Calculus 3 / Multivariable Calculus[1], Part 2 of 2

Towards and through the vector fields

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A short table of contents

[1] Recorded February–March 2021. Published on `www.udemy.com` on 2021-03-XX.

An extremely detailed table of contents; the videos (titles in green) are numbered

In blue: problems solved on an iPad (the solving process presented for the students; active problem solving)

In red: solved problems demonstrated during a presentation (a walk-through; passive problem solving)

In magenta: additional problems solved in written articles (added as resources).

C4 Multiple integrals

(Chapter[2] 14)

S1 Introduction to the course

1 Introduction to the course. Extra material: this list with all the movies and problems.

S2 Repetition (Riemann integrals, sets in the plane, curves)

2 Riemann integrals: repetition part 1 (definition, notation, and terminology).

3 Riemann integrals: repetition part 2 (integrable and non-integrable functions).

4 Riemann integrals: repetition part 3 (properties and applications).

5 Riemann integrals: repetition part 4 (integration by inspection).

Method 1: by area; three examples: $\int\limits_{-4}^{2} 6\,dx$, $\int\limits_{-2}^{6} x\,dx$ and $\int\limits_{-1}^{1} \sqrt{1-x^2}\,dx$.

Method 2: odd functions; two examples: $\int\limits_{-4}^{4} \sin x\,dx$ and $\int\limits_{-1.6}^{1.6} (x-5x^3+2x^5)\,dx$.

6 Riemann integrals: repetition part 5 (computations).

Extra material: an article with some integrals which will be particularly important in double and triple integrals (trigonometrical functions).

7 Curves: repetition part 1 (general).

8 Curves: repetition part 2 (arc length).

9 Sets in the plane: repetition.

S3 Double integrals

You will learn: compute double integrals by iteration of single integrals.

10 Notation and applications.

11 Three ways of defining APR (axis-parallel rectangles).

12 Definition of double integrals on APR.

13 Definition of double integrals on compact domains.

14 Multiple integrals, generally.

15 Properties of double integrals.

16 Integration by inspection 1.

Example 1: Estimate by inspection: $\iint\limits_R dxdy$, $\iint\limits_R 5\,dxdy$ where $R = \{(x,y);\ -1 \leqslant x \leqslant 3,\ -4 \leqslant y \leqslant 1\}$.

[2] Chapter numbers in Robert A. Adams, Christopher Essex: *Calculus, a complete course.* 8th or 9th edition.

Example 2: Estimate by inspection: $\iint\limits_{x^2+y^2\leqslant a^2} \sqrt{a^2-x^2-y^2}\,dA$.

Example 3: Estimate by inspection: $\iint\limits_{T}(1-x-y)\,dA$ where T is the triangle with vertices in $(0,0),(1,0),(0,1)$.

17 Functions odd w.r.t. x and odd w.r.t. y.

18 Integration by inspection 2.

19 Integration by inspection, Problem 1.

Problem 1: Let $D=\{(x,y);\ |x|+|y|\leqslant 1\}$. Estimate $\iint\limits_{D}(x^3\cos y^2+3\sin y-\pi)\,dA$.

Extra material: notes with solved problem 1.

20 Integration by inspection, Problem 2.

Problem 2: Let $D=\{(x,y);\ -2\leqslant x\leqslant 2,\ 0\leqslant y\leqslant\sqrt{4-x^2}\}$. Estimate $\iint\limits_{D}(x+3)\,dA$.

Extra material: notes with solved problem 2.

21 Integration by inspection, Problem 3.

Problem 3: Let D denote the parallelogram with vertices in $(2,2),\ (1,-1),\ (-2,-2),\ (-1,1)$. Estimate $\iint\limits_{D}(x+y)\,dA$.

Extra material: notes with solved problem 4.

22 Integration by inspection, Problem 4.

Problem 4: Let $D=\{(x,y);\ |x|+|y|\leqslant\pi\}$. Show that $\iint\limits_{D}\sin(x+y)\,dxdy=0$.

23 Integration by iteration, Fubini's theorem on APR.

24 Fubini, Problem 1.

Problem 1: Compute $\iint\limits_{R}x^3y^2\,dA$ where $R=\{(x,y);\ 0\leqslant x\leqslant 1,\ \ 0\leqslant y\leqslant 2\}$. Show two methods.

Extra material: notes with solved problem 1.

25 Fubini, Problem 2.

Problem 2: Compute $\iint\limits_{R}y\cos(xy)\,dA$ where $R=\{(x,y);\ 0\leqslant x\leqslant 1,\ \ 0\leqslant y\leqslant\pi/2\}$.

Extra material: notes with solved problem 2.

26 Fubini, Problem 3.

Problem 3: Compute $\iint\limits_{R}e^{x+y}\,dA$ where $R=\{(x,y);\ 0\leqslant x\leqslant 1,\ \ 0\leqslant y\leqslant 1\}$.

Extra material: notes with solved problem 3.

27 A very, very important computational trick.

Extra material: notes.

28 Fubini, Problem 4.

Problem 4: Compute $\iint\limits_{R}e^{xy}(1+xy)\,dxdy$ where $R=\{(x,y);\ 0\leqslant x\leqslant 1,\ \ 1\leqslant y\leqslant 2\}$.

Extra material: notes with solved problem 4.

29 Fubini, an example where the order matters.

Let $D=\{(x,y)\in\mathbb{R}^2: 1\leqslant x\leqslant 3,\quad 0\leqslant y\leqslant 1\}$. Compute $\iint\limits_{D}\frac{x}{(1+xy)^2}dxdy$.

30 x-simple and y-simple domains.

31 Fubini's theorem for x-simple and for y-simple domains.

Example: Compute $\iint\limits_{R}(x^2+y^2)\,dxdy$ where R is a triangle with vertices in $(1,1),\ (1,0),\ (0,1)$.

Example: Compute (in two ways) $\iint_D x^2y\,dxdy$ where $D = \{(x,y);\ \ 0 \leqslant x \leqslant 2,\ \ 0 \leqslant y \leqslant x\,\}$.

32 Fubini general version, Problem 1.

Problem 1: Compute $\iint_R 2xy\,dxdy$ where R is a triangle with vertices in $(0,0),\ (2,-2),\ (2,4)$.

33 Fubini general version, Problem 2.

Problem 2: Compute $\iint_R xy\,dxdy$ where $R:\ x^2 \leqslant y \leqslant x$. Show two methods.

Extra material: notes with solved problem 2.

34 Fubini general version, Problem 3.

Problem 3: Compute $\iint_R \frac{x}{y} \cdot e^y\,dxdy$ where $R:\ x^2 \leqslant y \leqslant x$.

Extra material: notes with solved problem 3.

35 Fubini general version, Problem 4.

Problem 4: Compute $\iint_R \frac{x}{1+y^2}\,dxdy$ where $R = \{(x,y);\ x \geqslant 0,\ x^2 \leqslant y \leqslant 1\}$. Show two methods.

Extra material: notes with solved problem 4.

36 Fubini general version, Problem 5.

Problem 5: Compute $\iint_R x^3y^2\,dxdy$ where $R = \{(x,y);\ x \geqslant 0,\ x^2 - y^2 \geqslant 1,\ x^2 + y^2 \leqslant 9\}$.

Extra material: notes with solved problem 5.

37 Fubini general version, Problem 6.

Problem 6: Compute $\int_0^1 \left(\int_{3y}^{3} e^{x^2}\,dx \right) dy$.

Extra material: notes with solved problem 6.

38 Fubini general version, Problem 7.

Problem 7: Compute $\iint_R e^{y^3}\,dxdy$ where $R:\ 0 \leqslant x \leqslant 1,\ \ \sqrt{x} \leqslant y \leqslant 1$.

Extra material: notes with solved problem 7.

39 Fubini general version, Problem 8.

Problem 8: Compute $\iint_R \ln x\,dxdy$ where R is the set in the first quadrant, between the line $2x + 2y = 5$ and the hyperbola $xy = 1$.

Extra material: notes with solved problem 8.

Extra material: an article with more solved problems on double integrals.

- ⋆ **Extra problem 1**: Let $f(x,y) = xy$ and

$$D = \{\ (x,y) \in \mathbb{R}^2 \mid -1 \leqslant x \leqslant 1, \quad 0 \leqslant y \leqslant 1 + x^4, \quad 0 \leqslant y + x\ \}.$$

  Compute the double integral of function f over the domain D. Sketch D.

- ⋆ **Extra problem 2**: Given a rectangular box with the bottom $D:\ -1 \leqslant x \leqslant 1,\ -2 \leqslant y \leqslant 2$ in the plane $z = 0$ and the top in the plane $z = 7$. We cut off the upper part of the box with the surface of the paraboloid $z = 6 - x^2 - y^2$. Compute the volume of the solid obtained in this way.

- ⋆ **Extra problem 3**: Compute the double integral

$$\iint_D e^{x^2}\,dxdy$$

  where D is the triangle with vertices in $(0,0)$, $(1,1)$ and $(1,-1)$.

⋆ **Extra problem 4**: Compute the double integral

$$\iint_D (1-2x)\, dxdy$$

where $D = \{(x,y);\;\; x^2 \leqslant y \leqslant \sqrt{x}\}$. Draw the domain D.

S4 Change of variables in double integrals

You will learn: compute double integrals via variable substitution (mainly to polar coordinates).

40 Why change variables? Similarities and differences between Calc2 and Calc3.

41 Jacobian and the change in area element after substitution.

42 One formula for both substitutions.

43 Inverse substitution.

Problem 1: Compute the volume above the xy-plane and under the surface $z = 1 - x^2 - y^2$.

Extra material: an article with some integrals which will be particularly important in double and triple integrals (trigonometrical functions). Article from Movie 6 completed with applications to double integrals.

44 Direct substitution.

Problem 2: Compute the integral

$$\iint_D e^{x+y} dxdy$$

where $D = \{\, (x,y);\;\; |x| + |y| \leqslant a\,\}$ for some $a > 0$.

Extra material: notes with solved problem 2.

45 Change of variables, Problem 3.

Problem 3: Compute the integral

$$\iint_D \frac{x^2 e^{x^2+y^2}}{x^2+y^2} dxdy$$

over the half disk $x^2 + y^2 \leqslant 1, \quad y \geqslant 0$.

46 Change of variables, Problem 4.

Problem 4: Compute the double integral

$$\iint_D \ln(1 + x^2 + y^2)\, dxdy,$$

where $D = \{(x,y);\; 1 \leqslant x^2 + y^2 \leqslant 2\,\}$.

Extra material: notes with solved problem 4.

47 Change of variables, Problem 5.

Problem 5: Compute the volume of the solid between the surfaces $z = x^2 + y^2$ and $z = \frac{4}{3} - \frac{x^2}{3} - \frac{y^2}{3}$.

Extra material: notes with solved problem 5.

48 Change of variables, Problem 6.

Problem 6: Compute the integral

$$\iint_D (x^4 - y^4) dxdy$$

where D is the domain in the xy-plane between the four curves: $x^2 - y^2 = 1$, $x^2 - y^2 = 2$, $xy = 1$ and $xy = 3$.

49 Change of variables, Problem 7.

Problem 7: Compute the area of the domain D between the four curves: $xy = 1$, $xy = 4$, $y = x$ and $y = 2x$.

Extra material: notes with solved problem 7.

50 Double integrals, wrap-up.

Extra material: an article with more solved problems on change of variables in double integrals.

- ⋆ **Extra problem 1**: Compute the double integral

$$\iint\limits_D \arctan\left(\frac{y}{x}\right)\, dxdy,$$

where $D = \{(x, y);\ 1 \leqslant x^2 + y^2 \leqslant 4,\ 0 \leqslant y \leqslant x\}$.

- ⋆ **Extra problem 2**: Compute the double integral

$$\iint\limits_D x\, dxdy,$$

where $D = \{(x, y);\ 1 \leqslant x^2 + y^2 \leqslant 4,\ 0 \leqslant x \leqslant y \leqslant \sqrt{3}x\}$. Draw the domain D.

- ⋆ **Extra problem 3**: Compute the double integral

$$\iint\limits_D x^2\, dA,$$

where $D = \{(x, y);\ x^2 + y^2 \leqslant 1,\ y \geqslant 0\}$.

- ⋆ **Extra problem 4**: Compute the double integral

$$\iint\limits_D \frac{x}{x^2 + y^2}\, dxdy,$$

where $D = \{(x, y);\ x \geqslant 0,\ 1 \leqslant x^2 + y^2 \leqslant 4\}$. Draw the domain D.

S5 Improper integrals

You will learn: motivate if an improper integral is convergent or divergent; use the mean-value theorem for double integrals in order to compute the mean value for a two-variable function on a compact connected set.

51 Improper integrals, repetition from Calc2.

Examples: p-integrals $\int\limits_1^\infty \frac{1}{x^p}\, dx$, q-integrals $\int\limits_0^1 \frac{1}{x^q}\, dx$, a warning: $\int\limits_{-1}^1 \frac{1}{x}\, dx$ is divergent.

52 Improper double integrals.

Example: Show that the following improper integral is convergent:

$$\iint\limits_{x^2+y^2 \geqslant 1} \frac{1 + 2\sin(xy)}{(x^2 + y^2)^{3/2}}.$$

53 Calc 3 helps Calc 2. Problem 1.

Problem 1: Show that

$$\int\limits_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

54 Improper integrals, Problem 2.

Problem 2: Compute the improper double integral

$$\iint_D \frac{1}{(1+x^2)(1+y^2)} dxdy$$

where D is the first quadrant in the xy-plane.

55 Improper integrals, Problem 3.

Problem 3: Let $D = \{(x, y);\ 0 \leqslant x < y \leqslant 1\}$. Is the following integral convergent?

$$\iint_D \frac{1}{y-x}\, dxdy$$

56 Improper integrals, Problem 4.

Problem 4: Compute the integral

$$\iint_{x^2+y^2\leqslant 1} \ln(x^2+y^2)\, dxdy$$

Extra material: notes with solved problem 4.

57 Improper integrals, Problem 5.

Problem 5: Compute the integral

$$\iint_D \frac{dxdy}{1+x^2y^2}$$

where $D = \{(x, y);\ 1 \leqslant x \leqslant 2,\ y \geqslant 0\}$.

Extra material: notes with solved problem 5.

58 Improper integrals, Problem 6.

Problem 6: Compute the integral

$$\iint_D \frac{1}{x\sqrt{y}}\, dxdy$$

where D is the triangle with vertices in $(0, 0)$, $(1, 1)$ and $(1, 2)$.

Extra material: notes with solved problem 6.

59 Mean-Value Theorem for double integrals.

60 Mean value, Example 1.

Example 1: Compute the mean value of $f(x, y) = x^2 + y^2$ over $D = \{(x, y);\ 0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant a - x\}$.

Extra material: notes with solved Example 1.

61 Mean value, Example 2.

Example 2: Compute the mean value of $f(x, y) = \frac{1}{x}$ over $D = \{(x, y);\ 0 < x \leqslant 1,\ x^2 \leqslant y \leqslant \sqrt{x}\}$.

Extra material: notes with solved Example 2.

S6 Triple integrals

62 Triple integrals: notation, definition and properties.

63 Integration by inspection

Example 1: Show that

$$\iiint\limits_B (x^3y^2 + 5xz^2 \sin y - 6y^4 \sin z)\, dx = 0$$

where B is a ball centered in the origin.

Example 2: Compute the integral

$$\iiint\limits_B (3 + 2xy)\, dxdydz$$

where B is the upper half of the ball wit radius 2: $B = \{(x, y, z);\ x^2 + y^2 + z^2 \leqslant 4,\ \ z \geqslant 0\}$.

Extra material: notes with solved Example 2.

64 Fubini's Theorem

Example 3: Compute the integral

$$\iiint\limits_B (x^2 + y^2)\, dxdydz$$

where B is an APR: $B = [0, 1] \times [2, 4] \times [1, 4]$.

Extra material: notes with solved Example 3.

65 Triple integrals: Problem 1.

Problem 1: Compute the triple integral

$$\iiint\limits_B (1 - x^2 + 2z)\, dxdydz$$

where $B = \{(x, y, z);\ \ 0 \leqslant x \leqslant 3,\ 0 \leqslant y \leqslant 4,\ 0 \leqslant z \leqslant 1\}$ is an APR.

66 Triple integrals: Problem 2.

Problem 2: Compute the integral

$$\iiint\limits_B yz^2 e^{-xyz}\, dxdydz$$

where B is the unit cube $0 \leqslant x,\ y,\ z \leqslant 1$.

Extra material: notes with solved problem 2.

67 Triple integrals: Problem 3.

Problem 3: Compute the integral

$$\iiint\limits_T x\, dxdydz$$

where T is the tetrahedron between the planes $x = 1,\ \ y = 1,\ \ z = 1,\ \ x + y + z = 2$.

Extra material: notes with solved problem 3.

68 Triple integrals: Problem 4.

Problem 4: Compute the triple integral

$$\iiint\limits_B z\, dxdydz$$

where $B = \{(x, y, z);\ x^2 + y^2 \leqslant z^2,\ \ x^2 + y^2 + z^2 \leqslant 1,\ \ z \geqslant 0\}$.

69 Area and volume computed in different ways.

70 Volume of a tetrahedron.

S7 Change of variables in triple integrals

You will learn: compute triple integrals by Fubini's theorem or by variable substitution to spherical or cylindrical coordinates; compute the Jacobian for various kinds of change of variables.

71 Change of variables in triple integrals

72 Change of variables, Problem 1.

Problem 1 (Problem 4 from Video 68 one more time): Compute the triple integral

$$\iiint_B z\,dxdydz$$

where $B = \{(x,y,z);\ x^2+y^2 \leqslant z^2,\ \ x^2+y^2+z^2 \leqslant 1,\ \ z \geqslant 0\}$.

73 Change of variables, Problem 2.

Problem 2: Compute

$$\iiint_K z\,dxdydz$$

where B defines by the inequalities $z^2 \geqslant x^2+y^2,\ 0 \leqslant z \leqslant 1$.

74 Change of variables, Problem 3.

Problem 3: Find the volume of the region bounded from above by the paraboloid $z = 8 - x^2 - y^2$ and from below by the cone $z = 2\sqrt{x^2+y^2}$. Sketch the region.

Extra material: notes with solved problem 3.

75 Change of variables, Problem 4.

Problem 4: Compute the triple integral

$$\iiint_B \frac{1}{1+x^2+y^2+z^2}\,dxdydz$$

where B is the ball centered in the origin, with radius 2.

Extra material: notes with solved problem 4.

76 Change of variables, Problem 5.

Problem 5: Compute the triple integral of $f(x,y,z) = x+y$ over the solid B described by

$$B = \{(x,y,z);\ 0 \leqslant x+z \leqslant 2,\ \ 0 \leqslant x+y \leqslant 4,\ \ 1 \leqslant 5x+2y+z \leqslant 3\}.$$

Extra material: notes with solved problem 5.

77 Change of variables, wrap-up.

Extra material: an article with more solved problems on change of variables in triple integrals.

- ⋆ **Extra problem 1**: Compute volume of a ball with radius a for some positive a. Use spherical coordinates.
- ⋆ **Extra problem 2**: Compute the triple integral of $f(x,y,z) = xy^2z$ over the solid

$$B = \{(x,y,z);\ \ x^2+y^2 \leqslant 1,\ \ x \geqslant 0,\ \ y \geqslant 0,\ \ 0 \leqslant z \leqslant 3\}.$$

⋆ **Extra problem 3**: Compute the triple integral

$$\iiint_B \sin\left(\sqrt{x^2+y^2}\right)\cos(z)\,dxdydz$$

where $B = \{(x,y,z);\;\; 0 \leqslant z \leqslant 1,\, 0 \leqslant x^2+y^2 \leqslant 1\}$.

⋆ **Extra problem 4**: Compute the volume of the solid $B = \{(x,y,z) \in \mathbb{R}^3 : (x^2+y^2+z^2)^2 \leqslant y\}$.

S8 Applications of multiple integrals such as mass, surface area, mass centre

You will learn: apply multiple integrals for various aims.

78 Applications of multiple integrals: area and volume.

79 Applications of multiple integrals: mass.

80 Applications of multiple integrals: mass centre.
Example 1: The centroid of $B = \{(x,y,z);\;\; x^2+y^2+z^2 \leqslant a^2\}$ is in $(0,0,0)$.
Example 2: Compute the centroid of $D = \{(x,y);\;\; x^2 \leqslant y \leqslant \sqrt{x}\}$.
Extra material: notes with solved Example 2.

81 Applications of multiple integrals: surface area.
Example: Compute the area of the upper half of the sphere $x^2+y^2+z^2 = R^2$.

82 Surface area, Problem 1.
Problem 1: Compute the area of the piece of plane $z = 2x+2y$ inside the cylinder $x^2+y^2 = 1$.
Extra material: notes with solved Problem 1.

83 Surface area, Problem 2.
Problem 2: Compute the area of the surface $z = 4-x^2-y^2$ above the xy-plane.
Extra material: notes with solved Problem 2.

84 Surface area, Problem 3.
Problem 3: Compute the area of the conic surface $3z^2 = x^2+y^2,\;\; 0 \leqslant z \leqslant 2$.
Extra material: notes with solved Problem 3.

85 Surface area, Problem 4.
Problem 4: Compute the area of the surface $z = y^2$ above the triangle with vertices in $(0,0)$, $(0,1)$, $(1,1)$.
Extra material: notes with solved Problem 4.

C5 Vector fields

(Chapter 15)

S9 Vector fields

You will learn: about vector fields in the plane and in the space.

86 Different kinds of functions and their visualisation.

87 Vector fields, some examples.

88 Vector fields, definition, notation and domain.

89 Streamlines / field lines.
Example: Find equation of the field line through the origin for the plane vector field $\vec{F}(x,y) = (1, \sin x)$.

90 Streamlines, Problem 1.

Problem 1: Determine field lines for $\vec{F}(x,y) = (2x,\ 2y)$.

Extra material: notes with solved Problem 1.

91 Streamlines, Problem 2.

Problem 2: Determine field lines for $\vec{F}(x,y) = (y,\ -x)$.

Extra material: notes with solved Problem 2.

92 Streamlines, Problem 3.

Problem 3: Determine field lines for $\vec{F}(x,y) = (-x,\ y)$.

Extra material: notes with solved Problem 3.

93 Streamlines, Problem 4.

Problem 4: Determine field lines for $\vec{F}(x,y) = (y,\ x)$.

Extra material: notes with solved Problem 4.

94 Streamlines, Problem 5.

Problem 5: Determine field lines for $\vec{F}(x,y) = (e^x,\ \ e^{-x})$ and sketch the one passing through the origin.

Extra material: notes with solved Problem 5.

95 Streamlines, Problem 6.

Problem 6: Determine field lines for $\vec{F}(x,y,z) = e^{xyz}(x,\ \ y^2,\ \ z)$.

Extra material: notes with solved Problem 6.

S10 Conservative vector fields

You will learn: about conservative vector fields; use the necessary condition for a vector field to be conservative; compute potential functions for conservative vector fields.

96 Is each vector field a gradient to some function? Answer by computations.

Example: Vector fields $\vec{F} = (2x,\ 2y)$, $\vec{F} = (-x,\ y)$ and $\vec{F} = (y,\ x)$ are gradients to some functions; $\vec{F} = (y,\ -x)$ is not a gradient to any function.

Extra material: notes with some computation for the examples above.

97 Is each vector field a gradient to some function? Answer by geometry.

Example: $\vec{F} = (y,\ -x)$ is not a gradient to any function; a geometrical explanation.

98 Conservative vector fields and equipotential lines.

99 Schwarz's Theorem, a repetition.

100 Hessian vs Jacobian.

101 The necessary conditions for conservative vector fields.

102 Conservative vector fields, Example 1.

Example 1: Electrostatic vector field is conservative.

103 Conservative vector fields, Example 2.

Example 2: Gravitational vector field is conservative.

Extra material: notes with some computation for the examples above.

104 Conservative vector fields and their potentials, Problem 1

Problem 1: Determine whether the following vector fields are conservative or not: $\vec{F}(x,y) = (y-2x,\ x-1)$ and $\vec{G}(x,y) = (2x-y,\ x+1)$. If they are conservative, compute a potential.

Extra material: notes with solved Problem 1.

105 Conservative vector fields and their potentials, Problem 2

Problem 2: Determine whether the following vector fields are conservative or not: $\vec{F}(x,y) = (3x^2y+y^2,\ x^3+2xy+3y^2)$ and $\vec{G}(x,y) = (x+x^2y,\ \frac{1}{3}x^3+xy)$. If they are conservative, compute a potential.

Extra material: notes with solved Problem 2.

106 Conservative vector fields and their potentials, Problem 3.

Problem 3: Determine whether the following vector field is conservative or not. If it is conservative, compute a potential. $\vec{F}(x,y,z) = (y+z\sin x,\ \ x+e^z,\ \ ye^z-\cos x)$.

Extra material: notes with solved Problem 3.

107 Conservative vector fields and their potentials, Problem 4

Problem 4: Determine whether the following vector field is conservative or not. If it is conservative, compute a potential.

$$\vec{F}(x,y,z) = e^{x^2+y^2+z^2}(xz,\ \ yz,\ \ xy).$$

Extra material: notes with solved Problem 4.

Extra material: an article with more solved problems on conservative vector fields.

- ⋆ **Extra problem 1**: Show that the vector field $\vec{F}(x,y) = \big(\sin(x+y^2),\ \ 2y\cdot\sin(x+y^2)+1\big)$ is conservative by computing its potential.
- ⋆ **Extra problem 2**: Consider the vector field

  $$\vec{F}(x,y) = (2x+e^y+\cos(x+y^2),\ xe^y+2y\cos(x+y^2)+1).$$

  Show that the field is conservative by determining its potential.
- ⋆ **Extra problem 3**: Show that the field $\vec{F} = (3x^2y^2z+2xy,\ \ 2x^3yz+x^2+1,\ \ x^3y^2)$ is conservative and determine its potential.

S11 Line integrals of functions

You will learn: calculate line integrals of functions and use them for computations of mass, arc length and surface area.

108 Line integrals, notation.

109 Line integrals of functions: definition, applications and properties.

110 Line integrals of functions, Problem 1.

Problem 1: Compute the line integral

$$\int_\gamma y\,ds,$$

where γ is the half circle $\gamma = \{(x,y)\,;\ \ x^2+y^2=1,\ \ y\geqslant 0\,\}$. What geometrical and physical interpretations does this integral have?

Extra material: notes with solved Problem 1.

111 Line integrals of functions, Problem 2.

Problem 2: Determine the value of

$$\int_\gamma xy\,ds$$

where γ is the intersection of the cylinder $x^2+y^2=a^2$ (for some $a>0$) and the plane $z=x$, starting at $(0,a,0)$ and ending at $(a,0,a)$.

Extra material: notes with solved Problem 2.

112 Line integrals of functions, Problem 3.

Problem 3: Curve C is the intersection between surfaces $x^2+z^2=1$ and $y=x^2$. Determine the total mass of the curve if the density in the point (x,y,z) is expressed by $\rho(x,y,z)=\sqrt{1+4x^2z^2}$.

Extra material: notes with solved Problem 3.

113 Line integrals of functions, Problem 4.

Problem 4: Curve C is the part of the intersection between surfaces $z=2-x^2-2y^2$ and $z=x^2$ which is situated in the first octant $(x,y,z\geqslant 0)$. Determine the total mass of the curve if the density in the point (x,y,z) is expressed by $\rho(x,y,z)=xy$.

Extra material: notes with solved Problem 4.

S12 Line integrals of vector fields

You will learn: calculate line integrals of vector fields and use them for computations of work and area; three methods for computation of line integrals of vector fields.

114 Line integrals of vector fields, notation, definition and application.

115 Line integrals of vector fields, properties.

116 Line integrals of vector fields, Problem 1, from definition.

Problem 1: Let $\vec{F}(x,y)=(x,\ xy)$ and C be a curve with parametrisation $x(t)=t,\ \ y(t)=t^2,\ \ 0\leqslant t\leqslant 1$. Compute the line integral of the vector field $\vec{F}$ over the curve C.

Extra material: notes with solved Problem 1.

117 Line integrals of vector fields, Problem 2, from definition.

Problem 2: Let $\vec{F}(x,y)=(xy,\ x^2+y^2)$ and let C be the quarter of the unit circle from the point $(1,0)$ to the point $(0,1)$. Compute the line integral of the vector field over the curve C.

Extra material: notes with solved Problem 2.

118 Line integrals of vector fields, Problem 3.

Problem 3: Let $\vec{F}(x,y)=(2x^2+3y,\ 2x+y)$ and let the curve C be given by its parametrisation:
$x(t)=2t,\ \ y(t)=t^3,\ \ 0\leqslant t\leqslant 1$. Compute the line integral of the vector field over the curve.

Extra material: notes with solved Problem 3.

119 Line integrals of vector fields, Differential formula.

120 Line integrals of vector fields, Differential formula, Problem 4.

Problem 4: Compute

$$\oint_{\gamma} x^2y^2dx+x^3ydy$$

where γ is a square with vertices in $(0,0),\ (1,0),\ (1,1),\ (0,1)$ oriented counterclockwise.

121 Fundamental Theorem for conservative vector fields.

Example: Compute the line integral of the electrostatic field $\vec{E}(x,y)=\frac{1}{x^2+y^2}(x,y)$ over any smooth curve starting in the points with coordinates $(a_1,\ a_2)$ and ending in the point $(b_1,\ b_2)$.

122 Path independence of line integrals.

123 Path independence, Problem 5.

Problem 5: Compute $\int_{\gamma}\vec{F}\cdot d\vec{r}$ if γ is a half circle $\vec{r}(t)=(2+\cos t,1+\sin t)$, $t\in[0,\pi]$ and $\vec{F}(x,y)=(y+2x,x)$.

124 Path independence, Problem 6.

Problem 6: Compute $\oint_C \vec{F} \cdot d\vec{r}$ and $\oint_C \vec{G} \cdot d\vec{r}$ if C is the unit circle $x^2 + y^2 = 1$ oriented counterclockwise and

$$\vec{F}(x,y) = \left(x^2 e^{x^3+y^3},\ y^2 e^{x^3+y^3}\right), \quad \vec{G}(x,y) = \left(x^2 e^{x^3+y^3} + y,\ y^2 e^{x^3+y^3} - x\right).$$

Extra material: notes with solved Problem 6.

125 Path independence, Problem 7.

Problem 7: Compute $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x,y) = (2xye^y,\ \ x^2(1+y)e^y)$ where C is any piecewise smooth curve starting in $(1,0)$ and ending in $(2,1)$. Show two different solutions.

Extra material: notes with solved Problem 7.

126 Path independence, Problem 8.

Problem 8: Determine the values of constants A and B for which the vector field

$$\vec{F}(x,y,z) = (Ax \ln z,\ \ By^2 z,\ \ \frac{x^2}{z} + y^3)$$

is conservative. If γ is the straight-line segment from $(1,1,1)$ to $(2,1,2)$, determine

$$\int_\gamma 2x \ln z \, dx + 2y^2 z \, dy + y^3 \, dz.$$

Extra material: notes with solved Problem 8.

127 Path independence, Problem 9.

Problem 9: Let $\vec{F}(x,y) = (e^x y,\ \ e^x + 2y)$. Show that the field is conservative. Compute the line integral

$$\int_C \vec{F} \cdot d\vec{r},$$

where C is the curve with parametrisation

$$x(t) = (1-t) \cdot \cos(t^3 + t), \quad y(t) = \frac{4 \arctan t^2}{\pi}, \qquad 0 \leqslant t \leqslant 1.$$

Extra material: notes with solved Problem 9.

128 Line integrals of conservative vector fields, a wrap-up.

S13 Surfaces

You will learn: understand surfaces described as graphs to two-variable functions $f : \mathbb{R}^2 \to \mathbb{R}$ and as parametric surfaces, being graphs of $\mathbf{r} : \mathbb{R}^2 \to \mathbb{R}^3$; determine whether a surface is closed and determine surfaces' boundary; determine normal vector to surfaces.

129 Why surfaces and what they are.

130 Different ways of defining surfaces.

Examples of surfaces: plane, sphere, lateral surface of a cylinder, lateral surface of a cone, paraboloid.

131 Boundary of a surface; closed and composite surfaces.

132 Normal vector and orientation of a surface.

133 Normal vectors to some important surfaces.

Normal vectors of surfaces: plane, sphere, lateral surface of a cylinder.

134 Surface element, both for surfaces defined as graphs of real-valued functions of two variables and for parametric surfaces.

S14 Surface integrals

You will learn: calculate surface integrals of scalar functions and use them for computation of mass and area.

135 Surface integrals: notation.

136 Surface integrals of functions: definition and applications.

137 Surface integrals of functions: computations and properties.

138 Surface integrals of functions, Problem 1.

Problem 1: Compute

$$\iint\limits_{Y} \sqrt{x^2+y^2+1}\, dS$$

where Y is the **helicoid** defined by:

$$\vec{r}:[0,1]\times[0,\,2\pi], \qquad \vec{r}(\rho,\theta)=(\rho\cos\theta,\ \ \rho\sin\theta,\ \ \theta).$$

Extra material: notes with solved Problem 1.

139 Surface integrals of functions, Problem 2.

Problem 2: Compute $\iint\limits_{Y} x\, dS$ where Y is the graph surface to $g(x,y)=x^2+y$ for (x,y) on rectangle $[0,1]\times[-1,1]$.

Extra material: notes with solved Problem 2.

140 Surface integrals of functions, Problem 3.

Problem 3: Compute $\iint\limits_{Y} x\, dS$ over the part of the parabolic cylinder $z=\frac{x^2}{2}$ which lies inside the cylinder $x^2+y^2=1$ in the first octant.

Extra material: notes with solved Problem 3.

141 Surface integrals of functions, Problem 4.

Problem 4: Compute $\iint\limits_{Y} z^2\, dS$ where Y is the unit sphere $x^2+y^2+z^2=1$.

Extra material: notes with solved Problem 4.

Extra material: an article with one more solved problem on surface integrals of functions.

- ⋆ **Problem**: Given the lateral surface of the cone $z=\sqrt{2(x^2+y^2)}$ with surface density $\rho(x,y,z)=y^2$. Determine the total mass of this part of the surface which lies under the plane $z=1+y$.

S15 Oriented surfaces and flux integrals

You will learn: determine orientation of a surface; determine normal vector field; choose orientation of a surface which agrees with orientation of the surface's boundary; calculate flux integrals and use them for computation of the flux of a vector field across a surface.

142 Orientation of a surface which agrees with orientation of its boundary.

143 Flux integrals: notation, definition, computations and applications.

144 Flux integrals: properties.

145 Flux integrals, Problem 1.

Problem 1: Compute the flux of the vector field $\vec{F}(x,y,z)=(x,\ \ 3y,\ \ x+3y)$ up through the surface

$z = 1 - x + y,\;\; 0 \leqslant x^2 + y^2 \leqslant 4,\;\; x, y > 0.$

Extra material: notes with solved Problem 1.

146 Flux integrals, Problem 2.

Problem 2: Compute the flux of the vector field $\vec{F}(x, y, z) = (x,\; 2y,\; 0)$ up through the surface $Y:\;\; \vec{r}(s, t) = (2s,\; 2t,\; 3s + t),\;\; 0 \leqslant s \leqslant 1,\;\; 0 \leqslant t \leqslant 1.$

Extra material: notes with solved Problem 2.

147 Flux integrals, Problem 3.

Problem 3: Compute the flux of the vector field

$$\vec{F}(x, y, z) = \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}, \frac{z}{x^2 + y^2} \right)$$

out through the lateral surface Y of the cylinder, i.e. the surface with normal vector pointing away from the z-axis.

$$Y:\;\; x^2 + y^2 = 2,\;\; -2 \leqslant z \leqslant 2.$$

Extra material: notes with solved Problem 3.

Extra material: an article with more solved problems on flux integrals.

- ⋆ **Extra problem 1**: Compute the flux of the vector field $\vec{F} = (x,\; y,\; 3)$ out of the domain $K = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leqslant z \leqslant 4\}$.
- ⋆ **Extra problem 2**: Compute the flux of the vector field $\vec{F}(x, y, z) = (2x,\; y,\; 0)$ down through the surface Y with the following parametric definition:

$$\vec{r}(s, t) = (3s^2,\; -3t^2,\; 2s + t) \quad \text{for} \quad 0 \leqslant s \leqslant 1, \quad 0 \leqslant t \leqslant 1.$$

- ⋆ **Extra problem 3**: Compute the flux of the vector field $\vec{F} = (x + y,\; z,\; 0)$ out of the sphere S with radius R and centre in the origin.

C6 Vector calculus

(Chapter16: 16.1–16.5)

You will learn: define and compute curl and divergence of (two- and three-dimensional) vector fields and proof some basic formulas involving gradient, divergence and curl; irrotational and solenoidal vector fields; apply Green's, Gauss's and Stokes's theorems, estimate when it is possible (and convenient) to apply these theorems.

S16 Gradient, divergence and curl (16.1–2)

148 Derivatives: gradient, rotation (curl), divergence.

Problem 1: Compute the divergence and curl of $\vec{F}(x, y, z) = (xy,\;\; y^2 - z^2,\;\; yz)$.

Extra material: notes with solved Problem 1.

149 Curl, an interpretation; irrotational vector fields.

Problem 1: Compute curl of the following plane vector fields:

$$\vec{F} = (-y\,,\, x\,,\, 0), \qquad \vec{G} = (y\,,\, 0\,,\, 0), \qquad \vec{B} = \left(-\frac{y}{x^2 + y^2}\,,\, \frac{x}{x^2 + y^2}\,,\, 0 \right).$$

150 Rotation (curl) of a 3D vector field, an example.

151 Divergence, an interpretation; solenoidal vector fields.

152 Product rules for gradient, divergence and curl.

153 Product rule for gradient.

Product rule for gradient: $\nabla(fg) = f\nabla g + g\nabla f$.

Extra material: notes with a proof of the product rule.

154 Product rule for divergence.

Product rule for divergence: $\nabla \cdot (f\vec{F}) = (\nabla f) \cdot \vec{F} + f(\nabla \cdot \vec{F})$.

Extra material: notes with a proof of the product rule.

155 Product rule for curl.

Product rule for curl: $\nabla \times (f\vec{F}) = (\nabla f) \times \vec{F} + f(\nabla \times \vec{F})$.

Extra material: notes with a proof of the product rule.

156 Curl of each vector field is solenoidal; vector potentials.

The rule: $\mathrm{div}(\mathrm{curl}\, \vec{F}) = 0$.

Show that the following vector field has vector potential: $\vec{G}(x, y, z) = (x^2 + yz,\ \ -2y(x+z),\ \ xy + z^2)$.

Extra material: notes with a proof of the rule above and with solution of the problem above.

157 Conservative vector fields are irrotational.

158 Laplacian.

S17 Green's theorem in the plane (16.3)

159 Green's theorem: our third fundamental theorem.

160 Green's theorem: formulation of the theorem.

161 Green's theorem: proof.

162 Green's theorem: three common issues and how to handle them.

163 Green's theorem: Problem 1.

Problem 1: Compute the line integral

$$\oint_C (-\sin y \cos y - e^{3x^2})dx + (2x \sin^2 y + \cos^4 y)dy$$

where C is the boundary of $D = \{(x, y);\ \ -1 \leqslant x \leqslant 2,\ \ x - 2 \leqslant y \leqslant 4 - x^2\}$ oriented counterclockwise.

Extra material: notes with solved Problem 1.

164 Green's theorem: Problem 2.

Problem 2: Compute the line integral

$$\oint_C (2xy - x^2 + y^2 \sin(xy^2))dx + (x + y^2 + 2xy \sin(xy^2))dy$$

where C is the boundary of $D = \{(x, y);\ \ x^2 \leqslant y \leqslant \sqrt{x}\}$ oriented counterclockwise.

Extra material: notes with solved Problem 2.

165 Green's theorem: Problem 3.

Problem 3: Compute the line integral

$$\oint_C (\sin x + 3y^2)dx + (2x - e^{-y^2})dy$$

where C is the boundary of $D = \{(x, y);\ \ x^2 + y^2 \leqslant a^2,\ \ y \geqslant 0\}$ oriented clockwise.

Extra material: notes with solved Problem 3.

166 Green's theorem: Problem 4.

Problem 4: Compute the line integral

$$\oint_C (x^2 - xy - x^3\cos^4 x)dx + (xy - y^2 - e^{y^4-1})dy$$

where C is the boundary of the triangle with vertices in $(0,0)$, $(1,1)$, $(2,0)$, oriented clockwise.

Extra material: notes with solved Problem 4.

167 Green's theorem: Problem 5.

Problem 5: Compute the line integral

$$\int_C (e^{x+y} - y)dx + (e^{x+y} - 1)dy$$

where C is the half arc of a circle, from the origin to $(1,0)$ in the first quadrant, oriented clockwise.

Extra material: notes with solved Problem 5.

168 Magnetic field and enclosing singularities.

169 Necessary and sufficient condition for (plane) conservative vector fields.

170 Area with help of Green's theorem.

Example: Let $\vec{F}(x,y) = (-4y,\ 2x+8)$ and let C be a curve with parametrisation $\vec{r}(t) = (t^2-4,\ t^3-4t)$ from $t=-2$ to $t=2$. Evaluate the work done by $\vec{F}$ along C. Use then Green's theorem to evaluate the area of the domain enclosed by C.

Extra material: notes with solved Example.

Extra material: an article with more solved problems on Green's theorem.

- ⋆ **Extra problem 1**: Compute the line integral

$$\oint_\gamma -y^3\,dx + x^3\,dy,$$

where γ is the positively oriented boundary of the circle sector $x^2+y^2 \leqslant 1,\quad 0 \leqslant y \leqslant x$.

- ⋆ **Extra problem 2**: Compute the line integral

$$\oint_C y^2dx + x^2dy,$$

where C is the boundary of the trapezoid with vertices in $(0,-1)$, $(1,-2)$, $(1,2)$, $(0,1)$, oriented counterclockwise.

- ⋆ **Extra problem 3**: Compute the area under the cycloid

$$\begin{cases} x = t - \sin t \\ y = 1 - \cos t \end{cases}, \quad t \in [0, 2\pi].$$

- ⋆ **Extra problem 4**: Compute the line integral

$$\oint_{\partial D} \vec{F}\cdot d\vec{r},$$

where ∂D is the boundary of $D = \{(x,y);\ 1 \leqslant x^2+y^2 \leqslant 4,\ 0 \leqslant x \leqslant y \leqslant \sqrt{3}x\}$ oriented counterclockwise and

$$\vec{F}(x,y) = (xy - \cos y + \tfrac{1}{2}e^{y^2} + x^3\sin x,\ \ x^2 + x\sin y + xye^{y^2} + y\cos y).$$

Draw the domain D.

S18 Gauss' theorem (Divergence Theorem) in 3-space (16.4)

171 Gauss' theorem: our fourth fundamental theorem.

172 Gauss' theorem: formulation of the theorem.

173 Gauss' theorem: proof.

174 Gauss' theorem: three common issues and how to handle them.

175 Gauss' theorem, Problem 1.

Problem 1: Compute the flux of the vector field $\vec{F}$ out across the APR B:

$B = \{(x, y, z);\ \ 0 \leqslant x \leqslant 3,\ 0 \leqslant y \leqslant 4,\ 0 \leqslant z \leqslant 1\}$

$\vec{F}(x, y, z) = (y^2 z + x + z\cos y,\ -x^2 y + e^{2x+3z},\ z^2 + 4xy - \sin y^2 + 15e^{2x+y})$.

Extra material: notes with solved Problem 1.

176 Gauss' theorem, Problem 2.

Problem 2: Let $\vec{F}(x, y, z) = (3sx^2 + xy,\ \ -ty^2,\ \ 2xz + 4)$ for some constants s, t.

1. Find s and t such that $\operatorname{div}\vec{F} = 0$ for all $(x, y, z) \in \mathbb{R}^3$.

2. Using the values of s and t from above, compute the flux of $\vec{F}$ outwards the surface

$$Y = \{(x, y, z) \in \mathbb{R}^3;\ \ x^2 + y^2 + 4z^2 = 5,\ \ z \geqslant 0\}.$$

Extra material: notes with solved Problem 2.

177 Gauss' theorem, Problem 3.

Problem 3: Compute the flux of $\vec{F}(x, y, z) = (x^2 + y^2,\ \ y^2 - z^2,\ \ z)$ in through the sphere $x^2 + y^2 + z^2 = a^2$.

Extra material: notes with solved Problem 3.

178 Gauss' theorem, Problem 4.

Problem 4: Compute the flux of the vector field $\vec{F}$ in through the surface Y:

$Y = \{(x, y, z);\ \ x^2 + y^2 = 9,\ \ -1 \leqslant z \leqslant 3\}$

$\vec{F}(x, y, z) = (xz,\ yz,\ z(1 - z))$.

Extra material: notes with solved Problem 4.

179 An example where Gauss' theorem cannot be applied.

180 Volume of a cone.

Extra material: an article with more solved problems on Gauss' theorem.

- ⋆ **Extra problem 1**: Let $\vec{F}(x, y, z) = \sin x^2\mathbf{i} + (y - 2xy\cos x^2 + 15x^3 z^2 - x\cos z)\mathbf{j} + (1 + y + z)\mathbf{k}$. Compute
$$\iint\limits_{Y} \vec{F} \cdot d\vec{S}$$
where Y is the part of the surface $z = 1 - x^2 - y^2$ for which $x \geqslant 0$ and $z \geqslant 0$. Use normal vectors pointing upwards.
- ⋆ **Extra problem 2**: Compute the flux of the vector field $\vec{F} = (xz^2,\ \ 2xy,\ \ z^2 + 2)$ out through the lateral surface of the cylinder $x^2 + y^2 = 1,\ 0 \leqslant z \leqslant 1$.
- ⋆ **Extra problem 3**: Compute the flux of the vector field $\vec{F} = (\cos y + e^z + xz^2,\ xz^3 + 2xy,\ x^5 y^7 + z^2 + 2)$ out through the unit sphere.

S19 Stokes' theorem (16.5)

181 Stokes' theorem: our fifth fundamental theorem.

182 Stokes' theorem: formulation.

183 Stokes' theorem: proof.

Extra material: an article with the proof of Stokes' theorem.

184 Stokes' theorem: how to use it.

185 Stokes' theorem: how it helps; Example 1.

Example 1: Let $\vec{F}(x,y,z) = (-2y + x^2 \sin x,\ -7z + e^{y^2},\ 5x - \cos^2 z)$ and let γ be an intersection curve between the cylinder $x^2+y^2 = 4$ and plane $z = x+4$, oriented counterclockwise seen from above. Compute the line integral $\oint\limits_{\gamma} \vec{F} \cdot d\vec{r}$.

186 Stokes' theorem: verification on an example (Example 2).

Example 2: Let $Y = \{(x,y,z) \in \mathbb{R}^3;\ x^2+y^2+z^2 = 9,\ z \geqslant 0\}$ with the counterclockwise oriented boundary $\gamma = \partial Y = \{(x,y,0);\ x^2+y^2 = 9\}$ and let field $\vec{F}$ be defined as $\vec{F} = (y, -x, 0)$. Then we have

$$\oint\limits_{\gamma} \vec{F} \cdot d\vec{r} = \iint\limits_{Y} (\mathrm{curl}\, \vec{F}) \cdot \hat{N}\, dS.$$

187 Stokes' theorem: Example 3.

Example 3: Curve γ is an intersection between the unit sphere and the plane $x+y+z = 0$. Compute the work W performed by the vector field

$$\vec{F}(x,y,z) = (2z - 3y,\ 3x - z,\ y - 2x)$$

while moving a particle along γ. Choose the orientation of γ.

188 Stokes' theorem: surface independence. Example 4.

Example 4: Given vector field $\vec{F}(x,y,z) = (x + yz - z^2,\ x + yz - z^2,\ x + yz - z^2)$. Compute

$$\iint\limits_{\Gamma} (\mathrm{curl}\vec{F}) \cdot \hat{N}\, dS,$$

where Γ is the surface $z = \sqrt{25 - x^2 - y^2},\ x^2+y^2 \leqslant 9$, oriented so that the normal has positive z-coordinate.

189 Stokes' theorem: surface integral of curl over closed surfaces around regular domains.

190 Simply connected sets in space.

191 Necessary and sufficient condition for conservative vector fields.

192 Stokes' theorem, Problem 1.

Problem 1: Compute $\oint\limits_{C} xy\, dx + yz\, dy + zx\, dz$ where

C is the triangle with vertices in $(1,0,0),\ (0,1,0),\ (0,0,1)$

oriented clockwise when observed from the point $(1,1,1)$.

Extra material: notes with solved Problem 1.

193 Stokes' theorem, Problem 2.

Problem 2: Given vector field $\vec{F}(x,y,z) = (3y,\ -2xz,\ x^2 - y^2)$. Compute

$$\iint\limits_{Y} (\mathrm{curl}\vec{F}) \cdot \hat{N}\, dS,$$

where Y is the half sphere $x^2+y^2+z^2=a^2, \ \ z \geqslant 0$, oriented upwards.

Extra material: notes with solved Problem 2.

194 Stokes' theorem, Problem 3.

Problem 3: Let $\vec{F}(x,y,z)=(2x+3y, \ x^3, \ z^2)$ and curve C is such a curve on the surface $z=x^2+y$ that its projection on the xy-plane is the rectangle with vertices in $(0,0), \ (0,2), \ (1,2), \ (1,0)$. The rectangle is oriented clockwise and C inherits this orientation. Compute $\oint\limits_C \vec{F}\cdot d\vec{r}$. Use Stokes' Theorem.

Extra material: notes with solved Problem 3.

195 Stokes' theorem, Problem 4.

Problem 4: Compute $\oint\limits_C \vec{F}\cdot d\vec{r}$ where $\vec{F}=(e^x-y^3, \ e^y+x^3, \ e^z)$

and C has parametrisation $\vec{r}(t)=(\cos t, \ \sin t, \ \sin 2t), \ \ 0 \leqslant t \leqslant 2\pi$.

Extra material: notes with solved Problem 4.

196 Stokes' theorem, Problem 5.

Problem 5: Let C be the intersection curve between the cylinder $x^2+y^2-x=0$ and the paraboloid $z=1-x^2-y^2$. Compute $\oint\limits_C \vec{F}\cdot d\vec{r}$ where $\vec{F}(x,y,z)=(y, \ 1, \ x)$. Orientation of C is defined as follows: starting in the point $(1,0,0)$, along the vector $(0,1,0)$.

Extra material: notes with solved Problem 5.

197 Stokes' theorem, Problem 6.

Problem 6: Given vector field

$$\vec{F}(x,y,z)=(y^2\cos(xz)+x^3\sin(xz)-e^{z^2}, \ \ x^3e^{yz^2}-y\sin(z^2y^3), \ \ e^{x+yz-z^2}).$$

Compute $\iint\limits_Y (\mathrm{curl}\vec{F})\cdot\hat{N}\,dS$, where Y is the surface $x^2+y^2+z^2=16, \ z\geqslant 0$, oriented upwards.

Extra material: notes with solved Problem 6.

198 Stokes' theorem for computations of surface integrals; vector potentials.

Extra material: an article with more solved problems on Stokes' theorem.

- ⋆ **Extra problem 1**: Compute

$$\iint\limits_S (\nabla\times\vec{F})\cdot d\vec{S},$$

where $\vec{F}(x,y,z)=(e^z, \ 4-x, \ x\cos y)$ and surface S is the part of the paraboloid $z=9-x^2-y^2$ which lies above the xy-plane, oriented with the normal pointing upwards from $(0,0,9)$.

- ⋆ **Extra problem 2**: Let $\vec{F}=(x,-y,0)$. Compute (using Stokes' theorem) the flux of the vector field $\vec{F}$ in across the surface $Y=\{(x,y,z)\in\mathbb{R}^3; \ z=x^2+y^2, \ z\leqslant 2\}$.

S20 Wrap-up Multivariable calculus / Calculus 3, Part 2 of 2

199 Calculus 3, Wrap-up.

200 Final words.