of matrices). These matrices correspond to line symmetries and to rotations about the origin in the plane. (*Note that for $\alpha = 0$ we get $R_\alpha = I$.*)

166 Geometry of 2-by-2 orthogonal matrices.

167 A 3-by-3 example.

Example 2: Following matrix is orthogonal: $A = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$.

Extra material: notes with solved Example 2.

168 Useful formulas for the coming proofs.

Extra material: notes from the iPad.

169 Property 1: Determinant of each orthogonal matrix is 1 or -1.

Example 3: Matrix corresponding to shear has determinant 1, but is not orthogonal.

170 Property 2: Each orthogonal matrix A is invertible and A^{-1} is also orthogonal.

Extra material: notes from the iPad.

171 Property 3: Orthonormal columns and rows.

172 Property 4: Orthogonal matrices are transition matrices between ON-bases.

173 Property 5: Preserving distances and angles.

Extra material: notes from the iPad.

174 Property 6: Product of orthogonal matrices is orthogonal.

Extra material: notes from the iPad.

175 Orthogonal matrices, Problem 1.

Problem 1: Determine constants a and b for which matrix A is orthogonal:

$$a)\quad A = \begin{bmatrix} \frac{\sqrt{3}}{2} & a \\ 2 & b \end{bmatrix}, \qquad b)\quad A = \begin{bmatrix} \frac{4}{5} & a \\ a & b \end{bmatrix}, \qquad c)\quad A = \begin{bmatrix} \frac{1}{3} & 1 & b \\ \frac{1}{2} & a & a \\ \frac{1}{2} & b & 1 \end{bmatrix}$$

Extra material: notes with solved Problem 1.

176 Orthogonal matrices, Problem 2.

Problem 2: Following matrix is orthogonal: $A = \begin{bmatrix} 3/\sqrt{11} & -1/\sqrt{6} & -1/\sqrt{66} \\ 1/\sqrt{11} & 2/\sqrt{6} & -4/\sqrt{66} \\ 1/\sqrt{11} & 1/\sqrt{6} & 7/\sqrt{66} \end{bmatrix}$.

Extra material: notes with solved Problem 2.

C4 Intro to eigendecomposition of matrices

S14 Eigenvalues and eigenvectors

You will learn: compute eigenvalues and eigenvectors for square matrices with real entries; geometric interpretation of eigenvectors and eigenspaces.

177 Crash course in factoring polynomials.

178 Eigenvalues and eigenvectors, the terms.

179 Order of defining, order of computing.

180 Eigenvalues and eigenvectors geometrically.

Examples: T_A with standard matrix $A = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$; line symmetries; scalings; rotations.

181 Eigenvalues and eigenvectors, Problem 1.

Problem 1: Show that $\mathbf{v}$ is an eigenvector for A:

$$A = \begin{bmatrix} 8 & -9 & 4 \\ 3 & -4 & 3 \\ -3 & 3 & 1 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Determine also the corresponding eigenvalue.

Extra material: notes with solved Problem 1.

182 How to compute eigenvalues. Characteristic polynomial.

Example: Determine the characteristic polynomial and eigenvalues of the matrix $A = \begin{bmatrix} 4 & 3 \\ 3 & -4 \end{bmatrix}$.

183 How to compute eigenvectors.

Example: Determine eigenvectors for $A = \begin{bmatrix} 4 & 3 \\ 3 & -4 \end{bmatrix}$.

Extra material: notes with solved Example.

184 Finding eigenvalues and eigenvectors: short and sweet.

185 Eigenvalues and eigenvectors for examples from Video 180.

Problem 2: Confirm with computations our geometrical observations from Video 180:
T_A with standard matrix $A = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$; line symmetries; scalings; rotations.

Extra material: notes with solved Problem 2.

186 Eigenvalues and eigenvectors, Problem 3.

Problem 3: Determine all the eigenvalues and eigenvectors for A:

$$A = \begin{bmatrix} 2 & 5 & -5 \\ 2 & 8 & -8 \\ 2 & 11 & -11 \end{bmatrix}.$$

Extra material: notes with solved Problem 3.

187 Eigenvalues and eigenvectors, Problem 4.

Problem 4: Determine all the eigenvalues and eigenvectors for A:

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 1 & 2 \\ -2 & 0 & 1 \end{bmatrix}.$$

Extra material: notes with solved Problem 4.

188 Eigenvalues and eigenvectors, Problem 5.

Problem 5: Determine all the eigenvalues and eigenvectors for A:

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}.$$

Extra material: notes with solved Problem 5.

189 Eigenvalues and eigenvectors, Problem 6.

Problem 6: Determine all the eigenvalues and eigenvectors for A:

$$A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}.$$

Extra material: notes with solved Problem 6.

190 Eigenvalues and eigenvectors, Problem 7.

Problem 7: Determine all the eigenvalues and eigenvectors for A:

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}.$$

Extra material: notes with solved Problem 7.

Extra material: an article with more solved problems on eigenvalues and eigenvectors.

- ⋆ **Extra problem 1**: Determine all the eigenvalues and eigenvectors for matrix

$$A = \begin{bmatrix} -4 & 2 & 6 \\ -6 & 4 & 6 \\ -3 & 1 & 5 \end{bmatrix}.$$

- ⋆ **Extra problem 2**: Determine all the eigenvalues for matrix

$$B = \begin{bmatrix} 6 & 10 & 2 \\ -3 & -1 & -3 \\ -4 & -10 & 0 \end{bmatrix}.$$

- ⋆ **Extra problem 3**: Determine all the eigenvalues and eigenvectors for matrix

$$A = \begin{bmatrix} 4 & 0 & -2 \\ 1 & 3 & -2 \\ 1 & 2 & -1 \end{bmatrix}.$$

S15 Diagonalization

You will learn: to determine whether a given matrix is diagonalizable or not; diagonalize matrices and apply the diagonalization for problem solving (the powers of matrices).

191 Why you should love diagonal matrices.

192 Similar matrices.

Matrices $A = \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ are similar.

193 Similarity of matrices is an equivalence relation (RST).

194 Shared properties of similar matrices.

195 Diagonalizable matrices.

Matrix $A = \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix}$ is diagonalizable.

196 How to diagonalize a matrix, a recipe.

197 Diagonalize our favorite matrix.

Problem 1: Diagonalize the matrix $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$.

198 Eigenspaces; geometric and algebraic multiplicity of eigenvalues.

199 Eigenspaces, Problem 2.

Problem 2: Let A be a 2×2 matrix and call a line through the origin of $\mathbb{R}^2$ **invariant** under A if $A\mathbf{v}$ lies on this line when $\mathbf{v}$ does. Find equations for all lines in $\mathbb{R}^2$, if any, that are invariant under the given matrix transformation:

$$a) \quad A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}, \quad b) \quad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad c) \quad A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}.$$

Extra material: notes with solved Problem 2.

200 Eigenvectors corresponding to different eigenvalues are linearly independent.

Extra material: notes with the proof.

201 A sufficient, but not necessary, condition for diagonalizability.

202 Necessary and sufficient condition for diagonalizability.

Example: Four examples showing that being defective and being singular are two independent features.

203 Diagonalizability, Problem 3.

Problem 3: Is A diagonalizable?:

$$A = \begin{bmatrix} 2 & 5 & -5 \\ 2 & 8 & -8 \\ 2 & 11 & -11 \end{bmatrix}.$$

If this is a case, find the transition matrix P.

Extra material: notes with solved Problem 3.

204 Diagonalizability, Problem 4.

Problem 4: Is A diagonalizable?:

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 1 & 2 \\ -2 & 0 & 1 \end{bmatrix}.$$

If this is a case, find the transition matrix P.

Extra material: notes with solved Problem 4.

205 Diagonalizability, Problem 5.

Problem 5: Is A diagonalizable?:

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}.$$

If this is a case, find the transition matrix P.

Extra material: notes with solved Problem 5.

206 Diagonalizability, Problem 6.

Problem 6: Is A diagonalizable?:

$$A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}.$$

If this is a case, find the transition matrix P.

Extra material: notes with solved Problem 6.

207 Diagonalizability, Problem 7.

Problem 7: Is $A = \begin{bmatrix} 1 & 0 \\ 9 & 4 \end{bmatrix}$ diagonalizable? If this is a case, find the transition matrix P.

Extra material: notes with solved Problem 7.

208 Powers of matrices.

209 Powers of matrices, Problem 8.

Problem 8: Let $A = \begin{bmatrix} 7 & -12 \\ 4 & -7 \end{bmatrix}$. Diagonalize, if possible, matrix A. Compute A^{2021} and A^{2022}.

Extra material: notes with solved Problem 8.

210 Diagonalization, Problem 9.

Problem 9: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the symmetry along the line $3x + 4y = 0$. Determine the standard matrix for T using an appropriate change of variables in which the matrix of T is easy to establish without computations.

Extra material: notes with solved Problem 9.

211 Sneak peek into the next course; orthogonal diagonalization.

Extra material: an article with more solved problems on diagonalization of matrices.

- ⋆ **Extra problem 1**: Determine whether the following matrix is diagonalizable:

$$A = \begin{bmatrix} -4 & 2 & 6 \\ -6 & 4 & 6 \\ -3 & 1 & 5 \end{bmatrix}.$$

- ⋆ **Extra problem 2**: Determine whether the following matrix is diagonalizable:

$$B = \begin{bmatrix} 6 & 10 & 2 \\ -3 & -1 & -3 \\ -4 & -10 & 0 \end{bmatrix}.$$

- ⋆ **Extra problem 3**: Determine matrix P such that $P^{-1}AP$ is a diagonal matrix and provide this diagonal matrix.

$$A = \begin{bmatrix} 4 & 0 & -2 \\ 1 & 3 & -2 \\ 1 & 2 & -1 \end{bmatrix}.$$

- ⋆ **Extra problem 4**: Determine whether the following matrix is diagonalizable:

$$B = \begin{bmatrix} -1 & 2 & 4 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{bmatrix}.$$

If B is diagonalizable, determine an invertible 3×3-matrix P such that $P^{-1}BP$ is a diagonal matrix.

- ⋆ **Extra problem 5**: For matrix $A = \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix}$ we have $A = P^{-1}DP$ with $P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and with $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Compute A^{99} and A^{100}.

S16 Wrap-up Linear Algebra and Geometry 2

212 Linear Algebra and Geometry 2, Wrap-up.

213 Yes, there will be Part 3!

214 Final words.

S17 Extras

You will learn: about all the courses we offer, and where to find discount coupons. You will also get a glimpse into our plans for future courses, with approximate (very hypothetical!) release dates.

B Bonus lecture.

Extra material 1: a pdf with all the links to our courses, and coupon codes.

Extra material 2: a pdf with an advice about optimal order of studying our courses.

Extra material 3: a pdf with information about course books, and how to get more practice.