Discrete Mathematics 3[1]

Mathematics from high school to university

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A short table of contents

A detailed table of contents follows (see next pages).

Books to read along with the course, with more practice problems (often suggested in the outline and in some videos, as I have found these books before I started to record the series):

1. *Discrete Mathematics, An Open Introduction*, 4th Edition (2024): Oscar Levin; School of Mathematical Science, University of Northern Colorado.
2. *Mathematics for Computer Science* (LibreTexts): Eric Lehman, F. Thomson Leighton, Albert R. Meyer; MITOpenCourseWare.
3. *Book of Proof*: Richard Hammack; Richmond, Virginia.

The first book is added as a resource to V1 in each part of the DM series, with kind permission of Professor Oscar Levin (given by e-mail on 2 April 2025); the second one is added as a resource to V1 in DM2 and DM3, with kind permission of the Authors (given by e-mail on 20 June 2025).

[1]Recorded April 2026 – August 2026. Published on `www.udemy.com` on 2026-08-XX.

An extremely detailed table of contents; the videos (titles in green) are numbered

In blue: problems solved on an iPad (the solving process presented for the students; active problem solving)

In red: solved problems demonstrated during a presentation (a walk-through; passive problem solving)

In magenta: additional problems solved in written articles (added as resources).

In dark blue: *Read along with this section*: references for further reading and exercises in *Discrete Mathematics* book by Oscar Levin (this book is added as a resource to Video 1).

S1 Introduction to the course

You will learn: about this course: its content and the optimal way of studying.

1 Introduction to the course.

Extra material: this list with all the movies and problems. **Extra material: formula sheet for DM.**

Extra material: books named on the previous page (The DM Book and *Mathematics for CS*).

2 Sequences in DM1 and DM2.

3 Graphs in DM1.

4 Applications of DM in DM1 and DM2.

S2 Some preliminaries for the sections devoted to sequences

You will learn: some basic algebraic techniques that are needed for the sections about sequences: solving systems of linear equations (2-by-2 and 3-by-3 determinants, Sarrus' rule, Cramer's rule), solving quadratic equations (with help of the discriminant, or by qualified guesses), some basics about polynomials, finding integer zeros of polynomials with integer coefficients, polynomial division, partial fraction decomposition, subspaces of vector spaces and their generators, span, some formulas for derivatives. My advice is that you don't watch this section before you know why you need this stuff; I will refer to the appropriate videos when I'm about to use some of the concepts and techniques above for working with sequences.

5 How to study this section and which courses cover this stuff in depth.

6 Solving systems of linear equations, two examples.

Example 1: We get a quick look back at the system (solved in V171 of DM2; comes back in V10 in DM3):

$$\begin{cases} 2x + 3y = -1 \\ 4x + y = 3. \end{cases}$$

Example 2: Solve the following system of equations (Example 4.3.7 from the DM Book; comes back in S6):

$$\begin{cases} a + b + c = 1 \\ 8a + 4b + 2c = 5 \\ 27a + 9b + 3c = 14 \end{cases}$$

Extra material: notes with solved Example 2.

7 Uniqueness of solutions of 2-by-2 systems of linear equations.

Example: Given four non-zero numbers $a_{11}, a_{12}, a_{21}, a_{22}$ such that $a_{11}a_{22} - a_{12}a_{21} \neq 0$ and two real numbers b_1 and b_2. Solve the following system of linear equations:

$$\begin{cases} a_{11}x + a_{12}y = b_1 \\ a_{21}x + a_{22}y = b_2 \end{cases}$$

We observe and appreciate the fact that this system has exactly one solution. If this system is *homogenous* (i.e., there are only zeros at the RHS) then this unique solution is $(x, y) = (0, 0)$.

Extra material: notes with solved Example.

8 Determinants of 2-by-2 and 3-by-3 matrices.

Fact: An n-by-n system of linear equations has a unique solution (regardless the RHS) iff the determinant of its coefficient matrix is different from zero.

Fact: Each such system that is *homogenous* (the RHS is all zeros) has only the *trivial* solution (all zeros).

9 Sarrus' rule for 3-by-3 matrices.

Example: Compute the following determinants with Sarrus' method:

$$\begin{vmatrix} 1 & 1 & 1 \\ 8 & 4 & 2 \\ 27 & 9 & 3 \end{vmatrix}, \quad \begin{vmatrix} 1 & 1 & 1 \\ 5 & 4 & 2 \\ 14 & 9 & 3 \end{vmatrix}, \quad \begin{vmatrix} 1 & 1 & 1 \\ 8 & 5 & 2 \\ 27 & 14 & 3 \end{vmatrix}, \quad \begin{vmatrix} 1 & 1 & 1 \\ 8 & 4 & 5 \\ 27 & 9 & 14 \end{vmatrix}$$

Extra material: notes with solved example.

10 Cramer's rule for solving 2-by-2 or 3-by-3 systems.

Example 1: solve the following system of linear equations using Cramer's rule:

$$\begin{cases} 2x + 3y = -1 \\ 4x + y = 3. \end{cases}$$

Example 2: Solve the 3-by-3 system from V6 using the new method.

11 What is a polynomial and how we know that two polynomials are equal.

12 Exact polynomial fitting, a second-degree example.

Example: Find values of a, b and c such that the graph of the polynomial $p(x) = ax^2 + bx + c$ passes through the points $(1, 2)$, $(-1, 6)$, and $(2, 3)$.

Extra material: notes with solved Example.

13 Exact polynomial fitting, a third-degree example.

Example: Find the cubic polynomial whose graph passes through the points $(-1, -2)$, $(0, -4)$, $(1, 0)$, and $(2, 16)$.

Extra material: notes with solved Example.

14 Solving quadratic equations using discriminants.

Examples: Solve the following equations and factor their LHSs:

a) $x^2 + 2x - 3 = 0$,

b) $4x^2 - 7x - 2 = 0$,

c) $x^2 + x - 1 = 0$.

Extra material: notes with solved Examples.

15 Possible number of real zeros for a polynomial.

Example: Possible number of real zeros for polynomials, depending on degree:

a) $ax + b$, $a \neq 0$: always **one** zero,

b) $x^2 + x + 1$: **no** zeros, $x^2 + 2x + 1$: a **double** zero, $x^2 + 2x - 3$: **two** different zeros,

c) $(x - 1)(x^2 + x + 1)$: **one** zero, $(x - 1)(x - 2)(x - 3)$: **three** zeros, $(x - 1)^2(x - 2)$: **three** zeros, a double and a single, $(x - 1)^3$: a **triple** zero,

d) $(x^2+x+1)^2$: **no** zeros, $(x-1)(x-2)(x^2+x+1)$: **two** zeros, $(x-1)^2(x^2+x+1)$: **double** zero, $(x-1)^2(x-3)^2$: **two double** zeros, $(x - 1)(x - 2)(x - 3)(x - 4)$: **four** different zeros.

16 Rational zeros of polynomials with integer coefficients.

Examples: Find all the rational zeros of the following polynomials:

a) x^2+x-2

b) x^2+x-6

c) x^2+x-5

d) x^3+2x^2-4x+5

e) $2x^3+x^2+x-1$

f) x^5+x+1.

17 Polynomial division and why we need it.

Examples: Factor the polynomials $p_1(x)=x^3-3x^2+x+2$ and $p_2(x)=x^4-5x^3+6x^2+4x-8$. The results will be used in S7. (These two examples illustrate two different reasons why we need polynomial division for factoring polynomials.)

Extra material: notes with solved Examples.

Extra material: an article with more solved problems on polynomial division. (These examples will be later used in S7 for solving linear recurrences.) In all the problems below, divide $p(x)$ by $d(x)$:

- ⋆ **Extra problem 1**: $p(x)=x^3-5x^2+8x-4$ and $d(x)=x-1$
- ⋆ **Extra problem 2**: $p(x)=x^4-2x^3-2x^2+6x-3$ and $d(x)=x-1$
- ⋆ **Extra problem 3**: $p(x)=x^3-10x^2+32x-32$ and $d(x)=x-4$
- ⋆ **Extra problem 4**: $p(x)=x^3-9x^2+27x-27$ and $d(x)=x-3$.

18 Partial fraction decomposition.

Examples: Verify that $\dfrac{1}{x}+\dfrac{-x+2}{x^2+1}=\dfrac{2x+1}{x(x^2+1)}$. Perform partial fraction decomposition on $\dfrac{9+x}{9-x^2}$.

19 Partial fraction decomposition, an example.

Example: Perform partial fraction decomposition (that will be used in S8) on

$$\frac{x-2x^2+2x^3}{(1-x-6x^2)(1-2x+x^2)}.$$

Extra material: notes with solved Example.

20 Linear spaces and their subspaces, an (optional) topic for S7.

21 Linearly independent generators of a linear space, span (optional for S7).

Exercise: Show that the vectors $\mathbf{e}_1=(1,0)$ and $\mathbf{e}_2=(0,1)$ are linearly independent and that they generate the entire $\mathbb{R}^2$. (This shows that the dimension of $\mathbb{R}^2$ is 2, which shouldn't be surprising.)

22 Some words about rates of growth and derivatives.

Instantaneous rate of change for functions $f:\mathbb{R}\to\mathbb{R}$. Derivatives of polynomial and exponential functions. The derivative of $f(x)=\dfrac{1}{1-x}$ (will be needed for S8).

S3 A general introduction to sequences

You will learn: various ways of defining sequences (by a closed/explicit formula, by recursion, a verbal description, with help of a picture, by giving a number of elements that suggest a rule), arithmetic operations on sequences ([element-wise] addition, subtraction, multiplication, scaling), sequence of partial sums, sequence of differences, the vector space of all sequences of real numbers (a preparation for S7).

Read along with this section: *DM Book*, Section 4.1: *Describing Sequences* (pp.311–326).

23 Sequences in Calculus versus sequences in DM.

24 Sequences as functions; various ways of defining sequences.

- ∗ By picture.
- ∗ Explicit, by a (closed) formula. $a_n = \frac{1}{n}$.
- ∗ By writing down **enough** terms, with or without a table. $2,\ 5,\ 8,\ 11,\ 14,\ 17,\ \ldots$; $1,\ 2,\ 4,\ \ldots$.
- ∗ By a recursive description, i.e. showing how to get the next elements from the previous one(s).
  $a_1 = 2,\ a_{n+1} = 3a_n$ for $n \in \mathbb{N}^+$.
- ∗ By verbal description.
  a_n is the nth prime number; b_n is the nth digit in the decimal expansion of π.

Extra material: notes from the iPad.

25 Determining formulas for sequences, an exercise.

Example 4.1.2 from the DM Book, p.413: Write the formulas for the following sequences according to the most natural guess (made by the Author):

1. $7,\ 7,\ 7,\ 7,\ 7,\ \ldots$
2. $3,\ -3,\ 3,\ -3,\ 3,\ \ldots$
3. $1,\ 5,\ 2,\ 10,\ 3,\ 15,\ \ldots$
4. $1,\ 2,\ 8,\ 16,\ 32,\ \ldots$
5. $1,\ 4,\ 9,\ 16,\ 25,\ \ldots$
6. $1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ \ldots$
7. $1,\ 3,\ 6,\ 10,\ 15,\ 21,\ \ldots$
8. $2,\ 3,\ 5,\ 7,\ 11,\ 13,\ \ldots$
9. $3,\ 2,\ 1,\ 0,\ -1,\ \ldots$
10. $1,\ 1,\ 2,\ 6,\ \ldots$

26 Recursion on example of factorial.

Example: Calculate 4! using the recursive definition given in the lecture.

27 Fibonacci sequence and modelling.

Example: For each $n \in \mathbb{N}^+$ determine the number x_n of subsets of the set $A_n = \{1, 2, 3, \ldots, n\}$ that do not contain any pair of consecutive natural numbers.

Note: Definitions of Fibonacci sequence can differ in different sources: enumeration can start from 0 or 1, and initial conditions can be different (the name can then be different), but the recurrence is always the same.

28 Fibonacci sequence: closed formula.

Exercise: The Fibonacci sequence is defined as: $F_0 = 0,\ F_1 = 1,\ F_{n+2} = F_n + F_{n+1}$ for $n \in \mathbb{N}$. Show that

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

Extra material: notes with solved Problem 2.

29 Fibonacci sequence: other examples of modelling.

Two different examples of modelling with help of Fibonacci sequence:

1. Growth of rabbit population (to read yourself; a link is given).
2. *Sanskrit prosody*: In the Sanskrit poetic tradition, there was interest in enumerating all patterns of long (L) syllables of 2 units duration, juxtaposed with short (S) syllables of 1 unit duration. Counting the different patterns of successive L and S with a given total duration results in the Fibonacci numbers: the number of patterns of duration n units is F_{n+1}. (Explained in the video.)

30 Modelling in combinatorics, some examples from our earlier courses.

1. Denote by x_n the number of all permutations of the elements of $\{1, 2, \dots, n\}$ for $n \geqslant 1$. Look at the proof from V263 in DM1 and show that (x_n) satisfies the recursive description $x_1 = 1,\ x_{n+1} = (n+1)x_n$ for $n \geqslant 1$.
2. Denote by x_n the number of all subsets of $\{1, 2, \dots, n\}$ for $n \geqslant 1$. Look at the proof from V265 in DM1 and show that (x_n) satisfies the recursive description $x_1 = 2,\ x_{n+1} = 2x_n$ for $n \geqslant 1$.
3. Read **Example 4.1.6** on pp.315–316 in the DM Book; the recurrence here is very similar to the one given above in (2).

31 Two examples of sequences of two variables.

Two examples from earlier courses:

1. Sequence $a_{n,k} = C(n, k)$ that shows the number of k-subsets of a set with n elements (binomial coefficients defined recursively; DM1 V246).
2. Sequence $a_{n,k} = S(n, k)$ that shows the number of unordered partitions of $[n]$ into k non-empty sets (Stirling numbers defined recursively; DM2 V36).

Extra material: an article with solved Exercise: Use the recursion (Theorem, V36 in DM2) to compute $S(7, 4)$.

32 Building a recursive formula based on a verbal description, the Tower of Hanoi.

Example (**The Tower of Hanoi**; the DM Book, p.311): There is a monastery in Hanoi, as the legend goes, with a great hall containing three tall pillars. Resting on the first pillar were n giant disks (or washers), all different sizes, stacked from largest to smallest. The monks of the monastery have been moving the disks for generations, attempting to move the entire stack of disks to another pillar. However, due to the size of the disks, the monks cannot move more than one at a time. Each disk must be placed on one of the pillars before the next disk is moved. And because the disks are so heavy and fragile, the monks may never place a larger disk on top of a smaller disk. What is the smallest number of moves required to complete the Tower of Hanoi puzzle with n disks (for each $n \geqslant 1$)?

33 Back to the Tower of Hanoi: guess and prove the closed formula.

Exercise: We have shown in V32 that the number a_n of moves needed for solving the Tower-of-Hanoi puzzle with n washers is described as

$$a_1 = 1, \quad a_n = 2a_{n-1} + 1 \quad (n \geqslant 2).$$

Try to guess the closed formula for (a_n) and prove your guess by induction.

Extra material: notes from the iPad.

34 Another one with some modelling.

Example: A person decides to become a millionaire. He began with nothing, and at the end of the first year he ended up with one dollar. After a second year, he had five dollars. Thereafter he made a rule that in the course of each year he would buy goods to the value of six times his net assets at the beginning of the previous year, and sell them for four times the value of his net assets at the beginning of the current year. Find a recursion for a_n (this person's net assets at the end of the nth year). Show that $a_n = 3^n - 2^n$ satisfies this recursion. (In S7, we will learn the method of solving such recursions and then we will learn that this is the only solution to this problem.) Using this formula, compute how many years it did take him to become a millionaire.

Extra material: notes from the iPad.

35 Triangular numbers and telescoping sums.

Example: The sequence of *triangular numbers* (T_n) is defined in the following way:

$$T_1 = 1, \quad T_{n+1} = T_n + n + 1 \quad (\text{for } n \in \mathbb{N}^+).$$

Find a closed formula for T_n using the telescoping technique.

You will get an illustration that explains the name of this sequence.

Extra material: notes with solved Example.

36 A fun problem about triangular numbers and telescoping sums.

Problem: Compute the sum of the reciprocals of the triangular numbers $T_1, T_2, \ldots, T_n$.

You get some references to *Calculus 2, part 2: Sequences and series.*

Extra material: notes with solved Problem.

37 Sequence of differences.

Exercise: Find the sequences of differences $d_n = a_{n+1} - a_n$ and $\rho_n = b_{n+1} - b_n$, where $a_n = n^3$ and $b_n = 3^n$.

Find the sequence of differences $\gamma_n = F_{n+1} - F_n$, where $F_1 = 1,\ F_2 = 1,\ F_n = F_{n-2} + F_{n-1}$ (for $n \geqslant 3$).

Extra material: notes with solved Exercise.

38 Sequence of partial sums.

Examples: The definition of partial sums and an illustration for $a_n = n^2$. The sequence of partial sums of the sequence of differences.

Exercise: Determine the sequence of differences $d_n = a_{n+1} - a_n$, where $a_n = n^2$. Verify that the sequence of partial sums of (d_n) is (S_n) such that $S_n = a_{n+1} - a_1$.

Extra material: notes with solved Exercise.

39 Arithmetic operations on sequences.

* Scaling of a sequence. $k = 2,\ \ a_n = \dfrac{1}{n},\ \ m_n = ka_n = \dfrac{2}{n}$.
* Sum / difference of two sequences. $b_n = (-1)^n \dfrac{1}{n},\ \ d_n = \dfrac{n+1}{n},\ \ m_n = b_n + d_n$.
* Product of two sequences. $e_n = \dfrac{4n}{n+1},\ \ f_n = \dfrac{3n+2}{2n+3},\ \ m_n = e_n f_n$.

40 Vertical and horizontal shifts of sequences.

Examples: Determine and depict the following shifts of $a_n = n^2$: **vertical** $b_n = a_n + 9$, $c_n = a_n - 6$ and **horizontal** $d_n = a_{n+5}$.

Exercise 4.1.7.5 (the DM Book on p.325): Write out the first few terms of the sequence $a_n = n^2 - 3n + 1$. Then find a closed formula for the sequence (starting with b_1): $0, 2, 6, 12, 20, \ldots$.

Example 4.1.8 (the DM Book on p.317–319): To analyse on your own.

41 **Optional**: Vector space of all the sequences numbered with all natural numbers.

We show that the set of all the sequences numbered with all natural numbers, with addition and scaling defined as in V39 (element-wise) and with the zero sequence $(z_n)_{n\in\mathbb{N}}$ (defined as $z_n = 0$ for all $n \in \mathbb{N}$) is a vector space.

Extra material: notes from the iPad.

42 Some examples of periodic sequences (much less relevant than the other stuff).

* The sequence $a_n = n \pmod 5$; DM1, V158
* The sequence $a_n = 3^n \pmod 7$; DM2, V160 and V183
* The sequence $a_n = 4^n \pmod 9$; DM2, V160 and V154.

43 Some identities involving Fibonacci sequence, Problem 1.

A remark: If (a_n) is a sequence and (S_n) is its sequence of partial sums, then $S_{n+1} = S_n + a_{n+1}$ for $n \geqslant 1$. In particular, it means that the sequence of differences of (S_n) is (a_{n+1}).

Problem 1: Find the formula for a partial sum of the sequence of Fibonacci numbers (**Method 1**: make a guess and confirm it with help of mathematical induction; **Method 2**: use a telescoping technique).

Extra material: notes with solved Problem 1.

44 Some identities involving Fibonacci sequence, Problem 2.

Problem 2: Find the formula for a partial sum of the sequence of squares of Fibonacci numbers (make a guess and confirm it with help of mathematical induction; you can get a nice illustration, too [**Problem 4.7.13** on p.388 in the DM Book]). Extra material: notes with solved Problem 2.

45 Some identities involving Fibonacci sequence, Problem 3.

Problem 3: Show that the Fibonacci sequence obeys the rule $F_{n+1}^2 - F_{n+1}F_n - F_n^2 = (-1)^n$ for all $n \in \mathbb{N}$.

Extra material: notes with solved Problem 3.

46 Some identities involving Fibonacci sequence, Problem 4.

Problem 4 (**Exercise 4.6.5.7** on p.383 in the DM Book): Show that the Fibonacci numbers satisfy the identity

$$F_n^2 + F_{n+1}^2 = F_{2n+1}$$

for all $n \geqslant 0$, by treating it as a special case (with $m = n$) of the identity (for all $m \geqslant 0$ and for all $n \geqslant 0$)

$$F_m F_n + F_{m+1} F_{n+1} = F_{m+n+1}.$$

47 A real treat for your soul and eyes: plenty of celebrities in Pascal's Triangle.

* The sequence $a_n = 2^n$ of powers of two is hidden in Pascal's Triangle
* The sequence of triangular numbers $T_n = \frac{n(n-1)}{2}$ (for $n \geqslant 2$) is a part of Pascal's Triangle
* The sequence of tetrahedral numbers is a part of Pascal's Triangle
* Fibonacci sequence is hidden in Pascal's Triangle.

S4 Arithmetic progressions and arithmetic sums

You will learn: the definition and properties of arithmetic progressions; partial sums of an arithmetic progression; monotonicity of sequences (generally, and specifically of arithmetic progressions).

Read along with this section: *DM Book*, Section 4.2: *Rate of Growth* (pp.327–337).

48 Arithmetic progressions, Exercise 1.

Practice Problem 4.2.6.6 (DM Book, p.335): Your summer job pays you \$1000 per week, with a raise of \$50 per week as a bonus for not being a quitter. How much will you make in the 10th week?

Exercise 1: Which of the following sequences are arithmetic progressions? For these sequences, determine a_{10}.

a) $\frac{1}{3},\ 1,\ \frac{5}{3},\ \frac{7}{3},\ \ldots,$

b) $4,\ -1,\ -6,\ -11,\ \ldots,$

c) $9,\ 12,\ 16,\ \ldots.$

Extra material: notes with solved Exercise 1.

49 Arithmetic sums, Exercise 2.

1. Derive the formula for the partial sum S_n for the sequence $a_n = n$;
2. Derive the formula for the partial sum S_n for the sequence $a_n = a + (n-1)d$;
3. **Practice Problem 4.3.6.1** (DM Book, p.349): Consider the sequence $10, 13, 16, 19, 22, \ldots$, where $a_1 = 10$.
   (a) What is the recursive definition for the sequence?
   (b) Give a closed formula for the nth term of the sequence.
   (c) Is 1555 a term in the sequence?
   (d) How many terms does the finite sequence $10, 13, 16, \ldots, 433$ have?
   (e) Find the sum: $10 + 13 + 16 + \ldots + 433$.
   (f) Use what you found above to find b_n, the nth term of $3, 13, 26, 42, 61, \ldots$

Exercise 2: Show that the sum of n consecutive odd integers beginning with 1 is n^2.

Extra material: notes with solved Exercise 2.

50 Monotone sequences, Exercise 3.

Nine examples of sequences; some of them are monotone (decreasing or increasing), other aren't.

Exercise 3: Show with help of a computation that the following sequences are increasing:

$$e_n = \frac{4n}{n+1} \quad \text{and} \quad f_n = \frac{3n+2}{2n+3}.$$

Extra material: notes with solved Exercise 3.

51 Monotonicity of arithmetic progressions.

S5 Geometric progressions and geometric sums

You will learn: the definition and properties of geometric progressions; partial sums of a geometric progression; monotonicity of geometric progressions.

Read along with this section: *DM Book*, Section 4.2: *Rate of Growth* (pp.327–337).

52 Geometric progressions, Exercise 1.

Exercise 1: Which of the following sequences are geometric progressions?:

a) $-\frac{2}{5},\ \frac{4}{5},\ -\frac{8}{5},\ \ldots,$

b) $1,\ 3,\ 6,\ 9,\ \ldots,$

c) $-32,\ 16,\ ,-8,\ 4,\ \ldots.$

For these sequences, write the formula for a_9.

Extra material: notes with solved Exercise 1.

53 Geometric sums, Exercise 2.

1. Derive the formula for the partial sum S_{n+1} for the sequence $a_n = aq^n$ from $n = 0$;
2. Four different ways of writing the sums $3\sum_{k=0}^{9}\left(-\frac{2}{5}\right)^k$.

Example: Compute $-3\cdot\left(\frac{2}{5}\right)^3 + 3\cdot\left(\frac{2}{5}\right)^4 - 3\cdot\left(\frac{2}{5}\right)^5 + \ldots - 3\cdot\left(\frac{2}{5}\right)^9$.

Exercise 2: Compute the following sums:

$$2 + 1 + \frac{1}{2} + \ldots + \frac{1}{128}, \quad e + e^2 + \ldots + e^{10}, \quad 1 - x + x^2 - x^3 + \ldots - x^9.$$

Extra material: notes with solved Exercise 2.

54 Monotonicity of geometric progressions.

We derive a complete description of monotonicity of geometric progressions.

55 The rate of change of a geometric progression.

We illustrate the principle that the rate of change of a geometric progression is proportional to the very progression on examples of $a_n = q^n$ for $q = 2$, $q = 3$, $q = \frac{1}{2}$, and $q = \frac{3}{5}$; we also relate the new results to the ones from V54 and (to some extent) to the content of V22.

56 A cute little problem involving integer numbers.

Exercise 4.2.7.2 (DM Book, p.335): Is there a pair of integers (a, b) such that a, x_1, y_1, b is part of an arithmetic sequence and a, x_2, y_2, b is part of a geometric sequence with x_1, x_2, y_1, y_2 all integers?

Extra material: notes with solved Exercise.

S6 Polynomial sequences

You will learn: how to deal with sequences that have constant sequence of (first, second, third, and so on) differences; you will learn that all such sequences are polynomial sequences and you will learn how to find their explicit formulas.

Read along with this section: *DM Book*, Section 4.3: *Polynomial Sequences* (pp.338–352).

57 Neither arithmetic nor geometric.

58 Relevant videos from S2; polynomial sequences that we have already seen.

Some examples of *polynomial sequences* and their closed formulas: natural numbers, triangular numbers, tetrahedral numbers, and generally: $b_n = \binom{n}{k}$ for a fixed $k \geqslant 2$ and for all $n \geqslant k$.

59 First, second, third, etc, differences versus first, second, third, etc, derivatives.

Some conclusions from the **Remark** in V43 and from the facts presented in V47:

* Triangular numbers (starting with 3) form the sequence of differences for tetrahedral numbers.
* Natural numbers (starting with 2) form the sequence of differences for triangular numbers.

An observation: Given $(a_n)_{n\in\mathbb{N}}$. If $d_n^k = d \neq 0$ for some $k \in \mathbb{N}^+$ and for all $n \in \mathbb{N}$, then:

* $d_n^m = 0$ for all $m \geqslant k+1$ and for all $n \in \mathbb{N}$.
* (d_n^{k-1}) is not constant (for $k = 1$ we define $d_n^0 = a_n$, thus the 0-differences are simply the original sequence).
* **Conclusion**: If (a_n) is Δ^k-constant and $k \geqslant 1$, then k is the **least** natural number such that (d_n^k) is constant.

60 The one with making up sequences.

Additional Exercise 4.3.7.3, p.350 in the DM Book: Make up sequences that have:

(a) 3, 3, 3, 3, 3, ... as its second differences.
(b) 1, 2, 3, 4, 5, ... as its third differences.
(c) 1, 2, 4, 8, 16, ... as its 100th differences.

61 A graphical illustration to the Reverse-and-Add method; some reading recommendations.

Example 4.3.3, p.341 in the DM Book: Find a closed formula for $(a_n)_{n\geqslant 0}$ which starts $2, 3, 7, 14, 24, 37, \ldots$. Assume a recurrence relation for the sequence is $a_n = a_{n-1} + 3n - 2$ and $a_0 = 2$. (This example is solved in the book with the Reverse-and-Add method: study it; I will show a solution based on a telescoping technique.)

Extra material: notes from the iPad.

62 A counting problem with a polynomial sequence as a result.

Example (the DM Book, p.338): A standard 8×8 chessboard contains 64 squares. Actually, this is just the number of unit squares. How many squares of all sizes are there on a chessboard? Start with smaller boards: 1×1, 2×2, 3×3, etc. Find a formula for the total number of squares in an $n \times n$ board. (See Video 68.)

63 The theorem that characterises polynomial sequences.

Exercise: Given any geometric sequence $a_n = q^n$ (where $q \neq 0, 1$). Show that (a_n) is **not** Δ^k-constant for any $k \in \mathbb{N}$. Show that all its sequences of differences are scalings of (a_n) (which is a big similarity with exponential functions $f(x) = a^x$ where $a > 0$ and $a \neq 1$; there $f'(x) = \ln a \cdot a^x$).

Extra material: notes from the iPad.

64 Necessary condition for polynomial sequences.

Examples: The sequence of **pentatope** numbers is Δ^4-constant. The sequence (Te_n) of **tetrahedral** numbers is Δ^3-constant. The sequence (T_n) of **triangular** numbers is Δ^2-constant. The sequence $a_n = n$ of **natural** numbers is Δ^1-constant.

Exercise: Show with a computation that the sequence $a_n = n^k$ is Δ^k-constant for $k = 1, 2, 3$. The facts derived in the next video will show that this (together with mathematical induction) will prove the necessary condition for polynomial sequences.

Extra material: notes from the iPad.

65 Linearity of the operation of taking the differences of consecutive terms.

Facts: The difference of a scaling (of a sequence) is equal to the scaling of the difference (of this sequence). The difference of a sum (of two sequences) is equal to the sum of the differences (of these sequences); this is naturally generalised to the sum of *any* number of sequences (by induction). Both facts combined together give: The difference of a linear combination (of two or more sequences) is equal to the linear combination of the differences (of these sequences). As differences of higher order (second, third, and so on) are also differences of sequences, these rules hold for the differences of any order. (**The same rules hold for derivatives; see Additional Exercise 4.5.7.23 on p.375 in the DM Book.**)

Extra material: notes from the iPad.

66 Sufficient condition for polynomial sequences.

A generalisation of the sum-of-powers formulas: Sketch a proof (strong induction in combination with Binomial Theorem) that for each $m \in \mathbb{N}^+$

$$S_n^{(m)} = \sum_{k=1}^{n} k^m = 1^m + 2^m + 3^m + \ldots + (n-1)^m + n^m$$

is a polynomial (of variable $n \in \mathbb{N}^+$) of degree $m+1$.

Sufficient condition: Sketch a proof (motivation) showing that if the sequence (d_n^k) of kth differences for a sequence (a_n) is a constant (non-zero) sequence, then (a_n) is a degree-k polynomial of variable n.

Extra material: notes from the iPad.

67 Ansatz and undetermined coefficients.

Example 4.3.6, p.344 in DM Book: Find a closed formula for the sequence: 3, 7, 14, 24, ... Assume $a_1 = 3$.

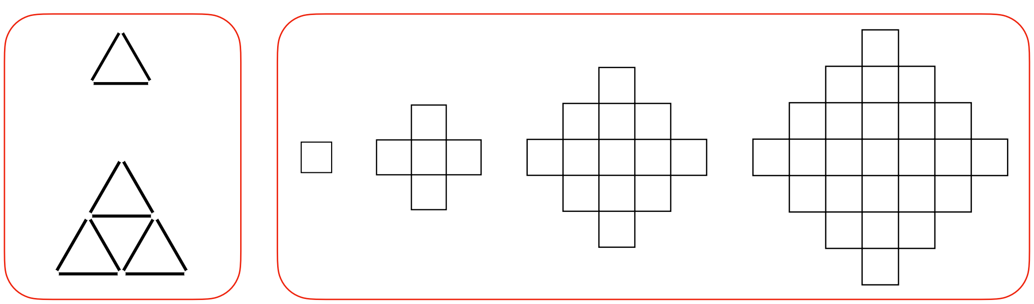
Extra material: notes from the iPad.

68 A bunch of problems with the same main determinant.

We revisit the square-counting problem from V62, and you get a bunch of problems to solve on your own.

69 The one with the toothpicks.

Additional Exercise 4.3.7.7, p.351 in the DM Book: If you have enough toothpicks, you can make a large triangular grid. Below, there are the triangular grids of size 1 and of size 2. The size 1 grid requires 3 toothpicks, the size 2 grid requires 9 toothpicks.



Figur 1: A bunch of toothpicks (for V69) and plenty of unit squares (for V70).

(a) Let t_n be the number of toothpicks required to make a size n triangular grid. Write out the first 4 terms of the sequence $t_1, t_2, \ldots$.

(b) Find a recursive definition for the sequence. Explain why you are correct.

(c) Is the sequence arithmetic or geometric? If not, is it the sequence of partial sums of an arithmetic or geometric sequence? Explain why your answer is correct.

(d) Use your results from part (c) to find a closed formula for the sequence. Show your work.

70 The one with plenty of unit squares.

Additional Exercise 4.3.7.8, p.351 in the DM Book: If you were to shade in an $n \times n$ square on graph paper, you could do it the boring way (with sides parallel to the edge of the paper) or the interesting way, as illustrated in the picture (in the text for V69).

The interesting thing here is that a 3×3 square now has area 13. Our goal is to find a formula for the area of an $n \times n$ (diagonal) square.

(a) Write out the first few terms of the sequence of areas (assume $a_1 = 1,\ a_2 = 5$, etc). Is the sequence arithmetic or geometric? If not, is it the sequence of partial sums of an arithmetic or geometric sequence? Explain why your answer is correct, referring to the diagonal squares.

(b) Use your results from part (a) to find a closed formula for the sequence. Show your work. Note that while there are lots of ways to find a closed formula here, you should use partial sums specifically.

(c) Find the closed formula in as many other interesting ways as you can. (**Method 1**: by partial sums, **Method 2**: graphical, **Method 3**: by undetermined coefficients.)

S7 Solving linear recurrence relations

You will learn: solving linear recurrence relations of order k with constant coefficients (mainly homogenous, but you will also see some simple examples of non-homogenous ones); some examples of counting problems that are modelled by linear recurrences; solution sets of linear recurrences with constant coefficients as subspaces of vector spaces of all sequences numbered from index 0; linearly independent solutions (optional).

Read along with this section: *DM Book*, Section 4.4: *Exponential Sequences* (pp.353–362).

71 Two main actors in this section; read Section 4.4.2 in the DM Book.

72 Our earlier examples that will be revisited in this section.

73 Some notation and terminology.

All the parts explained: A kth order linear recurrence relation with constant coefficients, homogenous or non-homogenous, with or without initial conditions/values.

74 Terminology, an exercise.

Which of the following recurrence relations are linear and with constant coefficients? Determine their order and all the coefficients:

1. $x_{n+2} = 2x_{n+1} - x_n$
2. $3x_{n+1} + x_n - 5x_{n+2} = 2x_{n+1} - x_n$
3. $x_{n+1} - x_n x_{n+2} = 1$
4. $x_{n+2} + 4x_n = n^2$ ($x_{n+2} + 4x_{n+1} = n^2$)
5. $5x_n^2 = x_n + x_{n+1}$
6. $nx_{n+1} + x_n - 5x_{n+2} = 8$
7. $9x_{n+1} + x_n - 5x_{n+2} = 8x_{n+1}$
8. $x_n - 5x_{n+5} + 7x_{n+1} = 3x_{n+4}$
9. $x_n + x_{n+1} + x_{n+2} + x_{n+4} = 5$
10. $\dfrac{x_n}{n} - 5x_{n+1} = x_{n+2}$
11. $\dfrac{x_n}{3} - 5x_{n+1} = x_{n+2}$
12. $3x_n + 5x_{n+2} = x_{n+2} - 7x_{n+1} + 5^n$

Which of the ones you have selected are homogenous and which ones non-homogenous?

75 First order homogenous linear recurrences, Example 1.0.

Example 1.0: Solve the first-order linear recurrence $x_{n+1} = qx_n$ for a given $q \in \mathbb{R}$. (Formulate a hypothesis and prove it by induction.) What is the general solution? What do you need to do in order to get a unique solution? What happens if $q = 0$?

Extra material: notes with solved Example 1.0.

76 Verifying recurrence relations, Example 1.1.

Example 1.1: Verify that $x_n = 2^n + 1$ satisfies the recurrence relation $x_n = 2x_{n-1} - 1$ with $x_1 = 3$.

Extra material: notes with solved Example 1.1.

77 Using iterations for getting the formula, Example 1.2.

Example 1.2: Solve the recurrence relation $x_n = x_{n-1} + n$ with the initial condition $x_0 = 4$.

Extra material: notes with solved Example 1.2.

78 Telescoping techniques, Example 1.3.

Example 1.3: Solve the recurrence relation $x_n = x_{n-1} + n$ with the initial condition $x_0 = 4$ using telescoping.

Extra material: notes with solved Example 1.3.

79 The prerequisites from S2.

80 Linear recursion of order 2: an introduction, Example 2.0.

Example 2.0: Define $x_n = 5x_{n-1} - 6x_{n-2}$. Write down the first five terms of this sequence and try to find a closed formula for x_n if:

* $x_0 = 1$, $x_1 = 2$,
* $x_0 = 1$, $x_1 = 3$,
* $x_0 = 2$, $x_1 = 5$.

Extra material: notes with solved Example 2.0.

81 Why we need polynomials in this section; characteristic equations (part 1).

We look at two earlier examples:

* The one from V28: Fibonacci sequence $x_0 = 1,\ x_1 = 1,\ x_{n+1} = x_n + x_{n-1}$ for $n \geqslant 2$.
* The one from V34: $x_1 = 1,\ x_2 = 5,\ x_{n+1} = 5x_n - 6x_{n-1}$ for $n \geqslant 2$.

82 Why we need polynomials in this section; characteristic equations (part 2).

Exercise: Write the characteristic equations to all the linear recurrences with constant coefficients that we identified in V74.

83 **Optional**: Continuous versus discrete, linear ODE versus linear recurrences.

84 Some very nice counting problems that are solved with help of our type of recurrences.

Investigate Problem on p.353 in the DM Book: You have a large collection of 1×1 squares and 1×2 dominoes. You want to arrange these to make a 1×15 strip. How many ways can you do this? What if the squares come in three different colors and the dominos come in four different colors? And why is this second question easier than the first? **Hint**. Start by creating a recurrence relation. How are the different 1×3 strips and 1×4 strips related to the 1×5 strips? (A similar problem will be solved in V91.)

85 Linear recursion of order 2: the meaning of linearity, Example 2.1.

Example 2.1: Define $x_n = a \cdot x_{n-1} + b \cdot x_{n-2}$, where $|a| + |b| > 0$ and $a^2 + 4b > 0$. The equation

$$r^2 - ar - b = 0$$

is called *the characteristic equation* (V81) of the recursion above. Do the following:

* show that if $x_n = q^n$ satisfies the recursive relation above, then q is a root of its characteristic equation,

* show that if (x_n) and (y_n) are two solutions to the recursive relation above, and α and β are any real numbers, then $z_n = \alpha x_n + \beta y_n$ also satisfies the same recursive relation. **Note**: This is the second meaning of the concept *linearity of the relation*: if two sequences satisfy the relation, then each linear combination of these sequences also satisfies the same relation. The first meaning was: each element of the sequence (except the first two elements) is a linear combination of two elements preceding it.
* write the characteristic equation for the recursion from Video 80: $x_n = 5x_{n-1} - 6x_{n-2}$ and explain the results from Video 80 in the light of our new findings.

Extra material: notes with solved Example 2.1.

86 Linear recursion of order 2: derivation of the closed formula.

Given sequence (F_n) defined as follows: $F_{n+2} = a \cdot F_{n+1} + b \cdot F_n$ for $n \geqslant 0$, where:

[A1] $|a| + |b| > 0$ (i.e., at least one of them must be non-zero)

[A2] $a^2 + 4b > 0$ (you will soon discover why!)

and F_0 and F_1 are given. Establish a closed formula for F_n for all natural n.

Extra material: an article with a derivation of the closed formula.

87 Linear recursion of order 2, Exercise 1.

Exercise 1: Define $x_n = x_{n-1} + 6x_{n-2}$ for $n \geqslant 2$. Find a closed formula for (x_n).

Extra material: notes with solved Exercise 1.

88 Linear recursion of order 2, with an initial condition, Exercise 2.

Exercise 2: Solve the recurrence relation $x_n = 7x_{n-1} - 10x_{n-2}$ with the initial conditions $x_0 = 2$ and $x_1 = 3$.

Extra material: notes with solved Exercise 2.

89 Linear recursion of order 2, with an initial condition, Exercise 3.

Exercise 3: Solve the recurrence relation $x_n = 2x_{n-1} + x_{n-2}$ with the initial conditions $x_0 = 1$ and $x_1 = 2$.

Extra material: notes with solved Exercise 3.

90 Back to Fibonacci, Exercise 4.

Exercise 4: Confirm the result obtained in V28 (closed formula for Fibonacci sequence).

Extra material: notes with solved Exercise 4.

91 Back to the problem from V84, Exercise 5.

Exercise 5 (**Additional Exercise 4.4.6.6** on p.361 in the DM Book): Let x_n be the number of $1 \times n$ tile designs you can make using 1×1 squares available in 4 colors and 1×2 dominoes available in 5 colors.

(a) First, find a recurrence relation to describe the problem. Explain why the recurrence relation is correct (in the context of the problem).

(b) Write out the first 4 terms of the sequence $x_1, x_2, \ldots$

(c) Solve the recurrence relation. That is, find a closed formula for x_n.

Extra material: notes with solved Exercise 5.

92 The case of a double root, Example 2.2.

Example 2.2: If a homogenous second-order recurrence relation with constant coefficients has a characteristic equation with a double root (We can assume WLOG that the relation is

$$x_{n+2} - 2ax_{n+1} + a^2 x_n = 0.)$$

then all the solutions are given by $x_n = C_1 a^n + C_2 n a^n$ for some $C_1, C_2 \in \mathbb{R}$. Confirm with a computation that $y_n = a^n$ and $z_n = na^n$ satisfy the recurrence.

Extra material: notes with solved Example 2.2.

93 Second-order linear recurrences with constant coefficients, homogenous, a recipe.

94 What about higher orders?

The theorem is formulated for linear recurrences of order $k \geqslant 2$, with constant coefficients, homogenous, and illustrated with **Example**: Characteristic polynomial $p(r) = (r-2)(r-3)^4(r-5)^2$ (deg 7, order 7 of RR).

A proof of the theorem is sketched for $k = 3$, for the case where all the roots of the characteristic equation are distinct real numbers. As I use **Vieta's formulas for third-degree polynomials**, you also get their proof.

Proofs of other versions are outside the scope of this course (most of the proofs are based on Linear Algebra); also complex roots are outside the scope of this course (require Trigonometry and Complex Numbers).

95 **Optional**: A fun explanation of the case of multiple roots for characteristic equation.

A verification (with help of derivatives) that if r_j is a multiple root of the characteristic equation of a linear recurrence with constant coefficients

$$x_{n+k} - a_{k-1}x_{n+k-1} - a_{k-2}x_{n+k-2} - \ldots - a_1x_{n+1} - a_0x_n = 0$$

then, except $x_n = r_j^n$, also $y_n = nr_j^n$ is a solution to the same recurrence and, if the multiplicity of the root is equal to $\alpha_j > 2$, so are $n^2r_j^n, \ \ldots, \ n^{\alpha_j-1}r_j^n$.

96 Homogenous case, an example from V17.

Example 3: Solve the linear homogenous recurrence $x_{n+3} = 3x_{n+2} - x_{n+1} - 2x_n$.

Extra material: notes with solved Example 3.

97 Homogenous case, another example from V17, and plenty of additional exercises.

Example 4: Solve the linear homogenous recurrence

$$x_{n+4} = 5x_{n+3} - 6x_{n+2} - 4x_{n+1} + 8x_n$$

with initial conditions $x_0 = 0, \ x_1 = -9, \ x_2 = -1, \ x_3 = 21$.

Extra material: notes with solved Example 4.

Extra material: an article with more solved problems on linear recurrences. (In these examples we use the computations performed in the article attached to V17.) In all the problems below, solve the linear recurrence:

- ⋆ **Extra problem 1**: $x_{n+3} = 5x_{n+2} - 8x_{n+1} + 4x_n$
- ⋆ **Extra problem 2**: $x_{n+4} = 2x_{n+3} + 2x_{n+2} - 6x_{n+1} + 3x_n$
- ⋆ **Extra problem 3**: $x_{n+3} = 10x_{n+2} - 32x_{n+1} + 32x_n$
- ⋆ **Extra problem 4**: $x_{n+3} = 9x_{n+2} - 27x_{n+1} + 27x_n$.

98 Non-homogenous case: qualified guesses.

We learn how to determine the solution set to the *non-homogenous* (nH) recurrence relation:

$$x_{n+k} - a_{k-1}x_{n+k-1} - a_{k-2}x_{n+k-2} - \ldots - a_1x_{n+1} - a_0x_n = f(n), \quad n \in \mathbb{N} \qquad (nH).$$

All the solutions are of the shape $x_n = x_n^{(p)} + x_n^{(h)}$ where $x_n^{(p)}$ is a *particular solution* to (nH) and $x_n^{(h)}$ is any *homogenous solution* to the homogenous part of (nH) denoted by (H).

Why is it so? Fix any solution $x_n^{(p)}$ to (nH). Then:

- ∗ If $x_n^{(h)}$ solves (H) then $x_n^{(p)} + x_n^{(h)}$ solves (nH).
- ∗ If y_n solves (nH) then $y_n - x_n^{(p)}$ solves (H).

Extra material: notes from the iPad.

99 Non-homogenous case, Exercise 6.

Exercise 6: Find an explicit formula for the term of sequence (x_n) defined by the recursion:

$$x_0 = 0, \quad x_1 = 1, \quad x_{n+2} - x_{n+1} - 6x_n = n \quad (n \geqslant 0).$$

This example will be revisited in Section 8, when we will use the result obtained in V19.

Extra material: notes with solved Exercise 6.

100 Non-homogenous case, Exercise 7.

Exercise 7: Solve the recursion:

- $x_{n+3} = 4x_{n+2} - 5x_{n+1} + 2x_n$ with initial values $x_0 = 0,\ x_1 = 2,\ x_2 = 3$
- $x_{n+3} = 4x_{n+2} - 5x_{n+1} + 2x_n + 3^n$
- $x_{n+3} = 4x_{n+2} - 5x_{n+1} + 2x_n + 6$
- What would you need to do if the RHS were 2^n?

Extra material: notes with solved Exercise 7.

Extra material: an article with more solved problems on non-homogenous recurrence relations.

- ⋆ **Extra problem 1**: Find a particular solution to $x_{n+2} - 5x_n = 2 \cdot 3^n$ for $n \geqslant 0$. (Also, find the entire solution set for this recurrence relation.)
- ⋆ **Extra problem 2**: Find a particular solution to $x_{n+1} - 2x_n + 3x_{n-4} = 6n$ for $n \geqslant 4$. (Here you won't be able to find the solution set; details are presented in the slides.)
- ⋆ **Extra problem 3**: Find the particular solution to $x_{n+2} - 5x_{n+1} + 6x_n = 2 \cdot 4^n$ for $n \geqslant 0$, with the initial conditions $x_0 = 3,\ x_1 = 9$.
- ⋆ **Extra problem 4**: Find the particular solution to $x_n - 2x_{n-1} = 2^n$ for $n \geqslant 1$, with $x_0 = 0$.
- ⋆ **Extra problem 5**: Find all the solutions to $x_n = 6x_{n-1} - 11x_{n-2} + 6x_{n-3} + 1$ for $n \geqslant 3$.
- ⋆ **Extra problem 6**: Find the particular solution to $x_{n+3} - 7x_{n+2} + 16x_{n+1} - 12x_n = 4^n \cdot n$ for $n \geqslant 0$, with the initial conditions $x_0 = -2,\ x_1 = 0,\ x_2 = 5$.

101 **Optional**: Some Linear-Algebra related remarks.

We look at our results in the light of Linear Algebra: Given a linear recurrence of order k, with constant coefficients, homogenous, and such that its characteristic equation has k distinct real roots. I claim that all the solutions $x_n = r_j^n$ (for $j = 1, 2, \ldots, k$) are linearly independent, i.e., they generate a subspace of dimension k in the space of all real-number sequences that start from 0 (see V8, V20, V21, and V41).

- Show that if $r_i \neq r_j$ are two real numbers different than zero, then the sequences $x_n = r_i^n$ and $y_n = r_j^n$ are linearly independent.
- Show that if r_i, r_j, r_m are three distinct real numbers different than zero, then the sequences $x_n = r_i^n$, $y_n = r_j^n$, and $z_n = r_m^n$ are linearly independent.

A concept related to the solution set of a non-homogenous recurrence: an *affine space*.

Extra material: notes from the iPad.

102 Modelling, solving recurrences, and interpreting the result, Problem 1.

Problem 1: For each $n \in \mathbb{N}^+$ define two sets: A_n is the set of all the positive integers with n digits (in decimal system) and X_n is the subset of A_n containing only numbers that have an odd number of zeros in their decimal representation. Determine the quotient x_n/a_n, where $x_n = |X_n|$ and $a_n = |A_n|$.

Extra material: notes with solved Problem 1.

103 The last one: a really cool problem from the DM Book, Problem 2.

Problem 2 (**Exercise 4.4.6.10** on p.361 in the DM Book): Here is a surprising use of sequences to answer a counting question: How many license plates consist of 6 symbols, using only the three numerals 1, 2, and 3 and the four letters a, b, c, and d, so that no numeral appears after any letter? For example, $31ddac$, 123321, and $ababab$ are each acceptable license plates, but $13ba2c$ is not.

(a) First answer this question by considering different cases: how many of the license plates contain no numerals? How many contain one numeral, etc.

(b) Now use the techniques of this section to show why the answer is $4^7 - 3^7$.

Extra material: notes with solved Problem 2.

S8 Generating functions

You will learn: the concept of a generating function for a sequence, with some arithmetic rules for working with it, some examples of generating functions of various sequences, and two examples of application.

Read along with this section: *DM Book*, Section 6.1: *Generating Functions* (pp.421–431).

104 Generatingfunctionology: yes, it is a word!

105 Arithmetic operations and what they do to generating functions.

Computational rules: If $(a_n)_{n=0}^{\infty}$ has generating function $A(x)$ and $(b_n)_{n=0}^{\infty}$ has generating function $B(x)$ then:

- The *sum* $A(x)+B(x)$ is the generating function of the sequence $s_n = a_n + b_n$.
- The *composition* $A(x^2)$ is the generating function of the sequence $a_0, 0, a_1, 0, a_2, 0, a_3, 0, a_4, 0, \ldots$.
- Function $\dfrac{A(x)-a_0}{x}$ is the generating function of the (*shifted*) sequence $a_1, a_2, a_3, \ldots$.
- The *product* $A(x)B(x)$ is the generating function of the sequence $c_n = \sum\limits_{k=0}^{n} a_k b_{n-k}$.
- (**Optional**) The *derivative* $A'(x)$ is the generating function of the sequence $a_1, 2a_2, 3a_3, 4a_4, \ldots$.

106 Generating functions of some types of sequences.

The following list shows generating functions for certain types of sequences:

- The *constant* sequence $1,\ 1,\ 1,\ 1,\ 1,\ \ldots$ has generating function $A(x) = \dfrac{1}{1-x}$.
- The *alternating* sequence $1,\ -1,\ 1,\ -1,\ 1,\ \ldots$ has generating function $A(x) = \dfrac{1}{1+x}$.
- An *alternating* sequence $1,\ 0,\ 1,\ 0,\ 1,\ \ldots$ has generating function $A(x) = \dfrac{1}{1-x^2}$.
- The *geometric* sequence $1,\ q,\ q^2,\ q^3,\ q^4,\ \ldots$ has generating function $A(x) = \dfrac{1}{1-qx}$.
- The sequence of *positive natural numbers* $1,\ 2,\ 3,\ 4,\ 5,\ \ldots$ has generating function $A(x) = \dfrac{1}{(1-x)^2}$.

107 Recurrence relations and generating functions, Example 0.

Example 0 (**Example 6.1.6** on p.428 in the DM Book): Solve the recurrence relation

$$a_n = 3a_{n-1} - 2a_{n-2} \qquad (n \geqslant 2)$$

with initial conditions $a_0 = 1,\ a_1 = 3$. Compare two methods: 1. as in S7, 2. with help of generating functions.

Extra material: notes with solved Example 0.

108 Recurrence relations and generating functions, Example 1.

Example 1: Determine the generating function for the (somewhat adapted) Fibonacci sequence

$$F_0 = 1, \quad F_1 = 1, \quad F_{n+2} - F_{n+1} - F_n = 0 \quad (n \geqslant 0).$$

Extra material: notes with solved Example 1.

109 Recurrence relations and generating functions, Example 2.

Example 2: Find an explicit formula for the term of sequence (a_n) defined by the recursion:

$$a_0 = 0, \quad a_1 = 1, \quad a_{n+2} - a_{n+1} - 6a_n = n \quad (n \geqslant 0).$$

This example was solved in V99; in this solution we use the result obtained in V19.

Extra material: notes with solved Example 2.

110 A fun counting problem, Problem 1.

Problem 1: How many integer-number solutions to

$$y_1 + y_2 + y_3 = n$$

satisfy $0 \leqslant y_1 \leqslant 2$, $1 \leqslant y_2 \leqslant 3$, and $1 \leqslant y_3 \leqslant 2$? (Compare to V33 and V40 in DM2.)

S9 Some fun problems about sequences

You will learn: OK, this section is just for fun: it shows you some really cool problems about sequences.

111 Fun Problem 1.

Problem 1: Find the formula for the partial sums for $a_n = \dfrac{1}{n+n^2}$.

112 Fun Problem 2.

Problem 2: Given sequence (a_n) defined as

$$a_1 = \frac{1}{2}, \quad a_n = \frac{a_{n-1}}{2na_{n-1}+1} \quad \text{for } \ n \geqslant 2.$$

Find the formula for the partial sums for (a_n).

Extra material: notes with solved Problem 2.

113 Fun Problem 3.

Problem 3: What is the 22nd positive integer n such that 22^n ends with 2 in the decimal system?

Extra material: notes with solved Problem 3.

114 Fun Problem 4.

Problem 4: The sequence (a_n) starts with $a_1 = 49$. For $n \geqslant 1$, the term a_{n+1} is obtained by adding 1 to the sum of the digits of a_n and then squaring the result. What is the value of a_{2026}?

Extra material: notes with solved Problem 4.

115 Fun Problem 5.

Problem 5: Sequence (a_n) is defined by the formula $a_n = 5n + 4$ for $n \in \mathbb{N}^+$. Determine the number of perfect squares in the set $\{a_k;\ 1 \leqslant k \leqslant 2026\}$.

Extra material: notes with solved Problem 5.

116 Fun Problem 6.

Problem 6: Prove that for all $n \in \mathbb{N}^+$

$$\sum_{k=1}^{n}(k^5 + k^7) = 2\left(\sum_{k=1}^{n} k\right)^4.$$

Extra material: notes with solved Problem 6.

117 Fun Problem 7.

Problem 7: Find a closed formula for the sequence (a_n) that is recursively defined by:

$$a_1 = 1, \quad a_{n+1} = \frac{3}{n}(a_1 + a_2 + \ldots + a_n) \quad (n \in \mathbb{N}^+).$$

Extra material: notes with solved Problem 7.

118 Fun Problem 8.

Problem 8: Let D_n (for $n \in \mathbb{N}^+$) denote the number of all derangements (permutations without any fixed points; see V42, V43, and V97 in DM2) of the set $\{1, 2, \ldots, n\}$. Motivate (a combinatorial proof) that the sequence (D_n) satisfies the recursion $D_1 = 0,\ D_2 = 1,\ D_n = (n-1)(D_{n-1} + D_{n-2})$ for $n \geqslant 3$.

Extra material: notes with solved Problem 8.

Extra material: an article with direct verification of the formula (from the closed formula for D_n derived and discussed in Videos 42, 43, and 97 in DM2).

119 Fun Problem 9.

Problem 9: Numbers α and β are the roots of the quadratic equation $x^2 - x - 1 = 0$. We define

$$S_n = 2026\alpha^n + 2027\beta^n \quad \text{for} \quad n \geqslant 0.$$

Express S_{n+2} in terms of S_n and S_{n+1} for $n \geqslant 0$.

Extra material: notes with solved Problem 9.

120 Fun Problem 20.

Problem 10: Compute φ^{12}, where $\varphi = \dfrac{1+\sqrt{5}}{2}$. (Hint: Show that $\varphi^n = F_n\varphi + F_{n-1}$, where (F_n) is the Fibonacci sequence $F_1 = 1,\ F_2 = 1,\ F_{n+2} = F_{n+1} + F_n$ for $n \in \mathbb{N}^+$.)

S10 A brief introduction to Graph Theory

You will learn: you get a very brief and elementary introduction to Graph Theory; for this one, I recommend reading Chapter 2 from the DM Book, as the concepts are quite easy to grasp (even though the theory itself is surprisingly complicated!) and I will concentrate on some more difficult parts in the video lectures and, as always, I'll deliver plenty of illustrations; this should give you a decent preparation to a future serious course devoted entirely to Graph Theory (I have no plans to create such a course, though, so you will have to look somewhere else for it...).

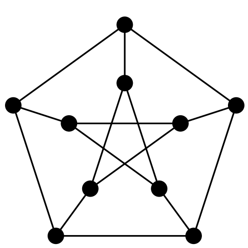
Read along with this section: *DM Book*, Chapter 2 (pp.99–190).

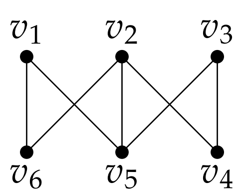
121 An unusual section: a soft introduction to a wonderful but rough subject.

122 Nine great names in Graph Theory.

123 P1: Some typical ways of representing graphs.

Preview Activity 2.1.1, p.101 in the DM Book: Retrieve as much information as possible from the following descriptions of graphs G_k ($k = 1, 2, 3, 4$). Try to find the simplest way of drawing the graphs G_2, G_3, and G_4.





vertex	adjacent to
a	b, c
b	a, f
c	a, d, e
d	c, e, f
e	c, d
f	b, d

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Figur 2: **(V123)**: Preview Activity 2.1.1, graphs G_1, G_2, G_3, and G_4.

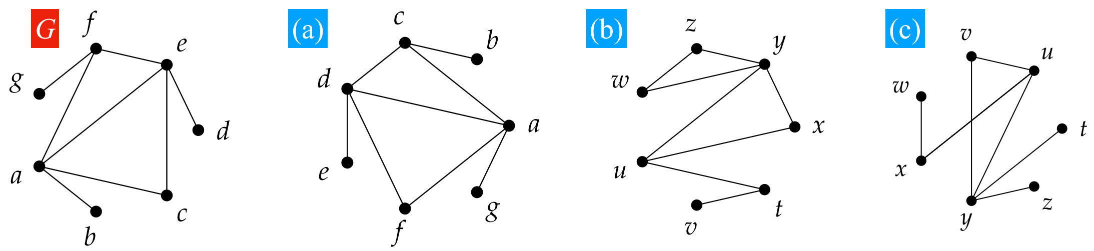
Extra material: notes from the iPad.

124 P1: A formal definition of a graph, subgraphs.

We look together at three graphical representations of the same graph (p.103) and at **Example 2.1.7** on p.107 in the DM Book (that shows the difference between subgraph and induced subgraph).

125 P1: Isomorphic graphs, Example 1.

Practical Problem 2.1.4.1, p.112 in the DM Book: Consider the graph G below.



Figur 3: **(V125)**: Practical Problem 2.1.4.1.

Which of the following graphs are isomorphic to G? Select all that apply.

126 P1: Isomorphic graphs, Example 2.

Additional Exercise 2.1.5.5 on p.113 in the DM Book: Consider the following two graphs:

G_1 : $\quad V_1 = \{a, b, c, d, e, f, g\}, \quad E_1 = \{\{a, b\}, \{a, d\}, \{b, c\}, \{b, d\}, \{b, e\}, \{b, f\}, \{c, g\}, \{d, e\}, \{e, f\}, \{f, g\}\}$

G_2 : $\quad V_2 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\},$

. $\quad E_2 = \{\{v_1, v_4\}, \{v_1, v_5\}, \{v_1, v_7\}, \{v_2, v_3\}, \{v_2, v_6\}, \{v_3, v_5\}, \{v_3, v_7\}, \{v_4, v_5\}, \{v_5, v_6\}, \{v_5, v_7\}\}$

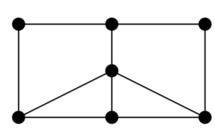
(a) Let $f : G_1 \to G_2$ be a function that takes the vertices of Graph 1 to vertices of Graph 2. The function is given by the following table:

x	a	b	c	d	e	f	g
$f(x)$	v_4	v_5	v_1	v_6	v_2	v_3	v_7

Does f define an isomorphism between Graph 1 and Graph 2?

(b) Define a new function g (with $f \neq g$) that defines an isomorphism between Graph 1 and Graph 2.

(c) Is the graph pictured below isomorphic to Graph 1 and Graph 2? Explain.



Figur 4: **(V126)**: Additional Exercise 2.1.5.5.

Extra material: notes with solved Example 2.

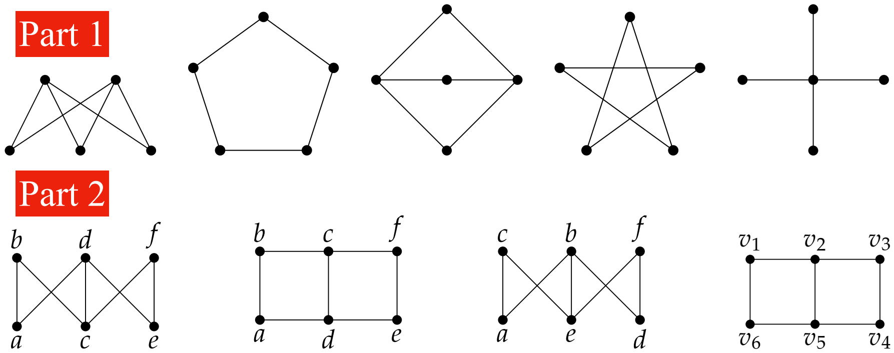
127 P1: Isomorphic graphs, Example 3.

Additional Exercise 2.1.5.16 on p.115 in the DM Book: Which (if any) of the graphs below (Part 1) are the same? These graphs are unlabelled. Usually we think of a graph as having a specific set of vertices. Which (if any) of the graphs below (Part 2) are the same?
Actually, all the graphs in the picture are just *drawings* of graphs. A graph is really an abstract mathematical object consisting of two sets V and E, where E is a set of 2-element subsets of V. Are the graphs below the same or different?

G_1 : $\quad V_1 = \{a, b, c, d, e\}, \quad E_1 = \{\{a, b\}, \{a, c\}, \{a, d\}, \{a, e\}, \{b, c\}, \{d, e\}\}$

G_2 : $\quad V_2 = \{v_1, v_2, v_3, v_4, v_5\}, \quad E_2 = \{\{v_1, v_3\}, \{v_1, v_5\}, \{v_2, v_4\}, \{v_2, v_5\}, \{v_3, v_5\}, \{v_4, v_5\}\}.$

Extra material: notes with solved Example 3.



Figur 5: **(V127)**: Additional Exercise 2.1.5.16.

128 P1: A more theoretical problem to solve.

Additional Exercise 2.1.5.11 on p.114 in the DM Book: We often define graph theory concepts using set theory. For example, given a graph $G = (V, E)$ and a vertex $v \in V$, we define

$$N(v) = \{u \in V;\ \ \{v, u\} \in E\}.$$

We define $N[v] = N(v) \cup \{v\}$. The goal of this problem is to figure out what all this means.

(a) Let G be the graph with vertices V and edges E given by:
$V = \{a, b, c, d, e, f\}, \quad E = \{\{a, b\}, \{a, e\}, \{b, c\}, \{b, e\}, \{c, d\}, \{c, f\}, \{d, f\}, \{e, f\}\}$.
Find $N(a)$, $N[a]$, $N(c)$, and $N[c]$.

(b) What are the largest and smallest possible values for $|N(v)|$ and $|N[v]|$ (the sizes of these sets) for the graph in part (a)? Explain.

(c) Give an example of a graph $G = (V, E)$ (probably different from the one above) for which $N[v] = V$ for some vertex $v \in V$. Is there a graph for which $N[v] = V$ for *all* $v \in V$? Explain.

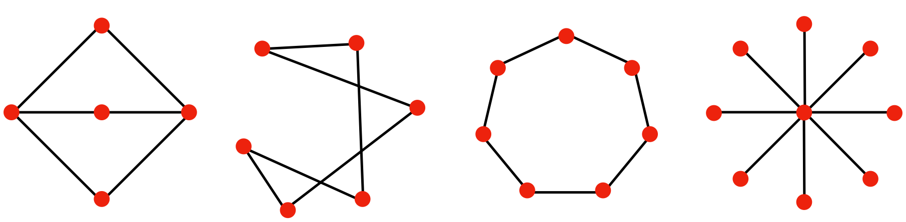
(d) Give an example of a graph $G = (V, E)$ for which $N(v) = \emptyset$ for some vertex $v \in V$. Is there an example of such graph for which $N[u] = V$ for some other $u \in V$ as well? Explain.

(e) Describe in words what $N(v)$ and $N[v]$ mean in general.

Extra material: notes with solved Problem.

129 P1: Bipartite graphs, an exercise.

Additional Exercise 2.1.5.7, p.114 in the DM Book: Which of the graphs below are bipartite? Justify your answers.



Figur 6: **(V129)**: Additional Exercise 2.1.5.7.

Conclusion: All the graphs C_{2n} are bipartite, but the C_{2n+1} are **not** bipartite.

130 P2: Playing with trees.

Practice Problem 2.2.6.5: A tree contains some number of leaves (degree-1 vertices) and four non-leaf vertices. The degrees of the non-leaf vertices are 8, 6, 5, and 3. How many leaves does the tree have?

131 P2: More playing with trees.

Proposition: If a finite tree has a vertex of degree $k \geqslant 2$, then it has at least k leaves.

132 P2: Spanning trees.

For which $n \in \mathbb{N}^+$ is there a graph with exactly n different (but possibly isomorphic) spanning trees?

133 P2: Who is the *minimal criminal* and what is he doing here?

134 P3: Proving or disproving planarity.

Induction proof of *Euler's Formula for Planar Graphs.*

Disproving planarity using a really heavy artillery: Show that the graph $K_{m,n}$ is planar iff $\min\{m, n\} \leqslant 2$.

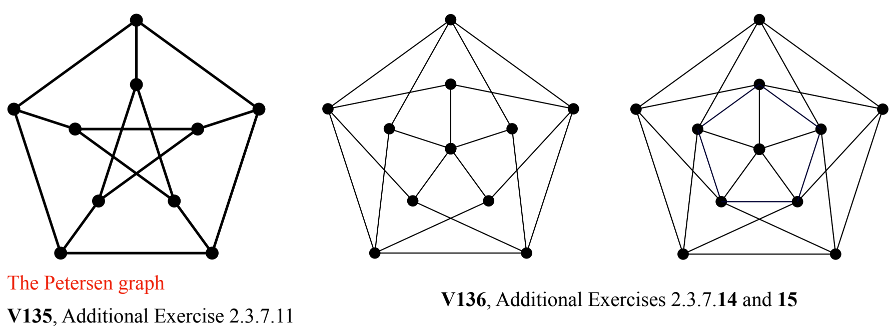
135 P3: Proving or disproving planarity, an exercise.

Additional Exercise 2.3.7.11, p.139 in the DM Book: Prove that the **Petersen graph** (below) is not planar.

Extra material: notes with solved Exercise.

136 P3: Disproving planarity, two exercises, two methods.

Additional Exercises 2.3.7.14 and 15, p.139 in the DM Book: Give a careful proof that the second graph in the picture below is not planar.



Figur 7: Graphs for **V135** (the first one) and for **V136** (the second and the third ones).

Explain why we cannot use the same sort of proof we did in Exercise 2.3.7.14 to prove that the third graph in the picture above is not planar. Then explain how you know the graph is not planar anyway.

Extra material: notes with solved Exercise.

137 P3: Regular polyhedra (Platonic solids).

All the details in the proof of Theorem 2.3.4 are carefully explained, with images and additional computations when necessary (to explain some Calculus-related details without using Calculus).

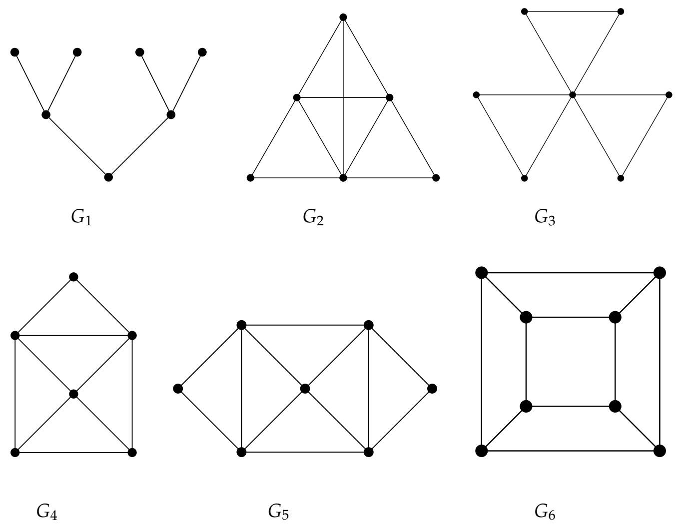
Extra material: an article with more solved exercises on graphs and their arithmetic characteristics (determining numbers of vertices etc); mainly planar graphs, but not only:

- ⋆ **Extra exercise 1**: Is it possible to construct a graph (simple, i.e., without multiple edges between vertices and without loops) with n vertices and with the sequence of degrees $n, n-1, n-2, \dots, 2, 1$?
- ⋆ **Extra exercise 2**: Show that it is impossible to construct a graph (simple, i.e., without multiple edges between vertices and without loops) whose vertices have different degrees pairwise.
- ⋆ **Extra exercise 3**: Given a bipartite graph G with vertices in two sets, A and B, such that the only edges in the graph can join elements of A with elements of B (no multiple edges allowed). Each vertex in A has degree seven and each vertex in B has degree three. There are $6p + 12$ vertices in A (where $p \in \mathbb{N}$ is fixed), i.e., $|A| = 6p + 12$. Determine the number of vertices in B.
- ⋆ **Extra exercise 4**: In a planar connected graph G all the vertices have degree 3. The graph has 186 edges (some of them are possibly multiple). Determine the number of regions, including the outer one, that the graph divides the plane into, when drawn in its planar form.

- ⋆ **Extra exercise 5**: A connected planar graph G has 6 faces, and each of its vertices has degree three. Determine the number of edges and the number of vertices in G.
- ⋆ **Extra exercise 6**: In a planar connected graph (or multigraph) G all the vertices have degree four. The number of edges is $8p + 116$ for some $p \in \mathbb{N}$. Determine the number of faces (including the outer one) of a planar representation of G.
- ⋆ **Extra exercise 7**: A planar connected graph G has vertices with degrees $7, 5, 5, 4, 4, 3, 3, 3, 3, 2, 2, 2, 1$. Determine the number of regions into which a planar representation of G divides the plane, including the outer region.

138 P4: Walking along all the edges, without repetitions; Euler trails and circuits.

Preview Activity on p.142 in the DM Book: Which of the graphs below have an Euler trail? Which have an Euler circuit? Confirm your answers with help of the Theorem on p.144.



Figur 8: **(V138)**: Preview Activity on p.142 in the DM Book.

Back to the bridges of Königsberg formulated in V27 of DM1.

Practice Problems 2.4.5: 1, 2, 3 on p.146: Examine the graphs (defined in various ways) w.r.t. Euler trials and circuits.

139 P4: Visiting all the vertices exactly once; Hamilton paths and cycles.

Additional Exercise 2.4.6.11 on p.148 in the DM Book: Is there anything we can say about whether a graph has a Hamilton path based on the degrees of its vertices?

(a) Suppose a graph has a Hamilton path. What is the maximum number of vertices of degree one the graph can have? Explain why your answer is correct.

(b) Find a graph that does not have a Hamilton path even though no vertex has degree one. Explain why your example works.

140 P4: Hamilton paths and cycles, an exercise.

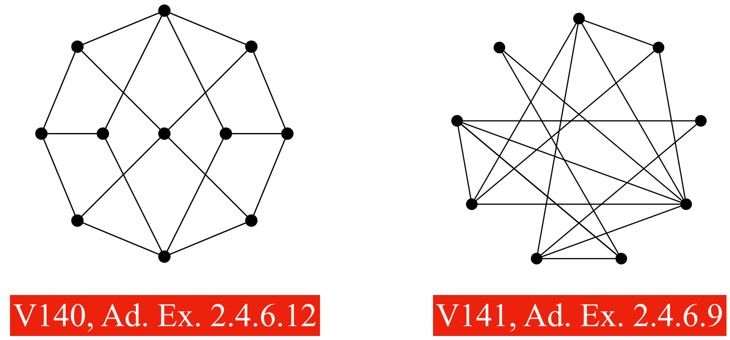
Additional Exercise 2.4.6.12 on p.148: Consider the following graph (see the picture).

(a) Find a Hamilton path. Can your path be extended to a Hamilton cycle?

(b) Is the graph bipartite? If so, how many vertices are in each *part*?

(c) Use your answer to part (b) to prove that the graph has no Hamilton cycle.

(d) Suppose you have a bipartite graph G in which one part has at least two more vertices than the other. Prove that G does not have a Hamilton path.



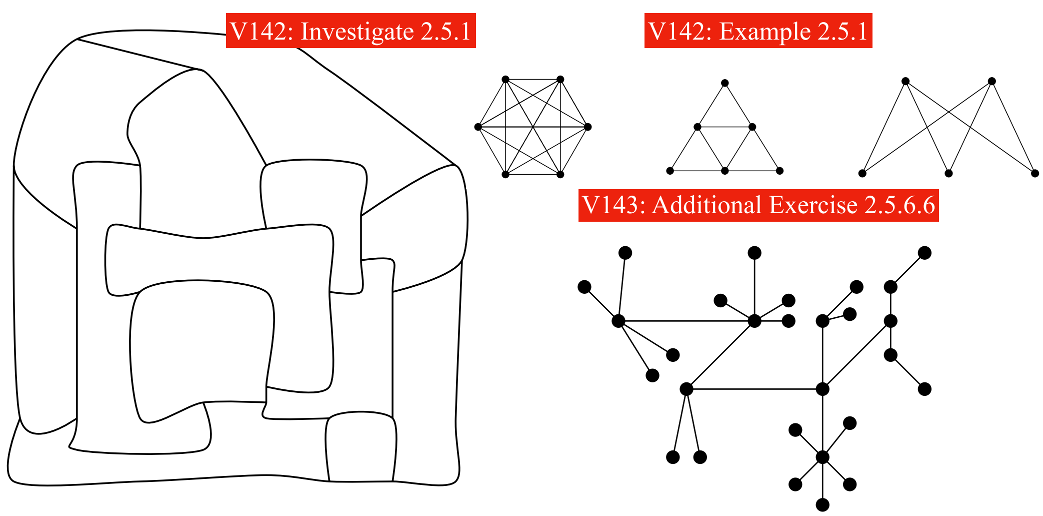
Figur 9: **(V140)** and **(V141)**: Additional Exercises 2.4.6.12 and 2.4.6.9.

141 P4: To euler, or to hamilton, that is the question.

Additional Exercise 2.4.6.9 on p.148: Above is a graph representing friendships between a group of students (each vertex is a student and each edge is a friendship). Is it possible for the students to sit around a round table in such away that every student sits between two friends? What does this question have to do with trails?

142 P5: Vertex coloring and chromatic number of a graph.

Example 2.5.1 on p.153 in the DM Book: Find the chromatic number of three graphs below.



Figur 10: **(V142)**: Investigate 2.5.1 and Example 2.5.1; **(V143)**: Additional Exercise 2.5.6.6.

Investigate 2.5.1 on p.150: Try to color the map (see the picture) with as few colors as possible. Draw the planar graph associated with this map. Make a proper coloring of its vertices. Translate this coloring into the coloring of the countries on the map. Now motivate your result (that it is a minimal coloring).

143 P5: Chromatic number, an exercise.

Additional Exercise 2.5.6.6 on p.160: Prove that the chromatic number of any tree is two. Recall, a tree is a connected graph with no cycles.

(a) Describe the procedure to color the tree in the picture.

(b) The chromatic number of C_n is two when n is even. What goes wrong when n is odd? [See V144.]

(c) Prove that your procedure from part (a) always works for any tree.

(d) Now, prove using induction that every tree has chromatic number 2.

144 P5: Chromatic number of some celebrity graphs.

Exercise: Determine the chromatic number of following graphs: P_n, K_n, C_n, W_n (the wheel graph: C_n with an extra vertex inside, with the rays towards the other vertices), $K_{n,m}$, any bipartite graph, the Petersen graph, the n-cube (a graph with 2^n vertices, each of them represented by a string of n 0s and 1s).

Extra material: notes with solved Exercise.

145 P5: Edge coloring and chromatic index of a graph.

Practice Problem 2.5.5.3: What is the chromatic *index* of: P_8, C_6, C_5, $K_{5,6}$, $K_{6,6}$. (K_8: see V158.)

146 P5: Some Ramsey stuff, a generalisation of Problem 27 in V57 of DM1.

Exercise: Color all the edges in K_{17} using three colors. Prove that regardless how you color, you will always get a monochromatic triangle.

147 P6: Graphs and relations, covered in DM1.

The topic of relations was covered in DM1 (see the references in the slides); the following items from Section 2.6 in the DM Book were covered:

- **Additional Exercise 2.6.8.4** on p.179 in the DM Book (DM1, V169)
- **Additional Exercise 2.6.8.5** on p.179 in the DM Book (DM1, V169)
- **Additional Exercise 2.6.8.6** on p.179 in the DM Book (DM1, V169)
- **Example 2.6.3** on pp.166–167 in the DM Book (DM1, V162)
- **Example 2.6.7** on p.169 in the DM Book (DM1, V185)

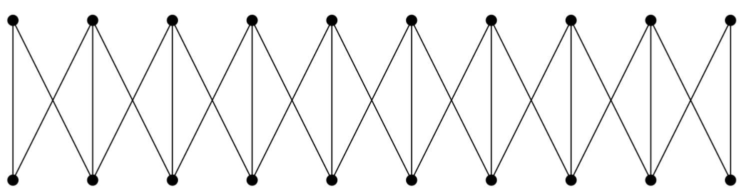
and many other stuff, taken from different sources (mainly my own high-school stuff).

148 P7: Matching in bipartite graphs.

We look together at **Investigate 2.7**, **Exercise 2.7.1 (b)**, and **Example 2.7.2** from the DM Book.

149 P7: Marriages, a matching problem.

Exercise 2.7.4 on p.184 in the DM Book: The two richest families in Westeros have decided to enter into an alliance by marriage. The first family has 10 sons, the second has 10 girls. The ages of the kids in the two families match up. To avoid impropriety, the families insist that each child must marry someone either their own age, or someone one position younger or older. In fact, the graph representing agreeable marriages looks like this:



Figur 11: **(V149)**: The graph representing agreeable marriages.

How many different acceptable marriage arrangements which marry off all 20 children are possible? Can you give a recurrence relation that fits the problem?

150 P8: Plenty of exercises and some advice for the future studies of DM.

151 P1: Counting isomorphic graphs with four vertices, Problem 1.

Problem 1: Find all non-isomorphic graphs with four vertices. For each of them, determine how many isomorphic counterparts it would have if we labeled the vertices with four different letters.

152 P2: Leaves in $(1,3)$-trees, Problem 2.

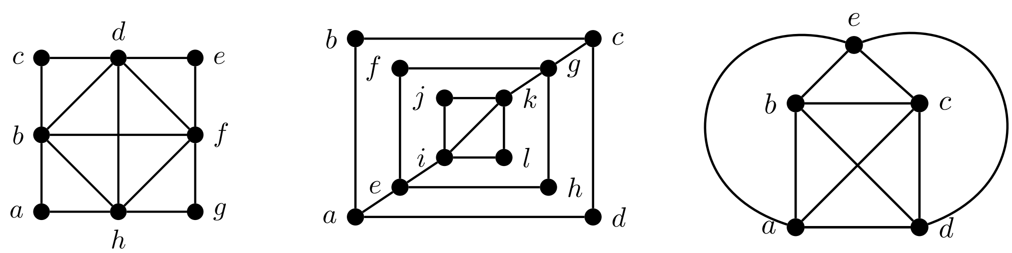
Problem 2: A $(1,3)$-tree has $n \geqslant 2$ vertices and all of them have the degrees 1 or 3. How many at most and at least leaves can such a tree have?

153 P2: Spanning trees of a bipartite graph, Problem 3.

Problem 3: Given $n \geqslant 3$. How many different spanning trees does $K_{2,n}$ have? How many of them are non-isomorphic?

154 P4: Euler and Hamilton, and planarity, Problem 4.

Problem 4: Given three graphs: G_1, G_2, and G_3 (see the picture). Determine for each of them if it:



Figur 12: **(V154)**: Problem 4, graphs G_1, G_2, and G_3 (from the left to the right).

(a) has an Euler trail

(b) has an Euler circuit

(c) has a Hamilton cycle

(d) is planar.

In cases where the answer is *yes*, indicate an Euler trail, an Euler circuit, a Hamilton cycle, or draw a planar representation of the graph. If the answer is *no*, motivate why.

155 P4: Euler and Hamilton, Problem 5.

Problem 5: Draw a connected graph with eight edges that:

- is eulerian but not hamiltonian
- is hamiltonian but not eulerian
- is neither eulerian nor hamiltonian.

156 P5: Vertex coloring in Tutte's graph, Problem 6.

Problem 6: Show that Tutte's graph (see V122) has chromatic number equal to 3.

157 P5: Edge and face coloring in Tutte's graph, Problem 7.

Problem 7: Tutte's graph (shown in V122) is a planar cubic graph without bridges. Its faces can be properly colored with four colors (shown in the slides). Prove that this graph (and other graphs with the same properties as just listed) is of Vizing class 1.

158 P5: Edge coloring in the complete graphs K_n, Problem 8.

Problem 8: Prove the following statement: The chromatic index of the complete graph K_n is

$$\chi'(K_n) = \begin{cases} n-1 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd.} \end{cases}$$

159 The Last One: One half of the Martian Friendship Theorem, Problem 9.

Problem 9: **The Friendship Theorem** (Erdős, Rényi, Sós): If, in a finite graph G that is *non-regular* (i.e., at least two of its vertices have different degrees), every two vertices have precisely one common neighbor, then some vertex of G is adjacent to all the vertices of G except itself.

Translation into Martian: If, in a group of people, every two people have precisely one common friend (but at least two of them have different numbers of friends), then the group includes a *politician*, someone who is a friend of everybody.

S11 Some applications of Discrete Mathematics

You will learn: some applications of DM to CS; it will not be much, just some (very modest) words about algorithm complexity, Chinese Remainder Theorem, and RSA encryption; you get a bunch of practice problems (with solutions), both in the videos and in other resources; you will also get some advice for your further studies of DM, and all the students are welcome to leave references to their favorite resources on the QA under the last lecture in this section.

160 What you will learn in this section.

161 Some examples of algorithms.

162 A very modest introduction to algorithm complexity (efficiency).

Example 0: Show that $3n^3 + 20n^2 + 5n + 11 \in O(n^3)$.

Example 1: Suppose that we have a machine which carries out one million operations per second, and an algorithm whose efficiency is $f(n)$. Compute the time needed for performing the algorithm for $n = 20$, $n = 40$, and $n = 60$, if the efficiency is (four different cases): $f_1(n) = n$, $f_2(n) = n^2$, $f_3(n) = n^3$, $f_4(n) = 2^n$.

163 Some pointers towards Calculus.

Standard Limits in the Infinity (see Calc1p1, V119): Let $a > b > 1$ and $k > m > 0$. Then:

$$n^n \gg n! \gg a^n \gg b^n \gg n^k \gg n^m \gg \ln n \qquad \text{where} \qquad a_n \gg b_n (> 0) \quad \Leftrightarrow \quad \frac{a_n}{b_n} \to \infty \quad \Leftrightarrow \quad \frac{b_n}{a_n} \to 0.$$

Example 0: Compare $a_n = n^n$, $b_n = 2^n$, $c_n = n^3$, $d_n = \sqrt{n}$, and $f_n = \ln n$ for small and large values of n.

Example 1: If $a_n = \dfrac{1.01^n}{n^{10}}$, then $a_n \approx 0$ for $n \leqslant 8,800$, and still $a_n \to \infty$ when $n \to \infty$! For $n = 10,000$ we have $a_n = 1,636$. All this shows that what happens for (relatively) small n does not matter.

164 More examples showing how your intuition can fail you.

Standard limits in the infinity: If $a > 1$ and $k > 0$ then:

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0, \qquad \lim_{n\to\infty} \frac{n^k}{a^n} = 0, \qquad \lim_{n\to\infty} \frac{n^k}{\ln n} = \infty.$$

(Proven in Calc1p1, V121.)

Example 0: Show that $2^n + 3n^5 + 12n^4 \in O(2^n)$.

Example 1: MANIM animation showing the following functions:

$$f_1(x) = \frac{x}{e^x}, \quad f_2(x) = \frac{x^2}{e^x}, \quad f_3(x) = \frac{x^3}{e^x}, \quad f_4(x) = \frac{x^4}{e^x}, \quad f_5(x) = \frac{x^5}{e^x}, \quad f_7(x) = \frac{x^7}{e^x}.$$

165 Bubble sort algorithm, a toy example.

Example: Use the *Bubble sort algorithm* to sort the numbers $(7, 3, 6, 4, 1)$ in the increasing order.

166 Back to Chinese Remainder Theorem (CRT).

Chinese Remainder Theorem (CRT): Let $n_1, n_2, \dots, n_k \in \mathbb{N}^+ \setminus \{1\}$ be any positive natural numbers that are *pairwise relatively prime* and $a_1, a_2, \dots, a_k \in \mathbb{Z}$ be any integers. The system of congruences

$$\begin{cases} x \equiv a_1 \pmod{n_1} \\ x \equiv a_2 \pmod{n_2} \\ \dots \\ x \equiv a_k \pmod{n_k} \end{cases}$$

has a solution, and the solution is unique modulo $N = n_1 \cdot n_2 \cdot \ldots \cdot n_k$.

The theorem can be formulated like this: For each $n_1, n_2, \dots, n_k \in \mathbb{N}^+ \setminus \{1\}$ (pairwise relatively prime) and for each $a_1, a_2, \dots, a_k \in \mathbb{Z}$ it is possible to find an integer number x that satisfies k conditions: x has remainder a_i in division by n_i (for $i = 1, 2, \dots, k$). All such x have the same remainder modulo $N = n_1 \cdot n_2 \cdot \ldots \cdot n_k$.

167 CRT continued: a constructive proof, Example 0.

With the assumptions as in CRT formulated in V166, the following number x solves the system:

$$x = a_1b_1N_1 + a_2b_2N_2 + \ldots + a_kb_kN_k,$$

where

$$N_i = \prod_{j\neq i} n_j = \frac{N}{n_i} \qquad \text{for } \ i = 1, 2, \ldots, k$$

and b_i are multiplicative inverses of N_i in $\mathbb{Z}_{n_i}$, i.e., such numbers that for $i = 1, 2, \ldots, k$

$$y_i = b_iN_i \equiv 1 \pmod{n_i}.$$

(Notation y_i as in Proof 2 in V166.) This solution is unique modulo N.

Example 0 Solve the problem from V174 in DM2 using the new formula:

$$\begin{cases} x \equiv 4 \ (\mathrm{mod}\ 5) \\ x \equiv 5 \ (\mathrm{mod}\ 6) \\ x \equiv 3 \ (\mathrm{mod}\ 7) \end{cases}$$

Compare with the result obtained earlier.

Extra material: notes with solved Example 0.

168 CRT: many birds, one stone, Example 1.

Example 1 Two examples, one computation:

$$\begin{cases} x \equiv 1 \ (\mathrm{mod}\ 3) \\ x \equiv 2 \ (\mathrm{mod}\ 4) \\ x \equiv 3 \ (\mathrm{mod}\ 5) \end{cases} \qquad\qquad \begin{cases} x \equiv 2 \ (\mathrm{mod}\ 3) \\ x \equiv 1 \ (\mathrm{mod}\ 4) \\ x \equiv 4 \ (\mathrm{mod}\ 5) \end{cases}$$

Notice that you only need to find b_1, b_2, b_3 once; you could use them for any other system with the same $\{n_1, n_2, n_3\}$ but with different RHS (a_i).

Extra material: notes with solved Example 1.

169 CRT in problem solving, Example 2.

Example 2: Use the CRT and Euler's Totient Theorem (ETT) to find two last digits of 14^{202}.

Extra material: notes with solved Example 2.

Extra material: notes with solved Exercise: Use the CRT and ETT to find two last digits of 18^{121}.

170 CRT in problem solving, Example 3.

Example 3: Use the CRT and ETT to find all the solutions to $2x^6 + 3x - 7 = 0$ in $\mathbb{Z}_{14}$.

Extra material: notes with solved Example 3.

Extra material: notes with solved Exercise: Use the CRT and ETT to solve $x^{20} + 3 = 0$ in $\mathbb{Z}_{14}$.

171 CTR, Cool Problem 1.

Cool Problem 1: Use the CRT to show that for each triplet of positive natural numbers p, q, r that are pairwise relatively prime the equation $x^p + y^q = z^r$ has a (positive) natural-number solution x, y, z.

Extra material: notes with solved Cool Problem 1.

172 CRT, Cool Problem 2.

Cool Problem 2: Use the CRT to show that the equation

$$x^{1994} + y^{1995} + z^{1996} = t^{1998}$$

has a (positive) natural-number solution x, y, z, t.

Extra material: notes with solved Cool Problem 2.

173 Some Abstract Algebra: number of isomorphism classes of finite abelian groups.

Examples of application of the structural theorem (given without a proof):

1. How many isomorphism classes are there of abelian groups with 4 elements?
2. How many isomorphism classes are there of abelian groups with 100 elements? Which of them describe groups isomorphic to $\mathbb{Z}_{100}$?
3. How many isomorphism classes are there of abelian groups with 24 elements? Which of them describe groups isomorphic to $\mathbb{Z}_{24}$?
4. Motivate that each abelian group with 30 elements is cyclic.
5. The multiplicative group $\mathbb{Z}_{12}^*$ (the group of units; see V204 in DM2) of all invertible elements of $\mathbb{Z}_{12}$ is abelian. Determine the product of cyclic groups (like in the theorem) this group is isomorphic to.

174 Quick arithmetic with help of the ring isomorphism from V213 in DM2.

Four examples of *quick arithmetic* in action ([1] and [2] are in the video, [3] and [4] in an attached article):

1. Go back to V147 and V213 in DM2, together with V167 in this course, and appreciate the computation of the inverse to 89 in $\mathbb{Z}_{210}$.
2. Create a value table for $f : \mathbb{Z}_{15} \to \mathbb{Z}_3 \times \mathbb{Z}_5$ and use it to compute $9 \cdot 13$, $9 + 13$, the inverse to 12 and the inverse to 13 in $\mathbb{Z}_{15}$.
3. Compute the determinant in $\mathbb{Z}_{30}$:
$$\begin{vmatrix} 27 & 8 & -10 \\ 5 & 19 & 13 \\ -26 & 25 & 14 \end{vmatrix}$$
4. Let $f : \mathbb{Z}_{315} \to \mathbb{Z}_5 \times \mathbb{Z}_7 \times \mathbb{Z}_9$ be the ring isomorphism. Determine $f(3)$ and $f(43)$.

   Determine $f^{-1}((1, 0, 0))$, $f^{-1}((0, 1, 0))$, and $f^{-1}((0, 0, 1))$.

   Use f and $^{-1}$ to compute $(186 + 212) \cdot 88^{-1} + 167$ in $\mathbb{Z}_{315}$.

175 RSA algorithm: how it works.

176 A proposition needed for the RSA encryption algorithm.

Proposition: If p and q are two prime numbers and $a \in \mathbb{Z}^+$ is not divisible by any of them, then

$$a^{(p-1)(q-1)} \equiv 1 \pmod{pq}.$$

Three proofs are presented: (1. FLT with elementary ending, 2. FLT with CRT at the end, 3. ETT).

177 RSA algorithm: why it works.

We show that $a^{ed} \equiv a \pmod{n}$, where a, e, d, n are as described in the algorithm in V175.

178 RSA algorithm: plenty of solved (toy) exercises.

Example: Describe the process of sending the secret key $a = 28$ using the RSA algorithm with the public key $(e, n) = (3, 33)$. (Note that it is a toy example; in real life you wouldn't be able to find d from the information about the public key, as it is extremely difficult to perform prime-factorisation of really large numbers $n = pq$.)

Extra material: an article with more solved exercises on RSA algorithm; solve exercises 1–4 after having seen this video, and exercises 5–8 after the next one; exercises 6–8 are a little bit less standard than exercises 1–5, but they still can be solved with help of the material that you have learnt in our DM trilogy, mainly solving Diophantine equations (finding inverses in the rings $\mathbb{Z}_m$) and computing powers of numbers modulo n:

- ⋆ **Extra exercise 1**: An RSA algorithm has the public key $n = 33$ and $e = 3$. Decrypt the secret key 2.
- ⋆ **Extra exercise 2**: An RSA algorithm has the public key $n = 33$ and $e = 9$. Decrypt the secret key 2.
- ⋆ **Extra exercise 3**: An RSA algorithm has the public key $n = 77$ and $e = 43$. Decrypt the secret key 2.
- ⋆ **Extra exercise 4**: An RSA algorithm has the public key $n = 187$ and $e = 89$. Decrypt the secret key 2.
- ⋆ **Extra exercise 5**: An RSA algorithm has the public key $n = 55$ and $e = 17$. Decrypt the secret key 5.

- ⋆ **Extra exercise 6**: We construct an RSA cryptosystem with help of primes $p = 7$ and $q = 11$. Choose an appropriate public key e restricted by $20 < e < 30$ and for this e determine the private key d.
- ⋆ **Extra exercise 7**: The public modulus in an RSA cryptosystem is $n = 143$. Choose an appropriate public key e that satisfies the restriction $12 \leqslant e \leqslant 20$. For this e, determine the private key d and encrypt the secret key $a = 2$.
- ⋆ **Extra exercise 8**: Let $p = 5$ and $p = 21$. We compute from this data $n = 105$ and $m = 80$. We choose the public key $e = 7$, which gives us the private key $d = 23$. Use these keys to encrypt the secret key $a = 3$ and then decrypt the result. Give the reason why you didn't get the original value of a after these operations.

179 RSA algorithm: a bigger example (but still just a toy).

Example (*Discrete Mathematics, a continuation course* by Kimmo Eriksson and Hillevi Gavel, pp.58–59): Kimmo sends a secret key to Hillevi. They use RSA encryption with Hillevi's public key $n = 517$ (with prime factorisation $p \cdot q = 47 \cdot 11$ that is known to Hillevi) and $e = 97$ (that is a prime number). Hillevi knows both p and q, so she can get both m and d. Show how she computes d. Kimmo sends her a secret key $a = 50$. He encrypts it with Hillevi's public key (e, n). Show all the steps in the algorithm, starting with the encryption and ending with the decryption of the message on Hillevi's side, with help of Hillevi's private key (d, n).

180 Our DM journey: from manual labor to playing with Big Guns.

The Last Problem (originally **Problem 13** from V42 in DM1, inspired by a question that I got from a student on 30 July 2026): Some integer gives remainder 2 in division by 3 and remainder 1 in division by 4. What is its remainder in division by 12?

181 So, what's next?

S12 Extras

You will learn: about all the courses we offer, and where to find discount coupons. You will also get a glimpse into our plans for future courses, with approximate (very hypothetical!) release dates.

B Bonus lecture.

Extra material 1: a pdf with all the links to our courses, and coupon codes.

Extra material 2: a pdf with an advice about optimal order of studying our courses.

Extra material 3: a pdf with information about course books, and how to get more practice.