Discrete Mathematics 1[1]

Mathematics from high school to university

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A short table of contents

A detailed table of contents follows (see next pages).

Books to read along with the course, with more practice problems (often suggested in the outline and in some videos, as I have found these books before I started to record the series):

1 *Discrete Mathematics, An Open Introduction*, 4th Edition (2024): Oscar Levin; School of Mathematical Science, University of Northern Colorado.

2 *Mathematics for Computer Science* (LibreTexts): Eric Lehman, F. Thomson Leighton, Albert R. Meyer; MITOpenCourseWare.

3 *Book of Proof*: Richard Hammack; Richmond, Virginia.

4 *Abstract Algebra: Theory and Applications* (LibreTexts): Thomas W. Judson; Stephen F. Austin State University.

This first book is added as a resource to V1 in each part of the DM series, with kind permission of Professor Oscar Levin (given by e-mail on 2 April 2025); the second one is added as a resource to V1 in DM2 and DM3, with kind permission of the Authors (given by e-mail on 20 June 2025).

[1] Recorded May–July 2025. Published on `www.udemy.com` on 2025-07-XX.

An extremely detailed table of contents; the videos (titles in green) are numbered

In blue: problems solved on an iPad (the solving process presented for the students; active problem solving)

In red: solved problems demonstrated during a presentation (a walk-through; passive problem solving)

In magenta: additional problems solved in written articles (added as resources).

In dark blue: *Read along with this section*: references for further reading and exercises in *Discrete Mathematics* book by Oscar Levin (this book is added as a resource to Video 1).

S1 Introduction to the course

You will learn: about this course: its content and the optimal way of studying it together with the book.

Read along with this section: Section 0.1: *What is Discrete Mathematics?* (pp.1–4).

1 Introduction to the course.

Extra material: this list with all the movies and problems.

Extra material: the book named on the previous page (*Discrete Mathematics* by Oscar Levin).

2 What is Discrete Mathematics.

3 This course and the book.

S2 Preliminaries: *paintbrushes and easels*

You will learn: some basics needed for understanding the new topics discussed in this course.

Read along with this section: Section 0.2: *Discrete Structures* (pp.5–12).

4 Mathematical symbols, old and new.

5 (P1 V75) A brief introduction to sets.

Concepts covered: set, belonging to the set, various methods of defining sets (verbal method, graphical method, special symbols, roster method {with braces}, set-builder method), Venn diagram, union of sets, intersection of sets, difference of sets, the empty set, a (proper) subset.

6 A reflection about mathematical structures.

7 Cartesian coordinate system and Cartesian product of two sets.

Example 1: Describe $A \times B$ if $A = \{0, 1, 2, 3\}$ and $B = \{1, 4, 5\}$.

Example 2: Describe $A \times B$ if $A = (2, 5]$ and $B = [1, 6)$.

8 Two important discrete sets.

9 **Optional, abstract, theoretical**: What is *discrete* metric.

10 Important properties of the set of natural numbers.

11 Computational rules.

12 (P1 V7) A visual motivation of commutativity.

13 (P1 V8) A visual motivation of associativity.

14 (P1 V9) A visual motivation of distributivity.

Show that $(k + l)(m + n) = km + kn + lm + ln$ for all natural numbers k, l, m, n.

Extra material: notes from the iPad.

15 Formal descriptions of mathematical theories.

16 (P1 V13) Integer numbers: addition, subtraction, and multiplication.

This lecture contains a repetition about integer numbers and their arithmetic. The main computation rules are listed, motivated, and illustrated with examples.

Example 1: You have earned 1 000 \$ and you spent 1 200 \$. How much money do you have left?

Example 2: Compute $-1+8-3-4+7+1-8$. Show two different methods of computing this sum.

Example 3: The temperature T rose from $-60°$ Celsius to $+40°$ Celsius. Compute ΔT.

Example 4: Compute the distance between the hot air balloon, which is 200m above the sea level, and the diver, who is 50m under the sea level.

17 Prime numbers and some divisibility rules.

Example: Factor the numbers 120 and 225 and find their gcd (greatest common divisor).

18 (P1 V14) Rational numbers as fractions.

This lecture contains a repetition about fractions and their arithmetic. The main computation rules are listed, motivated, and illustrated with examples.

Example 1: Express the improper fraction $\frac{23}{7}$ as a mixed number.

Example 2: What is the least common denominator for the fractions in $\frac{1}{6}-\frac{1}{12}+\frac{1}{9}+\frac{1}{4}+\frac{5}{36}+\frac{1}{2}+\frac{1}{3}$?

Example 3: One fourth of a field is meant for agriculture. Two thirds of this part is for planting strawberries. What part of the entire field will be a strawberry field?

Example 4: How many cans of soda do you need to get 2 liter soda? (One can contains $\frac{1}{3}$ liter.)

Example 5: Arrange the following numbers in increasing order (from the least to the greatest):

$$\frac{1001}{1000};\quad \frac{3}{7};\quad 1;\quad \frac{1}{2};\quad \frac{2}{3};\quad \frac{5}{4};\quad \frac{999}{1000};\quad \frac{1}{70000};\quad \frac{1}{3}.$$

Extra material: notes with solved Example 5.

19 Expressions, constants, variables, coefficients, etc.

20 (P1 V24 and V25) Order of operations (precedence rules), some examples.

Examples: Compute

(1) $3+[6(11-4+1)]/8\cdot 2$,

(2) $(-1)^2-1^2-(-2)$,

(3) $\dfrac{-5\cdot(-6)-7\cdot(-3)}{3\cdot 5+2}$

(4) $17\left(\dfrac{5}{283}-\dfrac{2}{175}\right)+13-\dfrac{85}{283}+4+\dfrac{34}{175}$.

The last example shows the importance of computational experience. Use the example of $5(4+7)$ to show that the Distributive Law $a(b+c)=ab+ac$ and PEMDAS don't contradict each other.

Extra material: notes with solved Examples.

21 (P1 V30) Evaluating expressions.

Example: Evaluate $E(x,y)=3xy-x^2-3y^3+x-12$ for:

$$(1)\;\; x=-1,\; y=-1, \qquad (2)\;\; x=1,\; y=-1, \qquad (3)\;\; x=0,\; y=-1.$$

Extra material: notes with solved Example.

22 (P1 V27) Simplifying expressions.

Example: Simplify the expression $\dfrac{(x+y)^2-(x-y)^2}{5xy}$ (where $x,y\neq 0$) as far as possible.

Extra material: notes with solved Example.

23 Index, sigma symbol, and pi symbol.

Exercises: Write the sums / products with (1–8) or without (9–16) the Sigma / Pi:

(1) $1+2+4+8+16+32$

(2) $x^2 - x^3 + \ldots - x^{19} + x^{20}$

(3) $e + e^2 + e^3 + \ldots + e^{10}$

(4) $1 - 3 + 9 - 27 + 81 - 243$

(5) $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7$

(6) $1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13$

(7) $\left(1-\frac{4}{1}\right)\left(1-\frac{4}{9}\right)\left(1-\frac{4}{25}\right)\left(1-\frac{4}{49}\right)\left(1-\frac{4}{81}\right)\left(1-\frac{4}{121}\right)$

(8) $\sqrt{a_1 a_2 a_3 \cdot \ldots \cdot a_{20}}$

(9) $\sum\limits_{k=1}^{7} k^3$

(10) $\sum\limits_{k=3}^{9} (2k-1)^2$

(11) $\sum\limits_{k=1}^{10} (-1)^k$

(12) $\sum\limits_{k=5}^{30} \left(\frac{1}{k} - b_k\right)$

(13) $\prod\limits_{k=1}^{5} k$

(14) $\prod\limits_{k=0}^{9} (2k-1)$

(15) $\prod\limits_{k=1}^{10} a_k$ (16) $\prod\limits_{k=5}^{30} (1-b_k)^{k-1}$.

Extra material: notes with solved Exercises.

24 Functions in Calculus and in Discrete Mathematics.

Example: Let $f : X \to Y$. Write the definition of its graph as the set of all the points $(x, f(x))$ where $x \in D_f$.

25 Sequences in DM1 and DM2.

26 (P1 V161 and V162) Relations in life and in mathematics.

a) parallelity relation, similarity relation: $l \parallel k$, $\triangle ABC \sim \triangle PQR$,

b) being divisor of: $2|8$, $5|35$, $\ldots$,

c) less than / greater than / $\ldots$: $x < y$, $x > y$, $x \leqslant y$, $x \geqslant y$,

d) equality relation: $x = y$, $A = B$, $(x_1, y_1) = (x_2, y_2), \ldots$,

e) being a subset of: $\mathbb{N} \subset \mathbb{Z}$, $\mathbb{N} \subset \mathbb{R}$, $\{1, 2, \pi, \sqrt{3}\} \subset \mathbb{R}$, $\ldots$.

27 Graph: a new concept (will be studied in DM2).

Example 1: Persons A and B know each other, persons A and C don't know each other, persons B and C do know each other. Make a picture illustrating their relation of knowing each other for persons A, B, and C.

Example 2: Depict the binary relation $\leqslant$ on the set $\{-2, -1, 0, 1, 2\}$ by putting an arrow from x to y if $x \leqslant y$.

Example 0: How it all started (1736, Leonard Euler and Bridges of Königsberg).

28 (P1, V211) Algebraic structures that will be discussed in DM2.

S3 A very soft start: *painting happy little trees*

You will learn: you will get the first glimpse into various types of problems and tricks that are specific to Discrete Mathematics: proving formulas, motivating formulas, deriving formulas, generalising formulas, a promise of Mathematical Induction, divisibility of numbers (by factoring, by analysing remainders / cases), various ways of dealing with problem solving (mathematical modelling, using graphs, using charts, using Pigeonhole Principle, using the Minimum Principle, [always] using logical thinking; strategies); you will also see some problems that we are not able to solve (yet) but which will be solved in Section 12 where we will learn more about counting; nothing here is based on big theories, just logical thinking; treat it as a *smörgåsbord* (Swedish buffet) of Discrete Mathematics.

29 Proving formulas, Problem 1.

Problem 1: Show that for all real numbers a and b holds $(a \pm b)^2 = a^2 \pm 2ab + b^2$. These formulas are known as *square of the sum* and *square of the difference*, and they form a basis for the method of *completing the square*.

Extra material: notes with solved Problem 1.

30 (P1 V23) Proving formulas, Problem 2.

Problem 2: Show that for all real numbers a and b holds $a^2 - b^2 = (a - b)(a + b)$. This formula is known as *difference of two squares* and is used for solving quadratic equations (for derivation of the formula for solutions).

Extra material: notes with solved Problem 2.

31 Proving formulas, Problem 3.

Problem 3: Show that for all real numbers a and b holds the following:

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3, \qquad (a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

These are generalisations of the formula in V29. You will see more of it in Section 11.

Extra material: notes with solved Problem 3.

32 Proving formulas, Problem 4.

Problem 4: Show that for all real numbers a and b holds the following:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2), \qquad a^4 - b^4 = (a - b)(a^3 + a^2b + ab^2 + b^3).$$

These are generalisations of the formula in V30.

Extra material: notes with solved Problem 4.

33 Motivating formulas, Problem 5.

Problem 5: Let a be any real number and k and n be two positive natural numbers. Motivate that $a^k \cdot a^n = a^{k+n}$, using the definition of power with positive natural exponent as repeated multiplication of the base by itself.

Extra material: notes with solved Problem 5.

34 Generalising formulas for more terms or factors, Problem 6.

Problem 6: Let a be any real number and $n_1, n_2, \ldots, n_k$ be k positive natural numbers. Motivate that

$$a^{n_1} \cdot a^{n_2} \cdot \ldots \cdot a^{n_k} = \prod_{i=1}^{k} a^{n_i} = a^{\sum_{i=1}^{k} n_i} = a^{n_1+n_2+\ldots+n_k}.$$

Extra material: notes with solved Problem 6.

35 Deriving formulas, Problem 7.

Problem 7: Let $n \in \mathbb{N}^+$. Derive the formula for the sum of all natural numbers starting at 1 and ending at n.

Extra material: notes with derivation of a formula for the sum of squares (and, partly, the sum of cubes).

36 Proving formulas, quite informally, Problem 8.

Problem 8: Prove the formula from V35 using the Induction Principle (V34) and the Minimum Principle (V10).

Extra material: notes with solved Problem 8.

37 Proving inequalities (directly), Problem 9.

Problem 9: Show that for any real numbers a and b we have $a^2 + b^2 \geqslant 2ab$, using the result from V29 and the fact that squares of all real numbers are non-negative.

Extra material: notes with solved Problem 9.

38 Proving inequalities (directly), Problem 10.

Problem 10: Show that if $x > 2$ and $y > 3$ then $2x + 5y > 19$. Perform a *direct proof*, i.e., starting from the assumptions arrive at the conclusion.

Extra material: notes with solved Problem 10.

39 Proving inequalities (indirectly), Problem 11.

Problem 11: Show that if $x > 2$ and $y > 3$ then $2x + 5y > 19$. Perform an *indirect proof*, i.e., see what would happen if, with given assumptions, the conclusion wasn't true.

Extra material: notes with solved Problem 11.

40 Back to divisibility: working with remainders, factoring numbers and expressions.

41 Proving divisibility: working with remainders, Problem 12.

Problem 12: Prove that if a natural number n is not divisible by 3, then $n^2 - 1$ is divisible by 3.

Extra material: notes with solved Problem 12.

42 Proving divisibility: working with remainders, Problem 13.

Problem 13: Some integer gives remainder 2 in division by 3 and remainder 1 in division by 4. What is its remainder in division by 12?

Extra material: notes with solved Problem 13.

43 Proving divisibility: working with remainders, Problem 14.

Problem 14: Prove that for each natural number n we have $6|n^3 - n$.

Extra material: notes with solved Problem 14.

44 Proving divisibility using factoring, Problem 15.

Problem 15: Prove that for each natural number n we have $6|n^3 - n$.

Extra material: notes with solved Problem 15.

45 Proving divisibility using factoring, Problem 16.

Problem 16: Prove that for each natural number n we have $30|n^5 - n$.

Extra material: notes with solved Problem 16.

46 Proving divisibility using factoring, Problem 17.

Problem 17: Prove that $2 + 2^2 + 2^3 + \ldots + 2^{99} + 2^{100}$ is divisible by 3.

Extra material: notes with solved Problem 17.

47 Proving divisibility using factoring, Problem 18.

Problem 18: Use the generalisation of the formula from V32 to motivate that $11^{10} - 1$ is divisible by 100.

Extra material: notes with solved Problem 18.

48 Logical thinking with help of a chart, Problem 19.

Problem 19: Person X can speak at least one language. Her friends say:

* A says: X speaks more than five languages.
* B says: X speaks less than three languages.
* C says: X speaks at the most four languages.
* D says: X speaks more than two languages.

How many languages does X speak if exactly one of her friends tells the truth and the three others are lying?

49 Logical thinking and strategies, Problem 20.

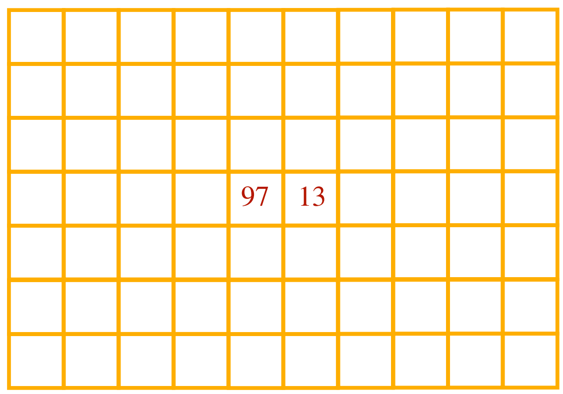
Problem 20: You have five coins that look identical, but two of them are fake. All the real coins weigh the same. One of the fake coins is lighter than a real one, and the second fake coin weighs more than a real one. You get a balance scale (one that doesn't show numbers, just compares weights). Find a strategy that finds the fake coins using the balance scale at most three times.

50 Logical thinking and strategies, Problem 21.

Problem 21: You have nine coins that look identical, but one of them is fake. All the real coins weigh the same. The fake coin has different weight than a real one. You get a balance scale (one that doesn't show numbers, just compares weights). Find a strategy that finds the fake coin using the balance scale at most three times.

51 Logical thinking, Problem 22.

Problem 22: Each square of the 7-by-10 board (see the picture) contains a number, but all the numbers except two (97 and 13) are covered. We know that the sum of all the entries in each 3-by-4 rectangle and in each 4-by-3 rectangle is zero. Determine the sum of all the 70 entries on the board.



52 Pigeonhole Principle, an easy but practical tool for reasoning.

Example: Show that in a group of 13 people we can find at least two people that were born in the same month.

53 Pigeonhole Principle, Problem 23.

Problem 23: Show that if we choose any five points inside an equilateral triangle with side length 6, then at least two of these points are on a distance less than or equal to 3 from each other.

54 Pigeonhole Principle and modulo, Problem 24.

Problem 24: Show that among any three natural numbers you will find such two (of them) that the difference of their squares is divisible by 3. Show that this is not true if we don't square our numbers. (Compare to V41.)

55 Pigeonhole Principle and modulo, Problem 25.

Problem 25: It is a problem regarding *integer* numbers. Show that:

(1) among any **three** numbers you will find such **two** that the difference of their *squares* is divisible by **4**.
(2) among any **four** numbers you will find such **two** that the difference of their *cubes* is divisible by **9**.
(3) among any **three** numbers you will find such **two** that the difference of their *4th powers* is divisible by **16**.

Extra material: notes with solved Problem 25.

56 Pigeonhole Principle, Problem 26.

Problem 26: In each of the 36 squares of a 6×6 board one writes -1, 0 or 1. We add all the numbers in each row to each other, all the numbers in each column to each other, and all the numbers in each of the two diagonals to each others, so that we get $6 + 6 + 2 = 14$ numbers. Is it possible to place -1, 0 or 1 in the squares in such a way that all these 14 numbers are different from each other?

Extra material: notes with solved Problem 26.

57 Pigeonhole Principle and graphs, Problem 27.

Problem 27: All the sides and all the diagonals of a hexagon are supposed to get colored with help of two colors (in such a way that each of these straight-line segments is entirely colored with one color from vertex to vertex). Show that, regardless how you color, you will always find a triangle (with the vertices in some of the hexagon's vertices) that is colored with one of the two colors only.

58 Mathematical modelling and graphs.

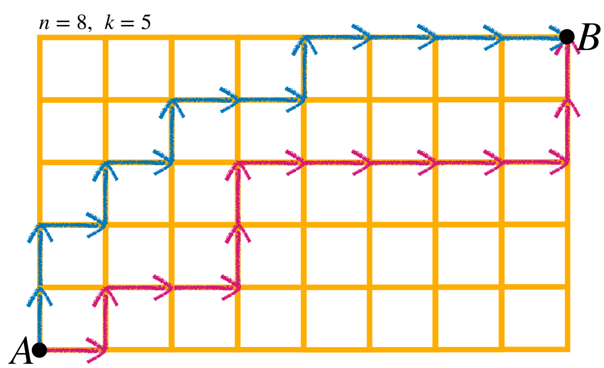
Problems solved with similar methods as Problem 27:

(1) Show that in a group of six people one can always find a subgroup of three that either all know each other or all are strangers to each other.

(2) We have six cities; no three of them are lying on the same straight line. Some of them are connected by a railway (that goes both ways), some of them aren't. Show that there exist such group of three (out of the given six) cities with this property that there is no railway connection at all between any pair of them, or that each pair from this group is connected by a railway.

59 Paths, Problem 28.

Problem 28: Given a $n \times k$ board / grid. We denote the lower left corner of the board by A and the upper right corner of the board by B. We try to determine the number of shortest paths (along the grid lines) from A to B, i.e., paths that only lead you horizontally to the right or vertically up (so no going back towards A). We show the solutions for $n = 1 = k$, for $n = 2, k = 1$, for $n = 1, k = 2$, and for $n = 2 = k$; for a general solution we will wait until Section 12.



60 Mathematical modelling and paths.

Problems solved with similar methods as Problem 28:

(1) Given $n, k \in \mathbb{N}^+$. In how many ways can you arrange n zeros and k ones in a sequence?

(2) Given $n, k \in \mathbb{N}^+$ and $n \geqslant k$. In how many ways can one choose an increasing (non-decreasing) sequence $(x_1, x_2, \ldots, x_k)$ of elements from the set $\{1, 2, \ldots, n\}$?

(3) Given $n, k \in \mathbb{N}^+$ and $n \geqslant k$. Determine the number of positive natural solutions to $x_1 + x_2 + \ldots + x_k = n$.

(4) Given $n, k \in \mathbb{N}^+$ and $n \geqslant k$. We have k boxes and n (indistinguishable) objects. In how many ways can one put all these objects in the boxes in such a way that no box is empty?

(5) Given $n, k \in \mathbb{N}^+$ and $n \geqslant k$. A train has k compartments. In how many ways can you distribute n passengers (that are *indistinguishable* in the meaning that only the *number* of people counts, not who is where in particular) over these compartments so that no compartment is empty?

61 Permutations: the nicest creatures to deal with in combinatorics.

Exercise: In how many ways can we arrange n different objects in a sequence? (permutations). Solve the problem for $n = 1$, $n = 2$, $n = 3$, and try to find the general answer. Notation: $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$ is called n *factorial*.

Extra material: notes from the iPad.

62 Mathematical modelling and permutations.

Exercise: How many bijections are there between two sets with n elements each?

63 Fixed points of permutations.

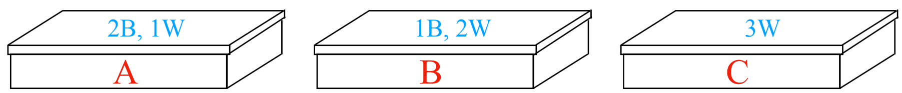
Exercise: Write down all the possible permutations of $\{1, 2, 3\}$ and mark all the fixed points.

You get the coolest geometrical interpretation of the permutations in the Exercise.

Extra material: notes from the iPad.

64 The last one, Problem 29.

Problem 29: We have three closed boxes. Each box contains three balls, and it has a label on the lid, telling you how many balls of each color (black [denoted B] or white [denoted W]) are inside, as in the picture below.



The problem is that each box got a label belonging to another box. We define an operation *one check* in the following way: *choose a box, pick (without looking) one random ball from this box, write down its color, close the box, and leave the ball outside the box.* Describe a strategy allowing you to determine the content of all the three boxes with help of two *checks* as defined above.

S4 Logic

You will learn: the meaning of the symbols used in logic; conjunction, disjunction, implication, equivalence, negation; basic rules of logic (tautologies) and how to prove them; two kinds of quantifiers: existential and universal; necessary and sufficient conditions. This section is almost identical to Section 7 in *Precalculus 1: Basic notions*; I have only removed the material covering the $\varepsilon - \delta$ definition of limit, because it is irrelevant for Discrete Mathematics. I have also added several new problems (Videos 87–93) that were *not* present in the Precalculus course.

Read along with this section: Chapter 1: *Logic and Proofs* (pp.13–61); Sections 1.4 and 1.5 from this chapter will be omitted now; they will be discussed in Section 10 of this course.

65 What is a statement and logical value; open and closed statements, sentences.

Example 1: Difference between statements and algebraic expressions; true and false statements.

66 (P1 V109) Unary logical connective: Negation NOT.

Example 2: Write the negations $\neg p$ for all the statements from Video 65. Determine whether they are false or true. Statements / sentences: 1. Number 7 is prime, 2. $98 = 26$, 3. $(-2)^2 > 0$, 4. $x^2 = 6$, 5. Function f is continuous, 6. $x = 26$, 7. $\sqrt{x} + \sin x > 0$.

Extra material: notes with solved Example 2.

67 (P1 V110) Binary logical connective 1: Conjunction AND.

Example 3: Determine whether the statement / sentence $p \wedge q$ is true or false:

a) p: $2 = 4$, q: $x^2 = 0$
b) p: $x^2 = 4$, q: $x < 0$
c) p: Number 7 is prime, q: $4 < 3$
d) p: $x - 5 < 0$, q: $x > -3$
e) p: Stockholm is the capital of Sweden, q: $6^2 = 36$
f) p: $x^2 - y^2 = (x - y)(x + y)$, q: $x = y$.

In case of open statements (sentences), determine for what values of the variables $p \wedge q$ is true.

Extra material: notes with solved Example 3.

68 (P1 V111) Binary logical connective 2: Disjunction OR.

Example 4: Determine whether the statement / sentence $p \vee q$ is true or false:

a) p: $2 = 4$, q: $x^2 = 0$

b) p: $x^2 = 4$, q: $x < 0$
c) p: Number 7 is prime, q: $4 < 3$
d) p: $x - 5 < 0$, q: $x > -3$
e) p: Stockholm is the capital of Sweden, q: $6^2 = 36$
f) p: $x^2 - y^2 = (x - y)(x + y)$, q: $x = y$.

In case of open statements (sentences), determine for what values of the variables $p \vee q$ is true.

Extra material: notes with solved Example 4.

69 (P1 V112) Connectives, Example 5.

a) Suppose p is false, q is false, s is true. Find the logical value of $(s \vee p) \wedge (q \wedge \neg s)$.
b) Suppose p is true, q is true, r is false, s is false. Find the logical value of $(s \vee p) \wedge (\neg r \vee \neg s)$.
c) Suppose p is true, q is true, s is false. Find the logical value of $(\neg s \vee p) \vee (q \wedge \neg s)$.
d) Suppose p is false, s is false, r is true. Find the logical value of $\neg[(s \wedge p) \vee \neg r]$.
e) Suppose p is false, q is true, s is true. Find the logical value of $(p \wedge \neg q) \vee \neg s$.
f) Suppose p is false, q is true, r is false. Find the logical value of $(p \vee \neg q) \vee r$.
g) Suppose p is true, q is true, r is true, s is false. Find the logical value of $(\neg p \vee s) \vee (s \wedge r)$.

Extra material: notes with solved Example 5.

70 (P1 V113) Binary logical connective 3: Implication IF THEN. A necessary condition.

Example 6: Necessary conditions:

a) $4|n \quad \Rightarrow \quad 2|n$.
b) If a quadrilateral is a square then it has four right angles.
c) $x = 2 \quad \Rightarrow \quad x^2 = 4$.
d) $x = 0 \quad \Rightarrow \quad \sin x = 0$.
e) $2x + 4 < 0 \quad \Rightarrow \quad x < -2$.

71 (P1 V114) Converse, inverse, and contrapositive statements / sentences.

Example 7: Consider the implication $x = 2 \Rightarrow x^2 = 4$. Formulate its *converse*, its *inverse*, and its *contrapositive*. Which of them are true?

Extra material: notes with solved Example 7.

72 (P1 V115) Binary logical connective 4: Equivalence IFF. A necessary and sufficient condition.

Example 8: Correct the necessary conditions in Video 70 so that they become sufficient conditions.

73 (P1 V116) Equivalence versus equality.

Example 9: Equal or equivalent?:

a) $x^2 - y^2$ $\quad (x - y)(x + y)$
b) $x + y > 0$ $\quad x > -y$
c) $x^2 + y^2 = 1$ $\quad y = \pm\sqrt{1 - x^2}$
d) $6 - 5 - 1$ $\quad \ln 1$
e) $x^2 - 2xy + y^2$ $\quad (x - y)^2$
f) $xy < 0$ $\quad \frac{x}{y} < 0$ (for $y \neq 0$).

74 (P1 V117) Tautology as a logical formula.

75 (P1 V118) How to verify tautologies.

Example 10: Prove both de Morgan's Laws: *The negation of a disjunction is the conjunction of the negations* and *The negation of a conjunction is the disjunction of the negations.*

76 (P1 V119) Tautologies, Problem 1.

Problem 1: Prove the following tautology: $(p \Rightarrow q) \Leftrightarrow \neg p \vee q$. Use this tautology to prove that the negation of implication is done according to the rule: $\neg(p \Rightarrow q) \Leftrightarrow (p \wedge \neg q)$. Show that the following tautology follows as a consequence of the previous ones: $(p \Rightarrow q) \;\Leftrightarrow\; [(p \wedge \neg q) \;\Rightarrow\; \bot]$.

Extra material: notes with solved Problem 1.

77 (P1 V120) Tautologies, Problem 2.

Problem 2: Prove that the contrapositive to a given implication is equivalent to this implication, but the converse and inverse are not. The two last cases are examples of formal fallacies, i.e. incorrect reasoning; the first one is an example of a valid, correct reasoning, and is used for proofs by contraposition which will be discussed in Section 10.

Extra material: notes with solved Problem 2.

78 (P1 V121) Tautologies, Problem 3.

Problem 3: Prove the distributivity of conjunction over disjunction: $p \wedge (q \vee r) \;\Leftrightarrow\; (p \wedge q) \vee (p \wedge r)$.

Extra material: notes with solved Problem 3.

79 (P1 V122) Tautologies, Problem 4.

Problem 4: Make a truth table for the following expression: $(p \vee q) \wedge \neg(\neg q \wedge r)$.

Extra material: notes with solved Problem 4.

80 (P1 V123) Tautologies, Problem 5.

Problem 5: Make a truth table for the following expression: $(p \Rightarrow q) \Rightarrow [(q \Rightarrow r) \Rightarrow (p \Rightarrow r)]$.

Extra material: notes with solved Problem 5.

81 (P1 V125) Existential quantifier THERE EXISTS.

Example 11: Determine whether the statements are true or false:

a) $\exists x \in \mathbb{R} \;\; x^2 < 0$,
b) $\exists! x \in \mathbb{R} \;\; x^2 \leqslant 0$,
c) $\exists! x \in \mathbb{R} \;\; x^2 > 0$,
d) $\exists x \in \mathbb{N} \;\; x + 5 = 3$,
e) $\exists x \in \mathbb{R} \;\; \exists y \in \mathbb{R} \;\; x + y = 0$,
f) $\exists x \in \mathbb{R} \;\; (x - 5 < 0 \;\wedge\; x > -3)$
g) $\exists x \in \mathbb{R} \;\; \exists y \in \mathbb{R} \;\; (x + y = 0 \;\vee\; x^2 < 0)$,
h) $\exists x \in \mathbb{R} \;\; (2x + 3 < 0 \;\wedge\; 3x - 6 > 4)$.

Extra material: notes with solved Example 11.

82 (P1 V126, p1) Universal quantifier FOR ALL. Order matters.

Example 12: Let $p(x, y) : x + y = 0$ and $q(x, y) : xy = 1$. Determine whether the following statements are true or false. Parts b) and c) show that the order of quantifiers matters (if they are different):

a) $\forall x \in \mathbb{R} \;\; \forall y \in \mathbb{R} \;\; p(x, y)$; $\quad \forall x \neq 0 \;\; \forall y \neq 0 \;\; q(x, y)$,
b) $\forall x \in \mathbb{R} \;\; \exists y \in \mathbb{R} \;\; p(x, y)$; $\quad \forall x \neq 0 \;\; \exists y \neq 0 \;\; q(x, y)$,
c) $\exists y \in \mathbb{R} \;\; \forall x \in \mathbb{R} \;\; p(x, y)$; $\quad \exists y \neq 0 \;\; \forall x \neq 0 \;\; q(x, y)$,
d) $\exists x \in \mathbb{R} \;\; \exists y \in \mathbb{R} \;\; p(x, y)$; $\quad \exists x \neq 0 \;\; \exists y \neq 0 \;\; q(x, y)$.

Extra material: notes with solved Example 12.

83 (P1 V126, p2) Universal quantifier FOR ALL. Precedence rules.

Example 13: Illustrate that the following two statements are **not** equivalent

$$\forall x \; (p(x) \vee q(x)) \quad \text{and} \quad \forall x \; p(x) \vee q(x).$$

Use statements $p(x) : x > -3$ and $q(x) : x < 5$.

Extra material: notes with solved Example 13.

84 (P1 V84 and V127) Negations: De Morgan's Laws for quantifiers, Problem 6.

Problem 6: Use logical symbols to write the definitions of monotone functions. Show that $f(x) = \frac{1}{x}$ is not decreasing on $\mathbb{R} \setminus \{0\}$.

85 (P1 V129) Logic, Problem 7.

Problem 7: Write the following sentences with mathematical symbols:

a) Natural number n is even.
b) Natural number n is a sum of squares of two integers.
c) Natural number n is prime.
d) Natural number n is not prime.
e) There is no largest element in the set of natural numbers.
f) There exists no real number whose square is negative.

Extra material: notes with solved Problem 7.

86 (P1 V130) Logic, Problem 8.

Problem 8: Translate *The sum of two positive integers is always positive* into a logical expression.

87 Logic, Problem 9.

Problem 9: Translate *Number n is divisible by 3, and the sum of squares of numbers a and b is divisible by 5* into a logical expression.

Extra material: notes with solved Problem 9.

88 Logical thinking, Problem 10.

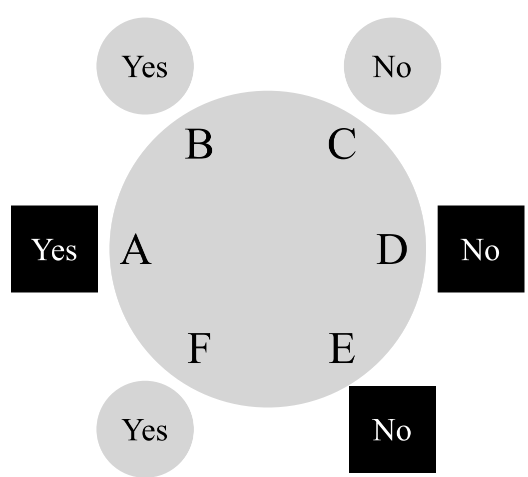
Problem 10: Determine all numbers $n, k \in \mathbb{N}^+$ for which exactly three of the following statements are true:

(1) $k | n + 1$
(2) $n = 2k + 5$
(3) $3 | n + k$
(4) $n + 7k$ is a prime number.

Extra material: notes with solved Problem 10.

89 Logical thinking, Problem 11.

Problem 11: Six people are sitting around the table as shown in the picture. The people sitting on square chairs form Team 1 and the people sitting on the round chairs form Team 2. All the members of one of the teams always lie, and of the other one always tell the truth. One of the persons is keeping a diamond, and everybody at the table knows who it is. You ask them *Is some of your neighbours to the left or to the right keeping the diamond?* Their answers are given in the picture. Who has the diamond?



90 Logical thinking, Problem 12.

Problem 12: Persons A, B, and C make some statements. Determine which of them tells the truth and which of them lies:

- A says: B lies.
- B says: C lies.
- C says: A lies and B lies.

Solve the problem using a chart.

91 Logical thinking and truth tables, Problem 13.

Problem 13: Persons A, B, and C make some statements. Determine which of them tells the truth and which of them lies:

- A says: B lies.
- B says: C lies.
- C says: A lies and B lies.

Solve the same problem as above, but this time using a truth table.

92 Logical thinking and truth tables, Problem 14.

Problem 14: Three friends: A, B, and C wrote a test. Determine which of them passed, knowing that:

1. If B passed the test, then A or C also passed the test.
2. If A passed the test, then B also passed the test.
3. Either C or B passed the test, but not both of them.
4. If B didn't pass the test, then A and C passed the test.

93 Einstein's riddle, Problem 15.

Problem 15 (Einstein's Riddle / Zebra Puzzle): The legends says that this problem was created by Albert Einstein. There are five houses of different colors next to each other. In each house lives a man. Each man has a unique nationality, an exclusive favorite drink, a distinct favorite brand of cigarettes and keeps specific pets. Use all the clues below to fill the grid and answer the questions: *Who drinks water? Who owns the zebra?*:

1. There are five houses.
2. The Englishman lives in the red house.
3. The Spaniard owns the dog.
4. Coffee is drunk in the green house.
5. The Ukrainian drinks tea.
6. The green house is immediately to the **right** of the ivory house.
7. The Old Gold smoker owns snails.
8. Kools are smoked in the yellow house.
9. Milk is drunk in the middle house.
10. The Norwegian lives in the first house.
11. The man who smokes Chesterfields lives in the house next to the man with the fox.
12. Kools are smoked in the house next to the house where the horse is kept.
13. The Lucky Strike smoker drinks orange juice.
14. The Japanese smokes Parliaments.
15. The Norwegian lives next to the blue house.

Source: according to Wikipedia:

`https://en.wikipedia.org/wiki/Zebra_Puzzle`

Life International, 1962. (My correction: in 6., it should be **right**, not *your right*, as Wikipedia says.)

S5 Sets

You will learn: the basic terms and formulas from the Set Theory and the link to Logic; union, intersection, set difference, subset, complement; cardinality of a set; Inclusion–exclusion principle. This section is almost identical to Section 8 in *Precalculus 1: Basic notions*.

Read along with this section: Subsection 0.2.2: *Sets* (p.5), Section 5.1: *Sets* (pp.389–402).

94 Primitive notions: set, belonging to a set, the empty set.

95 (P1 V135) The universe and its subsets.

96 (P1 V136) If you know logic, you know the set theory.

97 (P1 V137) Intersection defined with help of conjunction.

Example 1.1: Determine $A \cap B$ if $A = [-3, 3)$ and $B = (1, 5]$. Example 1.2 (**Optional**): $\bigcap_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n}) = \{0\}$.

Cool fact (**Optional**): old Polish universal quantifier.

98 (P1 V138) Union defined with help of disjunction.

Example 2.1: Determine $A \cup B$ if $A = [-3, 3)$ and $B = (1, 5]$. Example 2.2 (**Optional**): write down the solution set to the inequality $\sin x > 0$ (for $x > 0$), using the graph of $f(x) = \sin x$.

Cool fact (**Optional**): old Polish existential quantifier.

99 (P1 V139) Equality of sets defined with help of equivalence: Axiom of Extensionality.

100 (P1 V140) Subset defined with help of implication.

Example 3: The set $A = \{0, 1, 2, 3\}$ is not a subset of $B = \{1, 2, 3, 4, 5\}$.

101 (P1 V141) The empty set gives a false statement.

Example 4: The empty set is a subset of each set A.

102 (P1 V142) The universe gives a true statement.

103 (P1 V143) Set difference.

Example 5: Show that the set difference is **not** commutative using $A = \{0, 1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$.

104 (P1 V144) Symmetric difference.

Example 6: Compute symmetric difference of $A = \{1, 2, 3, 4, 6, 8\}$ and $B = \{1, 2, 4, 5, 7, 8, 9, 10\}$.

105 (P1 V145) Complement defined with help of negation.

Example 7: Describe A^c with respect to the universe U, where:

a) A: set of all the positive even integers, $U = \mathbb{N}$.
b) A: set of all the positive even integers, $U = \mathbb{R}$.
c) $A = (-1, 2]$, $U = \mathbb{R}$.
d) $A = \{-4, -1, 2, 3, 5\}$, $U = \{x \in \mathbb{Z};\ |x| \leqslant 5\}$.
e) $A = [-4, 3] \cap \mathbb{N}$, $U = \{x \in \mathbb{Z};\ |x| \leqslant 5\}$.
f) $A = [-4, 3] \cap \mathbb{N}$, $U = \{x \in \mathbb{R};\ |x| \leqslant 5\}$.

Extra material: notes with solved Example 7.

106 (P1 V146) The laws of set theory and the laws of logic.

107 (P1 V147) De Morgan's Laws, an illustration and proof.

Extra material: notes from the iPad.

108 (P1 V148) The distributive law, an illustration and proof.

Extra material: notes from the iPad.

109 (P1 V149) Set difference and symmetric difference, Problem 1.

Problem 1: Show that the symmetric difference of two sets can be expressed as $A \triangle B = (A \cup B) \setminus (A \cap B)$.

Extra material: notes with solved Problem 1.

110 Cartesian product of sets, some exercises.

Example 8: Describe $A \times B$ and $B \times A$ if $A = \{1, 2, 3, 4\}$ and $B = [1, 4)$.

Example 9: If $A \times B = \{(1, \diamondsuit), (\heartsuit, \circ), (\heartsuit, \diamondsuit), (1, d), (1, \circ), (\heartsuit, d)\}$, describe A and B.

Extra material: notes with solved examples.

111 (P1 V151) Power set of a given set; an example of Boolean algebra.

Example 10: Describe the power set 2^X, where:

a) $X = \{a, b\}$.

b) $X = \{1, 2, 3\}$.

c) $X = \{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$.

112 (P1 V152) Cardinality of sets.

113 (P1 V153) Cardinality of sets, Problem 2.

Problem 2: Let $|X| = n$. Show that cardinality of the power set 2^X is 2^n.

114 Equinumerous (equipotent) sets; cardinal numbers.

115 (P1 V158) Inclusion–exclusion principle.

Example 11: In a group of 30 students: 18 students like tea and 25 students like coffee.

a) How many students like both coffee and tea?

b) How many students like only coffee?

c) How many students like only tea? (We assume that each student likes *at least* one of them: coffee or tea.)

116 (P1 V159) Inclusion–exclusion principle, Problem 3.

Problem 3: In a group of 57 students: 42 students can speak English, 38 students can speak Swedish, 35 students can speak Polish. In the same group: 23 students can speak both Swedish and English, 20 students can speak both English and Polish, and 27 students can speak both Swedish and Polish. How many students in this group can speak all the three languages? How many students can speak just one language (if each speaks *at least* 1)?

Extra material: notes with solved Problem 3.

117 (P1 V160) Transposition law and subsets, Problem 4.

Problem 4: Illustrate the Transposition law for implications (formulated in Video 74, proven in Video 77) $(p \Rightarrow q) \;\Leftrightarrow\; (\neg q \Rightarrow \neg p)$ with help of Venn diagrams.

118 More solved problems in the DM Book, Problems 5 and 6.

Problem 5 (Problem 23 on page 401 in the DM Book): Let $A = \{2, 4, 6, 8\}$. Suppose B is a set with $|B| = 5$.

a) What are the smallest and largest possible values of $|A \cup B|$? Explain.

b) What are the smallest and largest possible values of $|A \cap B|$? Explain.

c) What are the smallest and largest possible values of $|A \times B|$? Explain.

Problem 6 (Problem 24 on page 401 in the DM Book): Let $X = \{n \in \mathbb{N};\ 10 \leqslant n < 20\}$. Find examples of sets with the properties below and very briefly explain why your examples work.

a) A set $A \subseteq \mathbb{N}$ with $|A| = 10$ such that $X \setminus A = \{10, 12, 14\}$.

b) A set $B \in \mathcal{P}(X)$ with $|B| = 5$.

c) A set $C \subseteq \mathcal{P}(X)$ with $|C| = 5$.

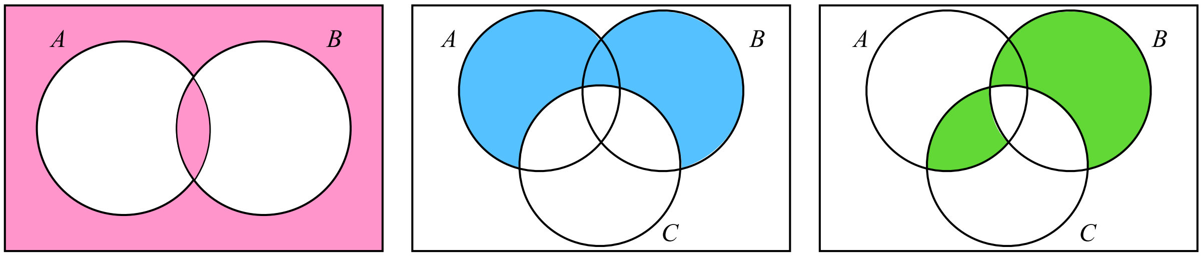
d) A set $D \subseteq X \times X$ with $|D| = 5$.

e) A set $E \subseteq X$ such that $|E| \in E$.

119 The one with three pictures and Boolean algebras, Problem 7.

Problem 7: Describe the colored parts in each of the Venn diagrams beneath.

Extra material: notes from the iPad, with a proof of a formula showing two descriptions of the first set.



S6 Functions

You will learn: about functions: various ways of defining functions; domain, codomain, range, graph; surjections, injections, bijections, inverse functions, inverse images; bijections and cardinality of sets; compositions of functions; some examples of monotone and periodic functions. You will get an information about other topics relevant for examining functions (in Calculus) and where to find them (in the Precalculus and Calculus series).

Read along with this section: Subsection 0.2.3: *Functions* (p.6), Section 5.2: *Functions* (pp.403–420).

120 What is a function: back to V24; terminological differences with the DM Book.

121 A list of more and less relevant topics.

122 Why some issues are less relevant for DM, and where to find them.

123 Various ways of defining functions, several examples.

Example 1: Social-security number (in Sweden).

Example 2: Determine the domain, codomain, and range for $f : X \to Y$, where $X = \{a, b, c, d\}$, $Y = \{1, 2, 3, 4\}$, and f is defined by the following table:

a	b	c	d
1	1	4	2

Try to illustrate this function using Venn diagrams. Mark the range as a subset of the target set. Plot the graph in the Cartesian product $X \times Y$.

Example 3: Each number is mapped on its double (various functions with different domains).

Example 4: Each integer is mapped on its remainder in division by 5.

124 How to recognise that a rule describes a function.

Examples from the previous lecture. We learn how to recognise that a rule is a function if represented:

a) with help of a table,
b) with help of Venn diagrams with arrows,
c) as a graph in the Cartesian product $X \times Y$ (*vertical-line test*).

125 Recursively defined functions, a very brief introduction.

We look at four examples on pages 407 and 408 in the DM Book. (These problems come back in Section 9 or Section 10, when we will perform a proof by induction, and later in DM2.)

Example: Plot the following function that is defined recursively: $f(0) = 1$, $f(n+1) = 2f(n)$ for $n \in \mathbb{N}$. Determine the explicit formula for this function (a formal proof will be conducted in Section 9 or Section 10).

126 Injections, surjections, and bijections.

127 How to detect an injection in various cases.

Case 1: $f : \mathbb{N} \to \mathbb{N}$ given by a formula: verify if from $f(n_1) = f(n_2)$ follows $n_1 = n_2$.

Case 2: $f : X \to Y$ given by its graph in the Cartesian product $X \times Y$: *horizontal-line test*.

Case 3: f given by a table: see if all the values in the second row are different from each other.

Case 4: f given by a verbal description: verify if you can go back to the arguments from the values.

Example 5.2.7 from the DM Book.

128 Function? Injection? Surjection? Some exercises.

Exercise 1: to each positive natural number we assign its positive natural divisors.

Exercise 2: to each positive natural number we assign the number of its positive natural divisors.

Exercise 3: to each positive natural number we assign the sum of its positive natural divisors.

Extra material: notes from the iPad.

129 Surjections, injections, and logical symbols, Problem 1.

Problem 1: Use logical symbols to write the definitions of an injection and surjection. Write with mathematical symbols that some function is not an injection / surjection.

Extra material: notes with solved Problem 1.

130 Surjections and injections, Problem 2.

Problem 2 (Problem 19 on page 418 in the DM Book):

Suppose $f : X \to Y$ is a function. Which of the following are possible? Explain.

a) f is injective but not surjective.
b) f is surjective but not injective.
c) $|X| = |Y|$ and f is injective but not surjective.
d) $|X| = |Y|$ and f is surjective but not injective.
e) $|X| = |Y|$, X and Y are finite, and f is injective but not surjective.
f) $|X| = |Y|$, X and Y are finite, and f is surjective but not injective.

131 Two classes of functions: identities and constant functions.

132 Inverse functions.

Example: Determine the inverse function to $f : X \to Y$, where $X = \{a, b, c, d\}$, $Y = \{1, 2, 3, 4\}$, and f is defined by the following table:

a	b	c	d
2	4	1	3

Use various ways of depicting this function and its inverse. Compare their graphs.

133 Compositions of functions.

Example: Let $A = \{a, b, c\},\ B = \{x, y, z\},\ C = \{r, s, t\}$. Let $f : A \to B$ and $g : B \to C$ be defined by:

$$f = \{(a, y),\ (b, x),\ (c, y)\} \quad \text{and} \quad g = \{(x, s),\ (y, t),\ (z, r)\}.$$

a) Find composition function $g \circ f : A \to C$.
b) Find $\mathrm{Im}(f)$, $\mathrm{Im}(g)$, $\mathrm{Im}(g \circ f)$ (Im, image, is another abbreviation for the range).

Extra material: notes with solved Example.

134 **Optional** Compositions of functions, a Precalculus example.

Example: Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x + 1$ and $g(x) = x^2 - 2$. Find the formula for the compositions $f \circ g$ and $g \circ f$. Comments? You get a nice illustration here.

Extra material: notes with solved Example.

135 Compositions with identities, an exercise.

Exercise 4: Show that compositions with appropriate identity functions don't change the given function. Illustrate it for the function $f : X \to Y$ with $X = \{1, 2, 3\}$ and $Y = \{a, b\}$, $f = \{(1, a),\ (2, a),\ (3, b)\}$; use both identity functions Id_X and Id_Y.

Extra material: notes with solved Exercise 4.

136 Compositions of functions, Problem 3.

Problem 3: Consider permutations:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 4 & 5 & 1 & 2 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 5 & 3 & 1 \end{pmatrix}.$$

Find composition $\tau \circ \sigma$ and σ^{-1}.

Extra material: notes with solved Problem 3.

137 Associativity of composition of functions.

Theorem: If f, g, and h are such functions that the composition $h \circ (g \circ f)$ is defined, then $(h \circ g) \circ f$ is also defined, and $(h \circ g) \circ f = h \circ (g \circ f)$, i.e., composition of functions (whenever defined) is *associative*.

Extra material: notes from the iPad.

138 Really cool stuff, where compositions meet geometry, Problem 4.

Problem 4: Find inverses to the permutations from Video 63 (that can be interpreted as own isometries of an equilateral triangle) and explain the geometric aspect of your findings. We will describe the group of permutations of three elements; we will analyse more groups like this one in DM2.

We show that $f^{-1} \circ f = \mathrm{Id}_X$ and $f \circ f^{-1} = \mathrm{Id}_Y$ for invertible functions $f : X \to Y$.

Extra material: notes with solved Problem 4.

139 (P1 V184) Inverse functions and compositions of functions, Problem 5.

Problem 5: Let $X = \{1, 2, 3, 4\}$ and $Y = \{5, 6, 7\}$.

a) Write down (explicitly) the Cartesian product $X \times Y$.
b) Give an example of a function $f : X \to Y$ that is surjective but not injective. Are there any injective functions $f : X \to Y$?
c) Give an example of a function $f : Y \to X$ that is injective but not surjective. Are there any surjective functions $f : Y \to X$?
d) Determine the inverse of the bijection $f : X \to X$ defined by: $f(1) = 3$, $f(2) = 4$, $f(3) = 1$, $f(4) = 2$. Verify that $f \circ f = \mathrm{Id}_X$ where $\mathrm{Id}_X(x) = x$ for all $x \in X$.

Extra material: notes with solved Problem 5.

140 Uniqueness of the inverse.

We formulate an equivalent definition of inverse function (in terms of compositions) to $f : X \to Y$ as such $g : Y \to X$ that $f \circ g = \mathrm{Id}_Y$ and $g \circ f = \mathrm{Id}_X$.

Theorem: Not every function has an inverse, but if an inverse exists, then it is unique.

141 A link between having an inverse and being a bijection.

Theorem: A function has an inverse if and only if it is a bijection.

Extra material: notes from the iPad.

142 (P1 V186 p1) Surjections, injections, and compositions, Problem 6.

Problem 6: Let $f : A \to B$ and $g : B \to C$ be two functions, and $h = g \circ f$ be their composition, i.e. $h(x) = g(f(x))$ for all $x \in A$. Answer the following questions and motivate your answers:

a) Assume that f and g are both surjective. Show that then h is surjective, too.
b) Assume that f and g are both injective. Show that then h is injective, too.

Extra material: notes with solved Problem 6.

143 (P1 V186 p2) Surjections, injections, and compositions, Problem 7.

Problem 7: Let $f : A \to B$ and $g : B \to C$ be two functions, and $h = g \circ f$ be their composition, i.e. $h(x) = g(f(x))$ for all $x \in A$. Answer the following questions and motivate your answers:

a) Assume that h and f are both surjective. Is then g necessarily surjective, too?

b) Assume that h and f are both injective. Is then g necessarily injective, too?

Extra material: notes with solved Problem 7.

144 (P1 V186 p3) Surjections, injections, and compositions, Problem 8.

Problem 8: Let $f : A \to B$ and $g : B \to C$ be two functions, and $h = g \circ f$ be their composition, i.e. $h(x) = g(f(x))$ for all $x \in A$. Answer the following questions and motivate your answers:

a) Assume that h and g are both surjective. Is then f necessarily surjective, too?

b) Assume that h and g are both injective. Is then f necessarily injective, too?

Extra material: notes with solved Problem 8.

145 (P1 V155) **Optional**: Finite, countable and uncountable sets.

146 (P1 V156) **Optional**: Comparing cardinalities of sets, Problem 9.

Problem 9: Construct surjections showing that the following sets are countable:

a) $\{4, 5, 6, 7, 8, \ldots\}$,

b) the set of all even natural numbers,

c) the set of all integers.

Extra material: notes with solved Problem 9.

147 **Optional, Advanced**: Comparing cardinalities of sets, Problem 10.

Problem 10: Show that the set of rational numbers is countable, i.e., because it is infinite, it has the same cardinality as the set of natural numbers.

Extra material: notes with solved Problem 10.

148 Images and inverse images of elements and subsets.

We look together at three examples from the DM Book: Examples 5.2.8, 5.2.9, and 5.2.10 on pp. 412 and 413.

We describe injections and surjections with help of the new concepts.

149 Inverse function versus inverse image, Problem 11.

Problem 11: Illustrate the difference between both concepts for the functions

a) $f : X \to Y$ with $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$, $f = \{(1, a),\ (2, a),\ (3, b)\}$,

b) $g : X \to Y$ with $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$, $g = \{(1, c),\ (2, a),\ (3, b)\}$.

Extra material: notes with solved Problem 11.

150 Images and inverse images of elements and subsets, Problem 12.

Problem 12 (Problem 15 on page 418 in the DM Book): Consider the set $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$, the set of all ordered pairs (a, b) where a and b are natural numbers. Consider a function $f : \mathbb{N}^2 \to \mathbb{N}$ given by $f((a, b)) = a + b$.

a) Let $A = \{(a, b) \in \mathbb{N}^2;\ a, b \leqslant 10\}$. Find $f(A)$.

b) Find $f^{-1}(\{3\})$ and $f^{-1}(\{0, 1, 2, 3\})$.

c) Give geometric descriptions of $f^{-1}(\{n\})$ and $f^{-1}(\{0, 1, \ldots, n\})$ for any $n \geqslant 1$.

d) Find $|f^{-1}(\{8\})|$ and $|f^{-1}(\{0, 1, \ldots, 8\})|$.

Extra material: notes with solved Problem 12.

151 Images and inverse images, Problem 13.

Problem 13 (Problem 28 on page 419 in the DM Book): Let $f : X \to Y$ be a function, $A \subseteq X$ and $B \subseteq Y$.

a) Is $f^{-1}(f(A)) = A$? Always, sometimes, or never? Explain.

b) Is $f(f^{-1}(B)) = B$? Always, sometimes, or never? Explain.

c) If one or both of the above do not always hold, is there something else you can say? Will equality always hold for particular types of functions?

Extra material: notes with solved Problem 13.

152 Images, unions, and intersections, Problem 14.

Problem 14 (Problem 29 on page 420 in the DM Book): Let $f : X \to Y$ be a function and $A, B \subseteq X$ be subsets of the domain.

a) Is $f(A \cup B) = f(A) \cup f(B)$? Always, sometimes, or never? Explain.

b) Is $f(A \cap B) = f(A) \cap f(B)$? Always, sometimes, or never? Explain.

Extra material: notes with solved Problem 14.

153 Inverse images, unions, and intersections, Problem 15.

Problem 15 (Problem 30 on page 420 in the DM Book): Let $f : X \to Y$ be a function and $A, B \subseteq Y$ be subsets of the codomain.

a) Is $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$? Always, sometimes, or never? Explain.

b) Is $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$? Always, sometimes, or never? Explain.

Extra material: notes with solved Problem 15.

154 Monotone (or: monotonic) functions, Example 1.

Example 1: The function $f : \mathbb{N} \to \mathbb{N},\ f(n) = 2n$, presented in V123, is strictly increasing.

155 Monotone (or: monotonic) functions, Example 2.

Example 2: We look at the solution to Problem 5.2.4.6 on p.512 in the DM Book and at our own functions $f : \{a, b, c\} \to \{1, 2, 3, 4\}$. (No, I will *not* show *all* of them! How many of such functions exist?). We examine how the relation between $|X|$ and $|Y|$ influences existence or absence of monotone functions $f : X \to Y$ (where we have some order on our *finite* sets X and on Y).

156 Monotone (or: monotonic) functions, Example 3.

Example 3: Ceiling $\lceil x \rceil$ (the *ceiling function*) and floor $\lfloor x \rfloor$ (the *greatest-integer* function or *floor function*) functions are non-decreasing.

157 **Optional**: A word about periodic functions.

Example 1: The function

$$f : \mathbb{Z} \to \{0, 1, 2, 3, 4\}, \quad f(n) = r \quad \text{if } n \equiv r \pmod 5$$

presented in V123 is periodic with the basic period $P = 5$.

Example 2: The *fractional-part* function $f : \mathbb{R} \to \mathbb{R}$ defined with help of the *greatest-integer* function (a.k.a. the *floor function*):

$$f(x) = \{x\} := x - \lfloor x \rfloor$$

is periodic with the basic period $P = 1$.

158 **Optional**: A word about even and odd functions.

Example: The function

$$f : \mathbb{Z} \to \{0, 1, 2, 3, 4\}, \quad f(n) = r \quad \text{if } n \equiv r \pmod 5$$

presented in V123 can be adapted in such a way that it becomes an even or odd function on $\mathbb{Z}$.

159 The last one, Problem 16.

Problem 16: Given the following functions $f, g : \mathbb{N} \to \mathbb{N}$:

$$f(n) = n^4, \qquad g(n) = r \quad \text{if } n \equiv r \pmod{16} \text{ and } r \in \{0, 1, \ldots, 15\}.$$

Find the compositions $f \circ g$ and $g \circ f$ and determine $|\text{Im}(f \circ g)|$ and $|\text{Im}(g \circ f)|$. Hint: see V55.

Extra material: notes with solved Problem 16.

S7 Relations

You will learn: about binary relations generally, and specifically about RST (Reflexive–Symmetric–Transitive) relations, equivalence classes, and about order (partial order) relations. This section is almost identical to Section 9 in *Precalculus* 1*: Basic notions*.

Read along with this section: Subsection 0.2.5: *Relations* (p.9), Section 2.6: *Relations and Graphs* (pp.163–180).

160 Three videos to watch (if you haven't yet).

161 Relations in the DM Book.

162 Relation: a formal definition and notation.

Example 1: We look together at Example 2.6.3 from the DM Book (p.166) and depict the relation in the Cartesian product of appropriate sets.

163 Various ways of depicting relations.

164 (P1 V164) How to plot relations.

Example 2: Let $X = \mathbb{R}$. Illustrate the following relations on X:

a) $xRy \quad \Leftrightarrow \quad x = y$,
b) $xRy \quad \Leftrightarrow \quad x < y$,
c) $xRy \quad \Leftrightarrow \quad x \leqslant y$,
d) $xRy \quad \Leftrightarrow \quad x > y$.

165 (P1 V165) How to plot relations, Problem 1.

Problem 1: Let $X = \{1, 2, 3, 4, 5\}$. Illustrate the following relations on X:

a) $xRy \quad \Leftrightarrow \quad x < y$,
b) $xRy \quad \Leftrightarrow \quad x|y$,
c) $xRy \quad \Leftrightarrow \quad y = x^2$,
d) $xRy \quad \Leftrightarrow \quad x \equiv y \pmod 2$.

166 (P1 V166) Relations on finite sets, and their graphs, Problem 2.

Problem 2: Draw the graphs depicting the relations from Problem 1.

Extra material: notes with solved Problem 2.

167 (P1 V167) RST relations / equivalence relations.

Problem 3: Which relations described in Example 0 (Video 26, Video 160, Video 162) are RST?

Extra material: notes with solved Problem 3.

168 (P1 V168) How to recognize an RST relation from its plot?

Problem 4: Which relations described in Videos 164 and 165 are RST?

169 (P1 V169) How to recognize an RST relation on a finite set from the graph?

Problem 5: Use the relations described in Video 165 and their graphs from Video 166 to verify the graph method of testing relations for being RST.

Extra material: notes with solved Problem 5. My solutions to Problems 4, 5, and 6 (p.179) in the DM Book.

170 (P1 V170) Relation congruence modulo n.

Extra material: notes with the proof that the relation modulo is an equivalence relation.

171 (P1 V171) Equivalence classes and partitions of a set.

Example 3: Analyse the relation modulo 5 and its equivalence classes.

Example 4: Describe equivalence classes in relations of parallelity of lines, similarity of triangles; vectors.

Extra material: notes with solved Example 4.

172 (P1 V172) More properties of relations: irreflexive, antisymmetric, strongly connected.

Example 5: Test the new properties for $<$, $>$, $\leqslant$, $\geqslant$ on $X = \mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}, \mathbb{R}$, and for $x|y$ on $\mathbb{N}^+$.

Extra material: notes with solved Example 5.

Extra material: notes from the iPad with an illustration: how various properties of relations look in their tables.

173 (P1 V173) Order relations.

174 (P1 V174) Partial orders and Hasse diagrams.

Example 6: Draw Hasse diagram for Example 10 c) from Video 111 (subset relation) and for relation of being divisor of described in Video 26 (use number 60).

175 (P1 V175) Relations, Problem 6.

Problem 6: Let $X = \{a, b, c, d\}$. Examine the properties of $R \subset X \times X$ if:

a) $R = \{(a, a), (b, b), (a, b)\}$,
b) $R = \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, a)\}$,
c) $R = \{(a, a), (b, b), (c, c), (a, b), (a, c), (b, c)\}$.

Extra material: notes with solved Problem 6.

176 (P1 V176) Relations, Problem 7.

Problem 7: The relation $R \subset \mathbb{N} \times \mathbb{N}$ is defined as:

$$\forall \ x, y \in \mathbb{N} \quad [xRy \quad \Leftrightarrow \quad (x \leqslant 5 \ \wedge \ y \leqslant 5 \ \wedge \ x = y) \ \vee \ (x > 5 \ \wedge \ y > 5 \ \wedge \ 2|x + y)].$$

Examine properties of this relation.

Extra material: notes with solved Problem 7.

177 (P1 V209) Defining the set of integers with help of an equivalence relation, Problem 8.

Extra material: notes from the iPad (showing that our relation is RST, and that the addition is well-defined). You also get some information about a construction of the set of rational numbers with help of an RST relation on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ (presented in detail in V210 of *Precalculus 1: Basic notions*).

178 (P1 V178) Relations, Problem 9.

Problem 9: Let $\mathcal{P}(\mathbb{N})$ be the set of all the subsets of $\mathbb{N}$ and let $\mathbb{E}$ denote the set of all the even natural numbers. We define the relation $R \subset \mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N})$ in the following way:

$$\forall \ X, Y \in \mathcal{P}(\mathbb{N}) \quad (XRY \quad \Leftrightarrow \quad X \cap \mathbb{E} = Y \cap \mathbb{E}).$$

Examine properties of this relation.

Extra material: notes with solved Problem 9.

S8 Functions as relations

You will learn: definition of a function as relation between sets: domain and co-domain; injections, surjections, bijections, inverse functions. This section is almost identical to Section 10 in *Precalculus 1: Basic notions.*

Read along with this section: Section 2.6: *Relations and Graphs* (pp.163–180).

179 (P1 V179) Some early signs.

180 (P1 V180) Each function is a relation, but not every relation is a function.

181 (P1 V181) Each relation is invertible!

Example: Find the inverses to the relations b) and c) from Video 165. Here $X = \{1, 2, 3, 4, 5\}$. Make a picture and write explicitly both R and R^{-1}: b) $xRy \quad \Leftrightarrow \quad x|y$, c) $xRy \quad \Leftrightarrow \quad y = x^2$.

182 (P1 V182) Relations and functions, Problem 1.

Problem 1: Relation R from $X = \{a, b, c, d\}$ to $Y = \{m, n, p\}$ is defined as

$$R = \{(a, m), (a, n), (b, m), (b, n)\}.$$

Determine the following for R:

a) its departure set and its target set,
b) its domain and its range,
c) its inverse R^{-1}.

Extra material: notes with solved Problem 1.

183 (P1 V183) Relations and functions, Problem 2.

Problem 2: Which of the following relations between $X = \{1, 2, 3, 4, 5\}$ and $Y = \{a, b, c, d\}$ are functions?:

a) $R_1 = \{(1, a), (2, b), (3, c), (3, d)\}$,
b) $R_2 = \{(1, a), (2, b), (2, c), (3, d)\}$,
c) $R_3 = \{(1, a), (1, b), (1, c), (3, d)\}$,
d) $R_4 = \{(1, a), (2, a), (4, d), (5, d)\}$.

Extra material: notes with solved Problem 2.

184 (P1 V185) Equality of functions, Problem 3.

Problem 3: Given $f : A \to B$ and $g : C \to D$. Verify if they are equal to each other if:

a) $A = \{1, 2, 3\},\ B = \{m, n, p, q\},\ C = \{1, 2, 3, 4\},\ D = \{m, n, p, q\},\ f = \{(1, m), (2, p)\},\ g = \{(1, m), (2, p)\}$,
b) $A = \{1, 2, 3\},\ B = \{m, n, p, q\},\ C = \{1, 2, 3\},\ D = \{m, n, p\},\ f = \{(1, m), (2, p)\},\ g = \{(1, m), (2, p)\}$,
c) $A = \{1, 2, 3\},\ B = \{m, n, p, q\},\ C = \{1, 2, 3\},\ D = \{m, n, p, q\},\ f = \{(1, m), (2, p)\},\ g = \{(1, m), (2, p), (3, q)\}$,
d) $A = \{1, 2, 3\},\ B = \{m, n, p, q\},\ C = \{1, 2, 3\},\ D = \{m, n, p, q\},\ f = \{(1, m), (2, p)\},\ g = \{(1, m), (2, p)\}$.

185 Compositions of relations, Example 1.

Example: We look together at Example 2.6.7 from the DM Book (p.169), which is a continuation of the example from V162; a new relation is defined (showing the professors that teach the courses) and we examine the composition of the old relation with the new one.

186 Compositions of relations, Example 2.

Example: Let $R = \{(n, n + 1);\ n \in \mathbb{N}\}$. Draw a graph of this relation, and of the composition $R \circ R$.

187 **Optional**: (P1 V187) Cancellation law for injective functions and why it is important.

Theorem: If $f : Y \to Z$ is injective, then it is **left cancellable**, i.e.

$$\forall X\ \ \forall g_1, g_2 : X \to Y \quad (f \circ g_1 = f \circ g_2 \quad \Rightarrow \quad g_1 = g_2).$$

Example: Solve two equations: $\ln(7x - 9) = \ln(x^2 - x - 29), \quad e^{7x-9} = e^{x^2-x-29}$.

188 Three theorems about compositions and inverses of functions.

a) If $f : X \to Y$, $g : Y \to Z$, $h : Z \to W$, then $(h \circ g) \circ f = h \circ (g \circ f)$ (proven in V137).
b) If $f : X \to Y$ and $g : Y \to Z$ are invertible then also $g \circ f$ is invertible and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.
c) If $f : X \to Y$ is invertible then also f^{-1} is invertible and $(f^{-1})^{-1} = f$.

Extra material: notes with the proof of Theorem.

189 Three theorems from the previous video translated for relations.

a) If $R \subset X \times Y$, $S \subset Y \times Z$, $T \subset Z \times W$, then $(T \circ S) \circ R = T \circ (S \circ R)$.
b) If $R \subset X \times Y$ and $S \subset Y \times Z$ then $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$.
c) If $R \subset X \times Y$ then $(R^{-1})^{-1} = R$.

S9 A very brief introduction to sequences

You will learn: you will get an extremely brief introduction to the topic of sequences: just enough for the next sections in this course (proofs and combinatorics; for the latter we only need *finite* sequences) and to complete the discussion of functions defined on the set of natural numbers (adding and scaling sequences, monotone sequences); the topic of sequences will be covered very thoroughly and from scratch in DM2.

Read along with this section: Sections 4.1: *Describing Sequences* (pp.311–326) and 4.2: *Rate of growth* (pp.327–337).

190 Sequences as functions defined on the set of natural numbers.

* Scaling of a sequence. $k = 2,\ \ a_n = \frac{1}{n},\ \ m_n = ka_n = \frac{2}{n}.$
* Sum of two sequences. $b_n = (-1)^n \frac{1}{n},\ \ d_n = \frac{n+1}{n},\ \ m_n = b_n + d_n.$

191 Explicit and recursive formulas defining sequences.

Problem 1 (Example 5.2.4 on page 407 from the DM Book): Consider the function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(0) = 0$ and $f(n+1) = f(n) + 2n + 1$ for $n \in \mathbb{N}$. Show that $f(n) = n^2$ for each $n \in \mathbb{N}$. (This problem comes back in Section 10, when we will perform a proof by induction, and later in DM2.)

Extra material: notes with solved Problem 1.

192 Monotone number sequences.

S10 Various proof techniques

You will learn: the meaning of words *axiom*, *definition*, *theorem*, *lemma*, *proposition*, *corollary*, *proof*; various types of proofs with some examples: direct proof, indirect proof (proof by contradiction, proof by contrapositive), proof by induction, proof by cases; proving or disproving statements (finding counterexamples).

Read along with this section: Sections 1.4: *Proofs* (pp. 62–82), 1.5: *Proofs about Discrete Structures* (83–95), 4.5: *Proof by Induction* (pp.363–376), 4.6: *Strong Induction* (pp.377–384).

193 Many types of proofs and where to find them.

* You get an information about more reading material about proofs.
* You have already got a soft introduction to proving stuff.
* Some good advice about conducting proofs.
* Do not despair, proving stuff *is* hard; there are even some simple statements that are still not proved...

194 (P1 V191) Examples of primitive notions and definitions.

195 (P1 V192) Axioms.

196 (P1 V193) Theorems, propositions, lemmas, corollaries.

197 Direct proofs.

198 Two proof-constructing exercises from the DM Book.

We look together at two problems from the DM Book: Problem 3 on p.93 and Problem 5 on p.94.

199 (P1 V195) Direct proof (deduction and reduction), Examples 1–3.

Example 1 (deduction): If an integer number n is even, then n^2 is also even.

Example 2 (deduction): If n is an integer such that $4|(n+1)$, then $8|(n^2-1)$.

Example 3 (reduction): The arithmetic mean of two non-negative numbers is greater than or equal to their geometric mean.

Extra material: notes from the iPad.

200 Proving and disproving statements containing quantifiers, Examples 4–7.

Prove the following statements:

(Ex4) For all $n \in \mathbb{Z}$ we have $n^2 \equiv 0 \pmod 3$ or $n^2 \equiv 1 \pmod 3$.

(Ex5) It is not true that $n^2 \equiv 1 \pmod 3$ for all integer n.

(Ex6) There exists $n \in \mathbb{Z}$ such that $n^2 \equiv 1 \pmod 3$.

(Ex7) It is not true that there exists $n \in \mathbb{Z}$ such that $n^2 \equiv 2 \pmod 3$.

201 Proving and disproving statements containing quantifiers, Examples 8–10.

Prove or disprove the following statements:

(Ex8) There exists a prime number p such that $p+2$ and $p+6$ are also prime numbers.

(Ex9) For each prime number p, both $p+2$ and $p+6$ are also prime numbers.

(Ex10) Define $p(n) := n^2 + n + 41$. The numbers $p(n)$ are prime for all natural numbers n.

Extra material: notes from the iPad.

202 (P1 V208 p1) Prove or disprove, Example 11.

Example 11: Prove or disprove the following formula. For all sets A, B, C holds:

$$A \times (B \cup C) = (A \times B) \cup (A \times C).$$

Extra material: notes from the iPad.

203 (P1 V208 p2) Prove or disprove, Example 12.

Example 12: Prove or disprove the following formula. For all sets A, B, C holds:

$$A \setminus (B \times C) = (A \setminus B) \times (A \setminus C).$$

Extra material: notes from the iPad.

204 (P1 V208 p3) Prove or disprove, Example 13.

Example 13: Prove or disprove the following formula. For all sets A, B, C holds:

$$A \times (B \setminus C) = (A \times B) \setminus (A \times C).$$

Extra material: notes from the iPad.

205 Existence theorems, some examples.

206 Proving stuff directly from axioms.

207 Proving uniqueness by indicating the only solution, Examples 14–16.

Show that there exists exactly one solution to each of the following equations:

(E14) $4x + 2 = 14$

(E15) $4 \cdot 2^x + 2 = 14$

(E16) $2^{4x+2} = 14$.

208 Proving uniqueness by indicating the only solution, Example 17.

Example 17: Show that there exists exactly one number $n \in \mathbb{N}^+$ for which exactly two of the following statements are true:

(1) Number $n + 51$ is the square of a natural number.

(2) The last digit of n is 1.

(3) Number $n - 38$ is the square of a natural number.

Extra material: notes from the iPad.

209 Proofs by cases, an introduction.

210 Proofs by cases, some examples from geometry.

(1) Prove by cases (*Precalculus 3: Trigonometry*, Video 25): **Inscribed Angle Theorem**: An angle inscribed in a circle is half of the central angle that subtends the same arc on the circle.

Corollary: **Thales' Theorem**: The angle subtended by a diameter is always $90°$, i.e., a right angle.

(2) Prove by cases (*Precalculus 3: Trigonometry*, Video 157): If ABC is any triangle, then:

- $|\sphericalangle BCA| < 90° \quad \Rightarrow \quad |CB|^2 + |AC|^2 > |BA|^2$,
- $|\sphericalangle BCA| = 90° \quad \Rightarrow \quad |CB|^2 + |AC|^2 = |BA|^2$,
- $|\sphericalangle BCA| > 90° \quad \Rightarrow \quad |CB|^2 + |AC|^2 < |BA|^2$.

This theorem is a corollary from The Cosine Rule, which is also proven with help of cases.

211 (P1 V199) Series of equivalences.

212 Two main types of indirect proofs.

213 Direct and indirect proofs in the DM Book.

214 Proof by contrapositive, Example 18.

Example 18: Prove by contraposition: If the product of two integers is an even number, then at least one of these numbers must be even. (A special case of this one will be treated as lemma for the proof that the square root of 2 is an irrational number, in Video 221.)

Corollary: If n^2 is an even number for some integer n, then n is even. (This is the converse statement to the one in Example 1 in V199.)

Extra material: notes from the iPad.

215 A preparation to the next video.

How to understand the following statement: Given n positive natural numbers: $k_1, k_2, \ldots, k_n$. Show that if

$$(*) \quad \frac{1}{k_1} + \cdots + \frac{1}{k_n} > \frac{n}{2},$$

then $k_i = 1$ for some i.

Extra material: notes from the iPad.

216 (P1 V197) Proof by contrapositive, Example 19.

Example 19: Prove by contraposition: Given n positive natural numbers: $k_1, k_2, \ldots, k_n$. Show that if

$$(*) \quad \frac{1}{k_1} + \cdots + \frac{1}{k_n} > \frac{n}{2},$$

then $k_i = 1$ for some i.

217 (P1 V198) The extended version of the Cancellation law from Video 187, Example 20.

Example 20: Function $f : Y \to Z$ is injective if and only if

$$\forall X \;\; \forall g_1, g_2 : X \to Y \quad (f \circ g_1 = f \circ g_2 \quad \Rightarrow \quad g_1 = g_2).$$

218 Proof by contradiction, Example 21.

Example 21: Prove by contradiction: Pigeonhole Principle (the DM Book, p.74).

219 (P1 V200) Proof by contradiction, Example 22.

Example 22: Prove by contradiction: Let a, b, c be three integers such that $a \neq 0$, $a|b$, and $a \nmid c$. Then $a \nmid (b+c)$.

Extra material: notes from the iPad.

220 (P1 V207 p1) Proof by contradiction, Example 23.

Example 23: Prove by contradiction (the statement from Video 97): $\bigcap_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n}) = \{0\}$.

Extra material: notes from the iPad.

221 (P1 V215) Proof by contradiction, Example 24.

Example 24: Prove by contradiction: The square root of 2 is an irrational number.

Extra material: notes from the iPad.

222 (P1 V207 p2) Proof by contradiction, Example 25.

Example 25: Prove by contradiction: There exist infinitely many prime numbers.

Extra material: notes from the iPad.

223 (P1 V202) Proof of the partition theorem from Video 171, Example 26.

Example 26: Prove by contradiction: *Theorem about partitions*: Let R be an equivalence relation on a set X. Then every element of X belongs to exactly one equivalence class.

224 (P1 V201) Two proofs of uniqueness, by contradiction, Example 27.

Example 27: Prove that in each group (see the definition in Video 28): the neutral element is unique, and the opposite (inverse) to each element is also unique.

Extra material: notes from the iPad.

225 The Principle of Mathematical Induction in three of our other courses.

226 (FI V2) What kinds of statements can be proven by induction.

227 (FI V3) Induction: this is how it works, Example 28.

Example 28 (formula from V35 and V36): Prove by induction that for each positive natural number n we have

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

228 Proving formulas, Examples 29 and 30.

Example 29: Prove the following formula (derived in the article attached to V35) for $n \in \mathbb{N}^+$:

$$S_n^{(2)} = \sum_{k=1}^{n} k^2 = 1 + 4 + 9 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Extra material: notes from the iPad. Example 30 (solved in the free course in V5): Prove that for each $n \in \mathbb{N}^+$:

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$

229 (FI V4) Both cases are necessary.

Examples showing two typical pitfalls in proofs by induction. Both theorems are obviously false:

a) Prove that all the natural numbers are equal to each other.
b) Prove that each natural number is equal to its own square.

230 Back to V125 and V191: Recursively defined functions, Example 31.

Example 31 (Example 5.2.4 on page 407 from the DM Book): Consider the function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(0) = 0$ and $f(n+1) = f(n) + 2n + 1$ for $n \in \mathbb{N}$. Show that $f(n) = n^2$ for each $n \in \mathbb{N}$.

Extra material: notes from the iPad.

231 Back to V125: Recursively defined functions, Examples 32–34.

Guess and prove by induction explicit formulas for the following functions:

(32) $f(0) = 3,\ f(n+1) = 2f(n)$ for all $n \in \mathbb{N}$
(33) $g(0) = 7,\ g(n+1) = g(n) + 2$ for all $n \in \mathbb{N}$
(34) $h(0) = 1,\ h(n+1) = (n+1) \cdot h(n)$ for all $n \in \mathbb{N}$.

Extra material: notes from the iPad.

232 (P1 V206 p1) Proof by strong induction, Example 35.

Example 35: Prove by strong induction: Let (a_n) be the sequence defined by: $a_1 = 1, a_2 = 8, a_n = a_{n-1} + 2a_{n-2}$ for all $n \geqslant 3$. Show that $a_n = 3 \cdot 2^{n-1} + 2 \cdot (-1)^n$ for all $n \in \mathbb{N}^+$.

Extra material: notes from the iPad.

233 Induction with two base cases, Examples 36–38.

Examples: Define (x_n) for $n \in \mathbb{N}$ such that $x_n = 5x_{n-1} - 6x_{n-2}$ for $n \geqslant 2$. Write down the first five terms of this sequence, try to find a closed formula for x_n, and prove this formula by induction with two base cases if:

(36) $x_0 = 1$, $x_1 = 2$,

(37) $x_0 = 1$, $x_1 = 3$.

Extra material: notes from the iPad. Example 38 (solved in the free course in V7): Find and prove the formula for a_n if $a_1 = 3$, $a_2 = 7$ and $a_n = 3a_{n-1} - 2a_{n-2}$ for $n = 3, 4, 5, \ldots$.

234 (P1 V206 p2) Proof by strong induction, Example 39.

Example 39: Prove by strong induction: **Fundamental Theorem of Arithmetic**: Every integer $n \geqslant 2$ can be factored into a product of primes.

Extra material: notes from the iPad.

235 Proving divisibility, Examples 40–42.

Example 40: For each natural number n we have $9 | 2^{2n+1} + 3n + 7$.

Extra material: notes from the iPad. Examples 41 and 42 (solved in the free course in V8): For each positive natural number n we have $6 | n^3 - n$ and $9 | 4^n + 15n - 1$.

236 Proving divisibility, Examples 43 and 44.

Example 43: For each natural number n we have $11 | 2^{6n+1} + 3^{2n+2}$.

Extra material: notes from the iPad.

Example 44: For each natural number n we have $133 | 11^{n+2} + 12^{2n+1}$.

237 Not necessarily for all natural numbers: an inequality, Example 45.

Example 45: Prove that for all natural numbers $n \geqslant 3$ we have $2^n > 2n$.

Extra material: notes from the iPad.

238 More inequalities to prove, Example 46.

Example 46: Prove that for all natural numbers $n \geqslant 4$ we have $n! \geqslant 2^n$.

Extra material: notes from the iPad.

239 More inequalities to prove, Example 47.

Example 47: Prove that for all natural numbers $n \geqslant 2$ we have

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} > \sqrt{n}.$$

Extra material: notes from the iPad.

240 When the base case is harder to prove than the induction step, Example 48.

Example 48: Let $a \neq 1$ be a positive number. Assume that we know that for all real numbers $x, y \in \mathbb{R}$ we have $a^{x+y} = a^x \cdot a^y$ (it is a generalisation of the formula motivated in V33 for natural exponents x and y; this was established in *Precalculus 4: Exponentials and logarithms*, Videos 33, 211, and 212). Prove that for all natural numbers $n \geqslant 2$ and for all n-tuples $(x_1, x_2, \ldots, x_n)$ we have

$$a^{x_1} \cdot a^{x_2} \cdot \ldots \cdot a^{x_n} = a^{x_1 + x_2 + \ldots + x_n}.$$

(It is a generalisation of the formula motivated in V34.)

Extra material: notes from the iPad.

241 When the base case is harder to prove than the induction step, Example 49.

Example 49: Let $a > 0$ and $a \neq 1$. Assume that we know that

$$\forall\, x, y \in \mathbb{R}^+ \quad \log_a(xy) = \log_a x + \log_a y.$$

Show that this formula can be generalised to:

$$\forall\, n \in \mathbb{N}^+ \quad \forall\, x_1, \ldots, x_n \in \mathbb{R}^+ \quad \log_a(x_1 \cdot \ldots \cdot x_n) = \log_a x_1 + \cdots + \log_a x_n.$$

(The formula for two variables is derived in Precalculus 4 (V118).)

Extra material: notes from the iPad.

242 More generalising formulas, Example 50.

Example 50 (de Morgan's laws): We know from V75, that for all pairs of statements p and q (regardless their logical value):

$$\neg(p \vee q) \;\Leftrightarrow\; \neg p \wedge \neg q,$$
$$\neg(p \wedge q) \;\Leftrightarrow\; \neg p \vee \neg q.$$

We also know that both conjunction and disjunction are associative. Show that these formulas can be generalised to any number of statements:

$$\forall\, n \in \mathbb{N}^+ \quad \forall\, p_1, \ldots, p_n \quad \neg(p_1 \vee \cdots \vee p_n) \;\Leftrightarrow\; \neg p_1 \wedge \cdots \wedge \neg p_n,$$
$$\forall\, n \in \mathbb{N}^+ \quad \forall\, p_1, \ldots, p_n \quad \neg(p_1 \wedge \cdots \wedge p_n) \;\Leftrightarrow\; \neg p_1 \vee \cdots \vee \neg p_n.$$

Extra material: notes from the iPad with solutions to the examples above (one in the video, one in an article).

243 (FI V10) A more difficult proof, Example 51.

Example 51: Show that the following formula holds for all real numbers a and b and for each $n \in \mathbb{N}^+$:

$$a^n - b^n = (a-b)\sum_{k=0}^{n-1} a^{n-1-k}b^k = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + a^2b^{n-3} + ab^{n-2} + b^{n-1}).$$

This formula was proven for $n = 2, 3, 4$ in V30 and V32. A proof for all $n \in \mathbb{N}^+$ is given in Video 31 in *Precalculus 1: Basic notions.*

244 Another difficult proof, Example 52.

Example 52: Show that the following formula holds for each integer $n \in \mathbb{N}^+$:

$$\prod_{k=1}^{n}\left(1 - \frac{4}{(2k-1)^2}\right) = \left(1 - \frac{4}{1}\right)\left(1 - \frac{4}{9}\right)\left(1 - \frac{4}{25}\right)\left(1 - \frac{4}{49}\right)\cdot\ldots\cdot\left(1 - \frac{4}{(2n-1)^2}\right) = \frac{1+2n}{1-2n}.$$

Extra material: notes from the iPad.

245 Proofs by induction, Wrap-up.

Examples showing that choosing a proof by induction is not always possible (for statements about natural numbers) or not the best choice:

a) Prove that none of the numbers S_n for $n \geqslant 2$ is integer, where

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}.$$

b) Prove that for each natural number n we have $n^2 + 1 \geqslant 2n$.

Extra material: an article with an advanced proof by induction, coming from the course **Precalculus 2: Polynomials and rational functions**: Theorem about Polynomial Division.

246 Induction versus Well-Ordering Principle, a controversy.

247 Some advice for very advanced theorem provers.

248 Indirect proofs, Problem 1.

Problem 1: All the squares in a 15-by-15 board are white. We want to color them with three colors (say: red, yellow, and green). Show that there exist at least two rows that have the same number of squares colored with the same color.

249 A kind of weird Pigeonhole Principle? Problem 2.

Problem 2: Prove that among each consecutive twelve positive integers one can find a number that is **not** the sum of ten 4th powers of integer numbers (Hint: see V55).

250 Pigeonhole Principle in a proof by contradiction, Problem 3.

Problem 3 (Erdős–Szekeres Theorem): Prove that every sequence of distinct n^2+1 real numbers contains a subsequence of length $n+1$ which is strictly increasing or strictly decreasing.

S11 Factorial, n choose k, and Binomial Theorem

You will learn: some important properties of the sigma symbol; the factorial function, binomial coefficients, Pascal's Triangle; the Binomial Theorem (theorem telling you how to raise a sum of two terms to any positive natural power) with a motivation and a formal proof; I will show you (**mostly without proving, as this is moved to DM2, where you will see plenty of proofs, both *regular* and combinatorial**) several binomial identities and where you can find even more of them; all this will be really important for the next section (Combinatorics).

Read along with this section: Section 3.1: *Pascal's Arithmetical Triangle*, pp.191–204.

251 Combinatorics: just enough to justify the course image, and to keep my promises.

252 Summation notation: a repetition and some properties.

Properties: Suppose $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are two sequences of real numbers, and m and p are such natural numbers that $m \leqslant p$. Then:

a) $\displaystyle\sum_{n=m}^{p}(a_n \pm b_n) = \sum_{n=m}^{p} a_n \pm \sum_{n=m}^{p} b_n$

b) $\displaystyle\sum_{n=m}^{p} c\,a_n = c\sum_{n=m}^{p} a_n$ for any real number c.

c) $\displaystyle\sum_{n=m}^{p} a_n = \sum_{n=m}^{j} a_n + \sum_{n=j+1}^{p} a_n$ for any natural number j such that $m \leqslant j < j+1 \leqslant p$.

d) $\displaystyle\sum_{n=m}^{p} a_n = \sum_{n=m+r}^{p+r} a_{n-r}$, for any natural number r.

Extra material: notes with proofs of the properties above.

253 Short and sweet about Pascal's Triangle.

254 What about powers of a difference?

255 Let's practice raising binomials to positive natural powers.

Exercise: Use Pascal's Triangle to compute:

a) $(1+2a)^3$,

b) $(x-2)^5$,

c) $(3x-y)^4$.

Extra material: notes with solved Exercise.

256 Factorial and binomial coefficients.

Compute the following:

a) $1!,\ 2!,\ 3!,\ 4!,\ 5!,\quad \dfrac{(k-1)!}{(k+2)!}$ for $k \geqslant 1,\quad \dfrac{10!}{7!}$

b) $\binom{5}{2}$, c) $\binom{7}{4}$.

Extra material: notes with solved exercises.

257 Short and sweet about the Binomial Theorem.

Apply the Binomial Theorem to compute $(x-2)^4$. Compute $\binom{4}{k}$ for $k = 0, 1, 2, 3, 4$.

Extra material: notes from the iPad.

258 How is it the same?

Show that $\binom{0}{0} = 1$. Let $n, k \in \mathbb{N}^+$ such that $k \leqslant n$. Show the following:

a) $\binom{n}{0} = 1 = \binom{n}{n}$,

b) $\binom{n}{1} = n = \binom{n}{n-1}$,

c) $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$,

d) $\binom{n}{k} = \binom{n}{n-k}$.

Use the results above to motivate that the binomial coefficients are always integer; the last one just shows that Pascal's Triangle has symmetric entries. The combination of $\binom{0}{0} = 1$ with a) and c) shows that both Pascal's Triangles are, indeed, the same.

Extra material: notes with solved exercises.

259 A formal proof of the Binomial Theorem: by induction.

Binomial Theorem: For all non-zero a and b, and for all $n \in \mathbb{N}^+$ holds: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$.

260 Plenty of (basic) binomial-coefficient identities: a promise for DM2.

261 **Advanced**: Where to find plenty of binomial-coefficient identities.

S12 Combinatorics: the art of counting, an introduction (TBC in DM2)

You will learn: some basic combinatorial concepts like permutation, variation, and combination; an explanation of the name of binomial coefficients (n *choose* k); counting subsets of a finite set; counting paths on a grid; some examples of combinatorial proofs. Much more follows in DM2, I promise.

Read along with this section: Chapter 3: *Counting*, pp.191–310 (without Section 3.7).

262 Combinatorics: the art of counting.

263 Back to permutations.

Let $n \in \mathbb{N}^+$. We have n different objects. In how many ways can we arrange these n different objects in a sequence? (This number is $n!$; we know it from V61. Now we prove it by induction.)

264 Variations and combinations; number of subsets with k elements of a set with n elements.

Let $n, k \in \mathbb{N}^+$ such that $k \leqslant n$. We have n different objects.

a) In how many ways can we choose k objects (order important) out of the given n objects? (variations)

b) In how many ways can we choose k objects (order **not** important) out of the given n objects? (combinations).

Sketch a reasoning motivating that the number of k-element subsets of a set with n elements is $C(n,k)$, using Pascal's identity (V260). This could be formally written as proof by induction w.r.t. n (for all $k = 0, 1, \ldots, n$).

265 Cardinality of the power set of a finite set, a formal proof.

Back to the problem from V113: Let $|X| = n$. Show that cardinality of the power set 2^X is 2^n.

266 Two counting problems promised in V118.

Problem 18 on p.401 in the DM Book: Let $A = \{1, 2, 3, \ldots, 9\}$. How many subsets of A contain exactly one element (i.e., how many *singleton* subsets are there)? How many *doubletons* (containing exactly two elements) are there?

Problem 19 on p.401 in the DM Book: Let $A = \{1, 2, 3, 4, 5, 6\}$. Find all sets $B \in \mathcal{P}(A)$ which have the property $\{2, 3, 5\} \subseteq B$.

267 Two proofs of the *counting-all-subsets* formula.

Let $n \in \mathbb{N}^+$. Show that

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n.$$

Use the Binomial Theorem (easy), and use a combinatorial reasoning (delightful).

268 Two proofs of the *even-and-odd-subsets* formula.

Let $n \in \mathbb{N}^+$. Show that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^{n-1}\binom{n}{n-1} + (-1)^n \binom{n}{n} = 0.$$

Use the Binomial Theorem (easy), and use a combinatorial reasoning (delightful).

269 A combinatorial proof of the Binomial Theorem.

Binomial Theorem: For all non-zero a and b, and for all $n \in \mathbb{N}^+$ holds: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$.

This theorem was motivated in V31, V253–258, and proven by induction in V259; now you get a combinatorial proof.

270 Counting paths.

Back to the problem from V59: What is the number of shortest paths from the point $(0, 0)$ to the point (n, k) along the grid lines.

271 Mathematical modelling with paths.

Back to the problems from V60.

272 Much more Combinatorics in DM2.

S13 Extras

You will learn: about all the courses we offer, and where to find discount coupons. You will also get a glimpse into our plans for future courses, with approximate (very hypothetical!) release dates.

B Bonus lecture.

Extra material 1: a pdf with all the links to our courses, and coupon codes.

Extra material 2: a pdf with an advice about optimal order of studying our courses.

Extra material 3: a pdf with information about course books, and how to get more practice.