Calculus 2, part 2 of 2: Sequences and series[1]

Single variable calculus

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A short table of contents

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An extremely detailed table of contents; the videos (titles in green) are numbered

In blue: problems solved on an iPad (the solving process presented for the students; active problem solving)

In red: solved problems demonstrated during a presentation (a walk-through; passive problem solving)

In magenta: additional problems solved in written articles (added as resources).

In dark blue: *Read along with this section*: references for further reading and more practice problems in:

- the *Calculus* book (chapters 9 and 10) by Gilbert Strang and Edwin *Jed* Herman:

  `https://math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)`

  `https://math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)/zz%3A_Back_Matter/30%3A_Detailed_Licensing`

  This book is added as a resource to Video 1, with kind permission of the LibreTexts Office (July 20th, 2023).

- the *Lecture Notes from UC Davis* by Professor John K. Hunter:

  `https://www.math.ucdavis.edu/~hunter/intro_analysis_pdf/intro_analysis.html`

  This book is added as a resource to Video 1, with kind permission of Professor John K. Hunter (October 6th, 2024).

10 Old and new methods for determining convergence of sequences.

11 Practicing rules of limits, Exercise 1.

Exercise 1 (Calculus book, p. 1093): Suppose that:

$$\lim_{n\to\infty} a_n = 1, \qquad \lim_{n\to\infty} b_n = -1, \qquad 0 < -b_n < a_n \text{ for all } n.$$

Using this information, evaluate each of the following limits, state that the limit does not exist, or state that there is not enough information to determine whether the limit exists:

* $\lim_{n\to\infty} (3a_n - 4b_n)$
* $\lim_{n\to\infty} (\frac{1}{2}b_n - \frac{1}{2}a_n)$
* $\lim_{n\to\infty} \frac{a_n + b_n}{a_n - b_n}$
* $\lim_{n\to\infty} \frac{a_n - b_n}{a_n + b_n}$

Extra material: notes with solved Exercise 1.

12 Indeterminate forms and standard limits, Exercise 2.

Exercise 2: Compute

$$\lim_{n\to\infty} \frac{-\ln n^3 + e^{-n} + 3n^3}{e^{-3n} + n^5}.$$

Extra material: notes with solved Exercise 2.

13 Indeterminate forms and conjugates, Exercise 3.

Exercise 3: Compute

$$\lim_{n\to\infty} (n - \sqrt{n^2 - 4n}).$$

Extra material: notes with solved Exercise 3.

14 Indeterminate forms and conjugates, Exercise 4.

Exercise 4: Compute

$$\lim_{n\to\infty} \sqrt{n}(\sqrt{n+1} - \sqrt{n}).$$

Extra material: notes with solved Exercise 4.

15 Indeterminate forms and arithmetic on extended reals, Exercise 5.

Exercise 5: Compute

$$\lim_{n\to\infty} \left(\frac{e^n + n}{2e^n + 3n}\right)^{3n}.$$

Extra material: notes with solved Exercise 5.

16 Squeeze Theorem, Exercise 6.

Exercise 6: Compute limits of the following sequences:

$$x_n = \frac{n}{\sqrt{n^2+n}}, \quad y_n = \frac{n}{\sqrt{n^2+1}}, \quad z_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}.$$

Show that the same method will not work for the sequence from V230 in Calc2p1:

$$a_n = \frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \cdots + \frac{n}{n^2+n^2},$$

and that the method from V230 wouldn't work for the current example.

Extra material: notes with solved Exercise 6.

17 Squeeze Theorem, Exercise 7.

Exercise 7: Given m positive numbers: $a_1, a_2, \ldots, a_m$. We denote by A the largest of these numbers. Show that

$$\lim_{n\to\infty} \sqrt[n]{a_1^n + a_2^n + \cdots + a_m^n} = A.$$

This is a generalisation of the exercise in V197 in Calc1p1.

Extra material: notes with solved Exercise 7.

18 An old promise from earlier courses.

Theorem: If the sequence (a_n) is s.t. $a_n \to 0$, $a_n \neq 0$ and $a_n > -1$ for all $n \in \mathbb{N}^+$, then

$$\lim_{n\to\infty} (1 + a_n)^{\frac{1}{a_n}} = e.$$

This theorem was first formulated in V109 in Calc1p1, but we left the proof for later (i.e., for now), because it was technical and boring (it still is, but now I have no choice...).

Extra material: notes with the proof.

19 Indeterminate forms *one to the infinity*, various situations.

Example: Study the following limits:

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e, \quad \lim_{n\to\infty} \left(1 + \frac{1}{n^2}\right)^n = 1,$$

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{n^2} = \infty.$$

Extra material: notes with solved Example.

Extra material: theorems and proofs from the course *Precalculus 4: Exponentials and logarithms.*

20 Indeterminate forms leading to a result with e, Exercise 8.

Exercise 8: Compute

$$\lim_{n\to\infty} \left(1 - \frac{3}{n}\right)^n.$$

Extra material: notes with solved Exercise 8.

21 Indeterminate forms leading to a result with e, Exercise 9.

Exercise 9: Compute

$$\lim_{n\to\infty} \left(\frac{n-1}{n+1}\right)^n.$$

Extra material: notes with solved Exercise 9.

22 Indeterminate forms leading to a result with e, Exercise 10.

Exercise 10: Compute

$$\lim_{n\to\infty} \left(\frac{n^2+1}{n^2-1}\right)^{n^2}.$$

Extra material: notes with solved Exercise 10.

23 A difficult one, Problem 1.

Problem 1: Compute

$$\lim_{n\to\infty} [(n+1)^n - n^{n+1}].$$

Extra material: notes with solved Problem 1.

S3 Weierstrass' Theorem: a continuation from Calc1p1

You will learn: here we continue (after Calc1p1) discussing monotone sequences and their convergence; the main tool is *Weierstrass' Theorem*, also called *Monotone Convergence Theorem*; after repetition of some basic facts, you will get a lot of solved problems that illustrate the issue in depth.

Read along with this section: **Calculus book**: Chapter 9.1 *Sequences*, pages: from 1073 (9.1.1) to 1097 (9.1E.7). **Lecture Notes from UC Davis**: Chapter 3 *Sequences*, pages: from 37 to 50.

24 A warm-up exercise, Exercise 1.

Exercise 1: Given number sequence $a_n = \dfrac{2n}{n^2+1}$. Determine whether this sequence is:

- bounded (both below and above),
- positive or negative (eventually), or neither,
- increasing or decreasing (eventually), or neither,
- convergent, divergent, divergent to an infinity.

Extra material: notes with solved Exercise 1.

25 Proving (eventual) monotonicity analysing quotients, Exercise 2.

Example: Illustrate with $a_n = \dfrac{1}{n}$ that Weierstrass' Theorem will not always give us the value of the limit.

Exercise 2: Answer the same questions as in V24 for $a_n = \dfrac{10^n}{n!}$.

Extra material: notes with solved Exercise 2.

26 Proving (eventual) monotonicity analysing quotients, Exercise 3.

Exercise 3: Answer the same questions as in V24 for $a_n = \dfrac{2^n}{n^n}$.

Extra material: notes with solved Exercise 3.

27 Proving (eventual) monotonicity, Exercise 4.

Exercise 4: Prove that the sequence (x_n) is increasing:

$$x_n = \left(1 + \frac{1}{2^n}\right)^{2^n}.$$

Extra material: notes with solved Exercise 4.

28 Proving (eventual) monotonicity by a little bit of Weierstrass cheating, Exercise 5.

Exercise 5: Answer the same questions as in V24 for (a_n) that is defined recursively by:

$$a_1 = 3, \; a_{n+1} = \sqrt{15 + 2a_n}.$$

Extra material: notes with solved Exercise 5.

29 Nested roots, Problem 1.

Problem 1: Given $c > 0$. Define the following sequence: $x_1 = \sqrt{c}$, $x_{n+1} = \sqrt{c + x_n}$ for $n \geqslant 1$. Compute the limit of this sequence. (This is a generalisation of the problem from V110 in Calc1p1.)

Extra material: notes with solved Problem 1.

30 Recursively defined sequence, Problem 2.

Problem 2 (Calculus book, p. 1086–1087): Using Weierstrass' Theorem (*the Monotone Convergence Theorem*), show that the following sequence converges and find its limit:

$$a_1 = 2, \qquad a_{n+1} = \frac{a_n}{2} + \frac{1}{2a_n} \quad \text{for } n \geqslant 1.$$

Extra material: notes with solved Problem 2.

31 Recursively defined sequence, Problem 3.

Problem 3: Prove that the sequence (a_n) is eventually decreasing if:

$$a_0 = 1, \qquad a_{n+1} = a_n \cdot \frac{n^2+15}{(n+1)(n+2)} \quad \text{for } n \in \mathbb{N}.$$

Extra material: notes with solved Problem 3.

32 Recursively defined sequence, Problem 4.

Problem 4: Prove that the sequence (a_n) is eventually monotone if:

$$a_0 = 1, \qquad a_{n+1} = a_n \cdot \frac{n-5}{(n+1)(n+2)} \quad \text{for } n \in \mathbb{N}.$$

Extra material: notes with solved Problem 4.

33 Using Riemann integrals, Problem 5.

Problem 5: Show that sequence (S_n) is increasing, but not bounded above:

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

using Riemann integrals. The terms of this sequence are called *harmonic sums*, and we have shown in *Precalculus 1: Basic notions* (V234) that this sequence diverges to infinity.

34 A version of Weierstrass' Theorem for unbounded monotone sequences.

Theorem: If (x_n) is increasing (non-decreasing) and *not* bounded above, then $\lim_{n\to\infty} x_n = \infty$.

Corollary: If (y_n) is decreasing (non-increasing) and *not* bounded below, then $\lim_{n\to\infty} y_n = -\infty$.

35 Coming closer to series, Problem 6.

Problem 6: Show that (b_n) is tending to infinity, by comparing its terms to the terms of (S_n) in Video 33:

$$b_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}.$$

Extra material: notes with solved Problem 6.

36 Coming closer to series: telescoping series, Problem 7.

Problem 7: Show that (a_n) is bounded above:

$$a_n = 1 + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}.$$

Extra material: notes with solved Problem 7.

37 Coming closer to series: comparing stuff, Problem 8.

Problem 8: Show that (b_n) is bounded above, by comparing its terms to the terms of (a_n) in Video 36:

$$b_n = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2}.$$

Extra material: notes with solved Problem 8.

38 The one with a product, Problem 9.

Problem 9: Prove that the sequence (a_n) is strictly increasing, and *not* bounded above:

$$a_n = \left(1+\frac{1}{1}\right)\cdot\left(1+\frac{1}{2}\right)\cdot\left(1+\frac{1}{3}\right)\cdot\ldots\cdot\left(1+\frac{1}{n}\right).$$

Extra material: notes with solved Problem 9.

39 Another one with a product, Problem 10.

Problem 10: Prove that the sequence (a_n) has a limit: $a_n = \dfrac{1 \cdot 3 \cdot \ldots \cdot (2n-1)}{2 \cdot 4 \cdot \ldots \cdot 2n}$.

Extra material: notes with solved Problem 10.

40 The one tending to e, Problem 11.

Problem 11: Prove that the sequence (b_n) is convergent to a number less than or equal to 3:

$$b_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}.$$

You know that this sequence tends to $e \approx 2.72$; compare this solution to the one in the article under V19.

Extra material: notes with solved Problem 11.

41 Monotonicity of geometric progressions using difference method.

The quotient method doesn't work: $a_n = 3 \cdot 2^{n-1}$, $b_n = -3 \cdot 2^{n-1}$, $c_n = 3 \cdot (\frac{1}{2})^{n-1}$, $d_n = -3 \cdot (\frac{1}{2})^{n-1}$.

S4 Using functions while working with sequences

You will learn: in this section we move to the new stuff: a functional approach to sequences, that we weren't able to study in Calc1p1, as the section about sequences came *before* the section about functions (in the context of limits and continuity); how to use (for sequences) the theory developed for functions (derivatives, l'Hôpital's rule, etc).

Read along with this section: **Calculus book**: Chapter 9.1 *Sequences*, pages: from 1073 (9.1.1) to 1097 (9.1E.7). **Lecture Notes from UC Davis**: Chapter 3 *Sequences*, pages: from 37 to 50.

42 Using functions while working with sequences.

Illustrate the results from the previous video using functional approach.

43 Using functions while working with sequences, Exercise 1.

Exercise 1: Illustrate the results from V24 using functional approach.

Extra material: notes with solved Exercise 1.

44 Using functions while working with sequences, Exercise 2.

Exercise 2 (Calculus, p. 1093): Find the limits of the following sequences, using l'Hôpital's rule when appropriate:

* $\lim\limits_{n\to\infty} \dfrac{n^2}{2^n}$, discuss monotonicity,
* $\lim\limits_{n\to\infty} \dfrac{(n-1)^2}{(n+1)^2}$, discuss monotonicity,
* $\lim\limits_{n\to\infty} \dfrac{\sqrt{n}}{\sqrt{n+1}}$, discuss monotonicity.

Extra material: notes with solved Exercise 2.

45 Using functions while working with sequences, Exercise 3.

Exercise 3 (Calculus, p. 1093): Find the limits of the following sequences, using l'Hôpital's rule when appropriate:

* $\lim\limits_{n\to\infty} \dfrac{\ln n^2}{\ln 2n}$,
* $\lim\limits_{n\to\infty} \ln \dfrac{n+2}{n^2-3}$,
* $\lim\limits_{n\to\infty} \ln \left(1 + \dfrac{1}{n}\right)$. Motivate that this sequence is decreasing.

Extra material: notes with solved Exercise 3.

46 Using functions while working with sequences, Exercise 4.

Exercise 4: Compute
$$\lim_{n\to\infty} \ln(3^n + \cos 3^n).$$

Extra material: notes with solved Exercise 4.

47 Using functions while working with sequences, Exercise 5.

Exercise 5: Compute
$$\lim_{n\to\infty} [\ln(3n^2 - 3) - \ln(5n^2 + 5)].$$

Extra material: notes with solved Exercise 5.

48 Using functions while working with sequences, Problem 1.

Problem 1: Show that $a_n = \sqrt[n]{n}$ is (eventually) decreasing. Use the formula (cancellation!) $n^{1/n} = e^{\frac{1}{n}\ln n}$ to compute its limit. (Another method was presented in Calc1p1, V99, where we computed this limit with help of Squeeze Theorem.) Finally, show that $n^{n+1} \gg (n+1)^n$.

49 A spoiler: using functions while working with series.

S5 New theorems and tests for convergence of sequences

You will learn: various tests helping us computing limits of sequences is some cases: Stolz–Cesàro Theorem with some corollaries, the ratio test for sequences; we will prove the theorems, discuss their content, and apply them on various examples.

Read along with this section: **Calculus book**: Chapter 9.1 *Sequences*, pages: from 1073 (9.1.1) to 1097 (9.1E.7). **Lecture Notes from UC Davis**: Chapter 3 *Sequences*, pages: from 37 to 50.

50 Stolz–Cesàro Theorem: formulation and proof.

Extra material: an article with a proof of Stolz–Cesàro Theorem.

Extra material: notes with a proof of the identity from the proof.

51 Applications of Stolz–Cesàro Theorem, Example 1.

Example 1: Let $a > 1$. Show, using Stolz–Cesàro Theorem, that
$$\lim_{n\to\infty} \frac{a^n}{n} = +\infty.$$

Extra material: notes with solved Example 1.

52 Applications of Stolz–Cesàro Theorem, Example 2 (Cauchy's Theorem).

Example 2: Show, using Stolz–Cesàro Theorem, that the following theorem (Cauchy's) is true: If sequence (a_n) has a limit, finite or infinite, then the sequence
$$A_n = \frac{a_1 + a_2 + \cdots + a_n}{n}$$
(i.e., A_n is the arithmetic mean of the first n elements of (a_n)) has the same limit.
Motivate, using the theorem above, that
$$\lim_{n\to\infty} \frac{1 + \sqrt{2} + \sqrt[3]{3} + \cdots + \sqrt[n]{n}}{n} = 1.$$

Finally, show that the converse is not true, i.e., construct an example where convergence of (A_n) as defined above does not imply convergence of (a_n).

Extra material: notes with solved Example 2.

53 A corollary from Cauchy's Theorem.

Corollary (from Cauchy's Theorem): If $(a_{n+1} - a_n) \to g$ then also $\frac{a_n}{n} \to g$. Show that the converse is not true.

Extra material: notes with a proof of the corollary.

54 Applications of the corollary from V53, Example 3.

Example 3: Use the corollary above to prove that

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0.$$

This was proven in Calc1p1, V121, and now we learn a new method for the same limit. We can also use the method from Calc1p2, V182 (l'Hôpital's rule)

Extra material: notes with solved Example 3.

55 Applications of Stolz–Cesàro Theorem, Problem 1.

Problem 1: Find, using Stolz–Cesàro Theorem, the following limit (that we computed in V231 of Calc2p1): for $p \in \mathbb{N}^+$, determine:

$$\lim_{n\to\infty} \frac{1^p + 2^p + \cdots + n^p}{n^{p+1}}.$$

Extra material: notes with solved Problem 1.

56 Using formulas or Riemann sums instead, Problem 2.

Problem 2: Show three special cases of the limit from the previous lecture, namely:

$$\lim_{n\to\infty} \frac{1 + 2 + \cdots + n}{n^2} = \frac{1}{2}, \quad \lim_{n\to\infty} \frac{1^2 + 2^2 + \cdots + n^2}{n^3} = \frac{1}{3}, \quad \lim_{n\to\infty} \frac{1^3 + 2^3 + \cdots + n^3}{n^4} = \frac{1}{4},$$

using Riemann sums (as in Calc2p1, V231), and by using the formulas derived in Calc1p1, V32–35.

57 Geometric means, Problem 3.

Problem 3: Given a convergent sequence (a_n) of strictly positive numbers ($a_n > 0$ and $a_n \to g \in \mathbb{R}$). Show that then the sequence

$$G_n = \sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n}$$

(i.e., G_n is the geometric mean of the first n elements of (a_n)) has the same limit.

Show that the converse is not true, i.e., construct an example where convergence of (G_n) as defined above does not imply convergence of (a_n).

Extra material: notes with solved Problem 3.

Extra material: proof of the great and powerful **AM–GM inequality (Cauchy in.)**: For all $n \in \mathbb{N}^+$ and for all $a_1, \ldots, a_n$ such that $a_i \geqslant 0$ for $i = 1, 2, \ldots, n$, the arithmetic mean of all these numbers is greater than or equal to (equality can only happen when all the numbers are equal to each other) their geometric mean, i.e.:

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geqslant \sqrt[n]{a_1 a_2 \cdot \ldots \cdot a_n}.$$

58 Corollary from the fact proven in V57.

Corollary: If (a_n) is a sequence of positive numbers, and $\frac{a_{n+1}}{a_n} \to g$, then also $\sqrt[n]{a_n} \to g$.

Extra material: notes with a proof of the corollary.

59 Applications of the corollary from V58, Example 4.

Example 4: Compute

$$\lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}}.$$

Extra material: notes with solved Example 4.

60 Ratio test for sequences.

Ratio test: If (a_n) is such a a sequence of non-zero numbers that the sequence of absolute values of the ratios of its consecutive terms is convergent and its limit is equal to $g \in \mathbb{R}$, i.e.:

$$\left|\frac{a_{n+1}}{a_n}\right| \to g,$$

then:

* if $g < 1$ then (a_n) is convergent and $a_n \to 0$
* if $g > 1$ then (a_n) is divergent and $|a_n| \to \infty$ [In this case we can also consider $g = \infty$.]
* if $g = 1$ then the test is inconclusive, so you need to look for a different method.

Extra material: an article with a proof of the ratio test.

61 Ratio test for sequences, Problem 4.

Problem 4: Given $a > 1$. Compute

$$\lim_{n\to\infty} \frac{n!}{a^n}.$$

Extra material: notes with solved Problem 4.

62 Ratio test for sequences, Problem 5.

Problem 5: Compute

$$\lim_{n\to\infty} \frac{(n+2)!}{(2n+1)!}.$$

Extra material: notes with solved Problem 5.

63 Ratio test for sequences, Problem 6.

Problem 6: Compute

$$\lim_{n\to\infty} \frac{(n!)^2}{(2n)!}.$$

Extra material: notes with solved Problem 6.

64 Ratio test for sequences: Case 3 is inconclusive.

Use the following sequences to illustrate that $g = 1$ in the ratio test really doesn't help:

$$a_n = 1, \quad b_n = (-1)^n, \quad c_n = \frac{1}{n}, \quad d_n = n.$$

Extra material: notes from the iPad.

65 Ratio test for sequences: the converse is not true.

Use the following sequences to illustrate that the converse to the ratio test is not true:

* $a_{2n-1} = a_{2n} = \frac{1}{2^n}$ for $n \geqslant 1$
* $b_{2n-1} = \frac{1}{(2n-1)^2}$, $b_{2n} = \frac{1}{2n}$ for $n \geqslant 1$.

Extra material: notes from the iPad.

S6 Solving recurrence relations

You will learn: solving linear recursions of order 2 (an introduction; more will be covered in *Discrete Mathematics*).

Read along with this section: **Calculus book**: Chapter 9.1 *Sequences*, pages: from 1073 (9.1.1) to 1097 (9.1E.7). **Lecture Notes from UC Davis**: Chapter 3 *Sequences*, pages: from 37 to 50.

66 Verifying recurrence relations, Exercise 1.

Exercise 1: Verify that $x_n = 2^n + 1$ satisfies the recurrence relation $x_n = 2x_{n-1} - 1$ with $x_1 = 3$.

Extra material: notes with solved Exercise 1.

67 Using iterations for getting the formula, Exercise 2.

Exercise 2: Solve the recurrence relation $x_n = x_{n-1} + n$ with the initial condition $x_0 = 4$.

Extra material: notes with solved Exercise 2.

68 Telescoping techniques, Exercise 3.

Exercise 3: Solve the recurrence relation $x_n = x_{n-1} + n$ with the initial condition $x_0 = 4$ using telescoping.

Extra material: notes with solved Exercise 3.

69 Linear recursion of order 2: an introduction, Exercise 4.

Exercise 4: Define $x_n = 5x_{n-1} - 6x_{n-2}$. Write down the first five terms of this sequence and try to find a closed formula for x_n if:

* $x_0 = 1$, $x_1 = 2$,
* $x_0 = 1$, $x_1 = 3$,
* $x_0 = 2$, $x_1 = 5$.

Extra material: notes with solved Exercise 4.

70 Linear recursion of order 2: the meaning of linearity, Exercise 5.

Exercise 5: Define $x_n = a \cdot x_{n-1} + b \cdot x_{n-2}$, where $|a| + |b| > 0$ and $a^2 + 4b > 0$. The equation

$$r^2 - ar - b = 0$$

is called *the characteristic equation* of the recursion above. Do the following:

* show that if $x_n = q^n$ satisfies the recursive relation above, then q is a root of its characteristic equation,
* show that if (x_n) and (y_n) are two solutions to the recursive relation above, and α and β are any real numbers, then $z_n = \alpha x_n + \beta y_n$ also satisfies the same recursive relation. **Note**: This is the second meaning linearity of the relation: if two sequences satisfy the relation, then each linear combination of these sequences also satisfies the same relation. The first meaning was: each element of the sequence (except the first two elements) is a linear combination of two elements preceding it.
* write the characteristic equation for the recursion from the previous video: $x_n = 5x_{n-1} - 6x_{n-2}$ and explain the results from the previous video in the light of our new findings.

Extra material: notes with solved Exercise 5.

71 Linear recursion of order 2: derivation of the closed formula.

Given sequence (F_n) defined as follows: $F_{n+2} = a \cdot F_{n+1} + b \cdot F_n$ for $n \geqslant 0$, where:

[A1] $|a| + |b| > 0$ (i.e., at least one of them must be non-zero)

[A2] $a^2 + 4b > 0$ (you will soon discover why!)

and F_0 and F_1 are given. Establish a closed formula for F_n for all natural n.

Extra material: an article with a derivation of the closed formula.

72 Linear recursion of order 2, Exercise 6.

Exercise 6: Define $x_n = x_{n-1} + 6x_{n-2}$ for $n \geqslant 2$. Find a closed formula for (x_n).

Extra material: notes with solved Exercise 6.

73 Linear recursion of order 2, with an initial condition, Exercise 7.

Exercise 7: Solve the recurrence relation $x_n = 7x_{n-1} - 10x_{n-2}$ with the initial conditions $x_0 = 2$ and $x_1 = 3$.

Extra material: notes with solved Exercise 7.

74 Linear recursion of order 2, with an initial condition, Exercise 8.

Exercise 8: Solve the recurrence relation $x_n = 2x_{n-1} + x_{n-2}$ with the initial conditions $x_0 = 1$ and $x_1 = 2$.

Extra material: notes with solved Exercise 8.

75 Back to Fibonacci, Exercise 9.

Exercise 9: Confirm the result obtained in V77 in Calc1p1 (closed formula for Fibonacci sequence).

Extra material: notes with solved Exercise 9.

S7 Applications of sequences and some more problems to solve

You will learn: various applications of sequences; more types of sequence-related problems that we haven't seen before (some problems here are really hard).

Read along with this section: **Calculus book**: Chapter 9.1 *Sequences*, pages: from 1073 (9.1.1) to 1097 (9.1E.7). **Lecture Notes from UC Davis**: Chapter 3 *Sequences*, pages: from 37 to 50.

76 Some words about applications of sequences.

77 **Optional**: Fibonacci and modelling.

Example: For each $n \in \mathbb{N}^+$ determine the number x_n of subsets of the set $A_n = \{1, 2, 3, \ldots, n\}$ that does not contain any pair of consecutive natural numbers.

78 Change factor, compound interest, etc in Precalculus 4.

79 Sequences in practical applications, Problem 1.

Problem 1 (Calculus book, p. 1096, Problem 59): A lake initially contains 2000 fish. Suppose that in the absence of predators or other causes of removal, the fish population increases by 6% each month. However, factoring in all causes, 150 fish are lost each month.

- Explain why the fish population after n months is modelled by $P_n = 1.06P_{n-1} - 150$ with $P_0 = 2000$.
- How many fish will be in the pond after one year? (Answer: about 1494.)

Extra material: notes with solved Problem 1.

80 Sequences in practical applications: decimal expansions of numbers, Problem 2.

Problem 2: Let (p_n) be any infinite sequence such that for all n we have $p_n \in \{0, 1, \ldots, 9\}$, i.e., all the elements are integer numbers between 0 and 9. Show that the sequence (s_n), where:

$$s_n = \frac{p_1}{10} + \frac{p_2}{10^2} + \cdots + \frac{p_n}{10^n}$$

is convergent. An example: If $p_1 = 1$, $p_2 = 4$, $p_3 = 1$, $p_4 = 5$, $p_5 = 9, \ldots$ are the digits in decimal expansion of π, then

$$\pi = 3 + \lim_{n\to\infty} s_n.$$

Motivate that $0.999(9) = 1$ (it was demonstrated in Video 16 of *Precalculus* 1).

Extra material: notes with solved Problem 2.

81 Sequences in practical applications: approximations of roots, Problem 3.

Problem 3 (Calculus book, p. 1096, Problem 54): Use Newton's method for approximating the value of $\sqrt{2}$.

Show that the sequence defined by:

$$a_0 = 2, \quad a_n = \frac{a_{n-1}}{2} + \frac{1}{a_{n-1}} \quad \text{for } n \geqslant 1$$

converges to the square root of 2.

Extra material: notes with solved Problem 3.

82 Cutting interval in half, Problem 4.

Problem 4: Define a sequence by

$$0 < a_0 < 1, \qquad a_{n+1} = \frac{a_n + 1}{2}.$$

Prove that the sequence is increasing and bounded above, and determine its limit by analysing the geometry of the situation. Confirm your result by a computation.

Extra material: notes with solved Problem 4.

83 Sequences in practical applications: an approximation of π, Problem 5.

Problem 5 (*Precalculus 3: Trigonometry*, V31):

Define $s_0 = 1$, $s_{n+1} = \sqrt{2 - \sqrt{4 - s_n^2}}$. Let $p_n = 3 \cdot 2^n \cdot s_n$. Motivate that (p_n) approximates π.

This problem can be compared to Problem 64 from the Calculus book on page 1097.

84 Sequences defined with help of Riemann integrals, Problem 6.

Problem 6: Show that $\lim_{n\to\infty} a_n = \infty$, where

$$a_n = \int_0^1 (x^2 + 2)^n \, dx.$$

Extra material: notes with solved Problem 6.

85 Sequences defined with help of Riemann integrals, Problem 7.

Problem 7: Show that $\lim_{n\to\infty} a_n = \infty$, where

$$a_n = \int_0^1 (x^2 + 1)^n \, dx.$$

Extra material: notes with solved Problem 7.

86 Sequences defined with help of Riemann integrals, Problem 8.

Problem 8: Show that $\lim_{n\to\infty} a_n = 0$, where

$$a_n = \int_0^1 x^n (1 - x)^n \, dx.$$

Extra material: notes with solved Problem 8.

87 Sequences defined with help of Riemann integrals, Problem 9.

Problem 9: Show that $\lim_{n\to\infty} a_n = 0$, where

$$a_n = \int_0^{\pi/2} \sin^n x \, dx.$$

Extra material: notes with solved Problem 9.

88 Time for some tests, Problem 10.

Problem 10: Prove that

$$\lim_{n\to\infty} \frac{2^n n!}{n^n} = 0.$$

Extra material: notes with solved Problem 10.

89 A cool trick, Problem 11.

Problem 11: Prove that

$$\lim_{n\to\infty}(1+n+n^2)^{1/n}=1.$$

Extra material: notes with solved Problem 11.

90 Using properties of logarithms, Problem 12.

Problem 12: Show that

$$\lim_{n\to\infty}\frac{\ln n!}{n\ln n}=1.$$

Extra material: notes with solved Problem 12.

91 2025, Problem 13.

Extra material: notes with solved Problem 13.

92 Subsequences of convergent sequences and the geometric sum, Problem 14.

Lemma: Each subsequence (a_{k_n}) of a convergent sequence (a_n) converges to the same limit, i.e.:

$$\lim_{n\to\infty}a_n=L\in\mathbb{R}\quad\Rightarrow\quad\lim_{n\to\infty}a_{k_n}=L.$$

Problem 14: Compute

$$\lim_{n\to\infty}\sqrt{2}\cdot\sqrt[4]{2}\cdot\sqrt[8]{2}\cdot\ldots\cdot\sqrt[2^n]{2}.$$

Extra material: notes with solved Problem 14.

93 A cool trigonometry trick, Problem 15.

Problem 15: Let φ be any number different from 0. Compute

$$\lim_{n\to\infty}\cos\frac{\varphi}{2}\cos\frac{\varphi}{2^2}\cdot\ldots\cdot\cos\frac{\varphi}{2^n}.$$

Extra material: notes with solved Problem 15.

94 Three divergent trigonometric sequences, Problem 16.

Problem 16: Show that

$$x_n=\cos n,\quad y_n=\sin n,\quad z_n=\tan n$$

are divergent. The last problem was already solved in Calc1p1, V188; the first two are solved visually in the Calculus book (p. 1095, Problems 44 and 45).

Extra material: notes with solved Problem 16.

95 Two remarks to Video 94.

S8 Cauchy sequences and the set of real numbers

You will learn: more (than in Calc1p1) about the relationships between monotonicity, boundedness, and convergence of number sequences; subsequences and their limits; limit superior and limit inferior (reading material only: Section 3.6 on pages 50–55 in the UC Davis notes); Bolzano–Weierstrass Theorem; fundamental sequences (sequences with Cauchy property), their boundedness and convergence; construction of the set of real numbers with help of equivalence classes of fundamental sequences of rational numbers; the definition of complete metric spaces.

Read along with this section: **Lecture Notes from UC Davis**: Chapter 3 *Sequences*, pages: from 50 to 60.

96 Real-number sequences and sequences in metric spaces.

97 Sequence-related stuff.

98 Sequence-related stuff: boundedness and monotonicity.

99 Sequence-related stuff: convergence and subsequences.

100 Sequence-related stuff: Cauchy property, fundamental sequences.

101 Real-number sequences versus sequences in metric spaces.

102 This is where we are heading.

103 Some relationships between properties that we already established.

Theorems that were proven in Calc1p1:

- Each convergent sequence is bounded (true for all metric spaces).
- Each subsequence of a convergent sequence is convergent (to the same limit) (true for all metric spaces).
- (Weierstrass' Th.) Each monotone and bounded sequence of real numbers is convergent (to its inf or sup).

Only the last theorem is restricted to real-number sequences; the first two are true in any metric space.

104 Cauchy property, fundamental sequences, and convergence.

Property 1: Each convergent sequence (even generally, in any metric space) is fundamental.

105 Cauchy property, fundamental sequences, and boundedness.

Property 2: Each fundamental sequence is bounded, but not each bounded sequence is fundamental.

106 Cauchy property, fundamental sequences: proximity of consecutive terms is not enough.

If $(a_{n+1} - a_n) \to 0$, then (a_n) doesn't need to be fundamental.

Extra material: notes from the iPad.

107 Monotone subsequences of real-number sequences.

Each sequence of real numbers has a monotone subsequence.

Examples of non-monotone sequences and their monotone subsequences:

$$a_{2n-1} = 0, \ \ a_{2n} = 1 \ \ (n = 1, 2, 3, \dots), \quad b_n = (-1)^n \cdot \frac{1}{n}, \quad h_n = (-1)^n + (-1)^{n+1} \cdot \frac{1}{n}.$$

Extra material: an article with a proof of the theorem.

108 Bolzano–Weierstrass' Theorem.

Bolzano–Weierstrass: Each bounded sequence of real numbers has a convergent subsequence.

109 The most important result about sequences of real numbers.

Theorem (Cauchy): Each fundamental (Cauchy) sequence of real numbers is convergent in $\mathbb{R}$.

Extra material: notes from the iPad.

110 (Another) construction of the set of real numbers.

Show that the following relation defined on the set of all the fundamental sequences of rational numbers

$$(x_n) \sim (y_n) \quad \Leftrightarrow \quad \lim_{n\to\infty} (x_n - y_n) = 0$$

is an RST relation. By Cauchy Theorem (V109) we know that each such a sequence is convergent in $\mathbb{R}$. Uniqueness of the limit (Calc1p1, V90) makes it possible to define each real number by the equivalence class under this relation of all the sequence of rational numbers converging to this number.

111 Accumulation points of a sequence as limits of its subsequences.

112 Back to a statement from V90 in Calc1p1.

If a real-number sequence is **bounded**, and it has exactly one accumulation point, then it is convergent in $\mathbb{R}$.

113 A new computation of an old limit.

Let $a_n \to a$. If $k \in \mathbb{N}^+ \setminus \{1\}$ then $\lim_{n\to\infty} \sqrt[k]{a_n} = \sqrt[k]{a}$. If k is even, then we need the restriction: $a_n \geqslant 0$ for all $n \in \mathbb{N}^+$; no restrictions needed for odd k.

Extra material: notes from the iPad.

114 **Optional**: Read about limit superior and limit inferior.

Example: Find the supremum, infimum, limit superior, and limit inferior for (a_n) if:

$a_1 = -3,\ a_2 = 0,\ a_3 = 1.5,\ a_4 = -1, \quad \text{for } n \geqslant 1: \ \ a_{4n+1} = \frac{1}{n},\ a_{4n+2} = -1,\ a_{4n+3} = 2 - \frac{1}{n},\ a_{4n+4} = 0.$

115 The Nested Intervals Theorem.

116 An application of nested intervals.

Let $a_0 = 0$ and for $n \in \mathbb{N}^+$

$$a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots + (-1)^{n-1}\frac{1}{n}.$$

Show that (a_n) converges. (**Note**: This is a partial sum of a so called *series with alternating signs*; they come back later in the course.)

117 What is the limit of the sequence from the previous lecture? Error estimation.

Show that the sequence from V116 is convergent to $\ln 2$.

The same method helps us motivate a fact presented to you in *Precalculus* 3: $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots$.

Extra material: notes from the iPad.

118 Complete metric spaces in the next course.

S9 Number series: a general introduction

You will learn: about series: their definition and interpretation, many examples of convergent and divergent series (geometric series, arithmetic series, p-series, telescoping series, alternating series); you will also learn how to determine the sum of series in some cases; we will later use these series for determining convergence or divergence of other series, that are harder to deal with.

Read along with this section: **Calculus book**: Chapter 9.2 *Infinite Series*, pages: from 1098 (9.2.1) to 1120 (9.2E.8); Chapter 9.5 *Alternating Series*, pages: from 1152 (9.5.1) to 1167 (9.5E.8). **Lecture Notes from UC Davis**: Chapter 4 *Series*, pages: from 61 to 90.

119 The definition of convergent and divergent series.

120 Three very simple examples; Cesàro sum for a series.

- The sequence $a_n = 0$ is summable, i.e., the corresponding series is convergent to a number: $\sum_{n=1}^{\infty} a_n = 0$.
- The sequence $a_n = 1$ is *not* summable; the corresponding series is divergent to plus infinity: $\sum_{n=1}^{\infty} a_n = +\infty$.
- The sequence $a_n = (-1)^n$ is *not* summable; its sequence of partial sums is divergent.

121 Examples of series we have seen when we worked with sequences.

Examples from videos: 33, 35, 36, 37, 40, 80, 116, 117.

122 From the sequence to its series, an exercise.

Exercise: Write the first four terms of the series on the basis of the known general term a_n:

(1) $a_n = \dfrac{3n-2}{n^2+1}$,

(2) $a_n = \dfrac{2+(-1)^n}{n^2}$,

(3) $a_n = \dfrac{1}{(3+(-1)^n)^n}$.

Extra material: notes from the iPad.

123 From the series to its sequence, an exercise.

Exercise: Find the formula for a_n for the following series, and use the sigma notation:

(1) $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots=$

(2) $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\cdots=$

(3) $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots=$

(4) $\frac{3}{4}+\frac{4}{9}+\frac{5}{16}+\frac{6}{25}+\cdots=$

(5) $\frac{2}{5}+\frac{4}{8}+\frac{6}{11}+\frac{8}{14}+\cdots=$

(6) $1+\frac{2}{2}+\frac{3}{4}+\frac{4}{8}+\cdots=$

(7) $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\frac{1}{42}+\cdots=$

(8) $1-\frac{3}{4}+\frac{5}{8}-\frac{7}{12}+\frac{9}{16}-\frac{11}{20}+\cdots=$

(9) $1-1+1-1+1-1+\cdots=$

(10) $1+\frac{1}{2}+3+\frac{1}{4}+5+\frac{1}{6}+\cdots=$

124 A necessary (but not sufficient) condition for convergence of series.

Theorem (necessary condition for convergence): If $\sum_{n=1}^{\infty} a_n$ is convergent, then $a_n \to 0$.

Contrapositive (a basic question for determining divergence): If $a_n \nrightarrow 0$ then $\sum_{n=1}^{\infty} a_n$ is not convergent.

Show that this condition is not sufficient; compare this to the necessary condition for convergence of improper integrals of the first kind.

125 The necessary condition for convergence, an exercise.

Exercise: Compute $\sum_{n=0}^{\infty} \frac{n}{n+2}$ or show that the series is divergent.

Which series from V123 don't satisfy the necessary condition for convergence (thus are divergent)?

126 For the sum of series it does matter from which index you start.

127 Heads, tails, and remainders.

Given a series $\sum_{k=1}^{\infty} a_k$ and $n \geqslant 1$: the nth *remainder* of the series is the sum of the elements in the nth tail, i.e., a new series: $R_n = \sum_{k=n+1}^{\infty} a_k$.

- ⋆ **Theorem 1**: If $\sum_{k=1}^{\infty} a_k$ is convergent (i.e., the sum S of series exists and is a number), then for all $n \in \mathbb{N}^+$ the series $\sum_{k=n+1}^{\infty} a_k$ is also convergent.
- ⋆ **Corollary**: If $\sum_{k=1}^{\infty} a_k$ is convergent then $\lim_{n\to\infty} R_n = 0$.
- ⋆ **Theorem 2** (converse to Th1): If for all $n \in \mathbb{N}^+$ the series $\sum_{k=n+1}^{\infty} a_k$ is convergent, then the series $\sum_{k=1}^{\infty} a_k$ is also convergent.

128 The Cauchy condition for series.

Cauchy condition for series: Series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if

$$\forall\ \varepsilon > 0 \quad \exists\ n_\varepsilon \quad \forall\ n \geqslant n_\varepsilon \quad \forall\ k \in \mathbb{N}^+ \quad |a_{n+1} + a_{n+2} + \cdots + a_{n+k}| < \varepsilon.$$

129 Arithmetic series: nothing interesting, really...

130 Geometric series: one of our celebrities.

131 Geometric series: various ways of writing down the same series.

Several examples of writing down the same sum: $3 - 3 \cdot \frac{2}{5} + 3 \cdot \left(\frac{2}{5}\right)^2 - 3 \cdot \left(\frac{2}{5}\right)^3 + 3 \cdot \left(\frac{2}{5}\right)^4 - 3 \cdot \left(\frac{2}{5}\right)^5 + \ldots$.

What should be done if the index doesn't start at zero: $-3 \cdot \left(\frac{2}{5}\right)^3 + 3 \cdot \left(\frac{2}{5}\right)^4 - 3 \cdot \left(\frac{2}{5}\right)^5 + \ldots$. Two different methods are given.

Extra material: an article with more solved problems on geometric series. In all cases, identify a and q, and use the well-known formula valid if $|q| < 1$:

$$\sum_{n=0}^{\infty} aq^n = \frac{a}{1-q}.$$

- ⋆ **Extra problem 1**: $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$.
- ⋆ **Extra problem 2**: $\sum_{n=0}^{\infty} 7 \cdot \left(\frac{2}{3}\right)^n$.
- ⋆ **Extra problem 3**: $\sum_{n=0}^{\infty} 5 \cdot \left(-\frac{2}{3}\right)^n$.
- ⋆ **Extra problem 4**: $\sum_{n=0}^{\infty} \frac{4}{3^n}$.
- ⋆ **Extra problem 5**: $\sum_{n=0}^{\infty} 2 \cdot \left(-\frac{3}{4}\right)^n$.
- ⋆ **Extra problem 6**: $\sum_{n=0}^{\infty} \left(-\frac{4}{5}\right)^n$.
- ⋆ **Extra problem 7**: $\sum_{n=0}^{\infty} \frac{3 \cdot (-1)^n}{4^n}$.
- ⋆ **Extra problem 8**: $\sum_{n=1}^{\infty} \frac{3}{4^{n-1}}$.
- ⋆ **Extra problem 9**: $\sum_{n=1}^{\infty} \frac{2}{7^{n-1}}$.
- ⋆ **Extra problem 10**: $\sum_{n=1}^{\infty} 9 \cdot \left(\frac{1}{4}\right)^n$.
- ⋆ **Extra problem 11**: $\sum_{n=1}^{\infty} \frac{4}{3^n}$.
- ⋆ **Extra problem 12**: $\sum_{n=1}^{\infty} \frac{8}{(-3)^n}$.
- ⋆ **Extra problem 13**: $2 - \frac{2}{7} + \frac{2}{49} - \frac{2}{343} + \ldots$
- ⋆ **Extra problem 14**: $-2 \cdot \frac{3}{5} + 2 \cdot \left(\frac{3}{5}\right)^2 - 2 \cdot \left(\frac{3}{5}\right)^3 + 2 \cdot \left(\frac{3}{5}\right)^4 - 2 \cdot \left(\frac{3}{5}\right)^5 + \ldots$.
- ⋆ **Extra problem 15**: $-\frac{7}{4} + \frac{7}{8} - \frac{7}{16} + \frac{7}{32} - \frac{7}{64} + \ldots$
- ⋆ **Extra problem 16**: $\sum_{n=0}^{\infty} \left(-\frac{4}{3}\right)^n$.

132 A geometrical illustration of summability of a geometric series.

Motivate with a computation and present a geometric interpretation of the fact that

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots=\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n=1.$$

133 The sum, difference, and scalings of series; a warning about the product.

Theorem: Let

$$\sum_{n=1}^{\infty} a_n = A, \qquad \sum_{n=1}^{\infty} b_n = B$$

be two convergent series. Then for each pair of real numbers α and β we have

$$\sum_{n=1}^{\infty}(\alpha a_n+\beta b_n)=\alpha A+\beta B.$$

134 The sum, difference, and scalings of series, an exercise.

Exercise: Compute $\sum_{n=0}^{\infty}\frac{3+2^n}{3^{n+2}}$ and $\sum_{n=0}^{\infty}\frac{1}{(3+(-1)^n)^n}$ or show that the series are divergent.

Extra material: notes from the iPad.

135 From harmonic series to p-series.

Theorem (p-series): Let $p\in\mathbb{R}$ be any number. The p-series $\sum_{n=1}^{\infty}\frac{1}{n^p}$

* is divergent to plus infinity if $p \leqslant 1$,
* is convergent if $p > 1$.

Example: the series $\sum_{n=1}^{\infty}\frac{1}{n\sqrt{n}}$, as a p-series for $p=\frac{3}{2}$, is convergent.

136 Alternating series.

Examples: alternating series:

(1) $\sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{1}{n}$ from V116 and V117.

(2) $\sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{1}{2n-1}$ from V117.

(3) $\sum_{n=0}^{\infty}(-1)^n\cdot q^n$, with $q>0$.

Theorem (alternating series): Suppose that (a_n) is a decreasing (non-increasing) sequence of non-negative numbers, i.e., $0\leqslant a_{n+1}\leqslant a_n$ (for all $n\in\mathbb{N}^+$) such that $\lim_{n\to\infty} a_n=0$. The following alternating series converges:

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_n=a_1-a_2+a_3-a_4+a_5-a_6+\ldots$$

137 Convergent alternating series and approximations.

Example: Approximate the alternating harmonic series to one decimal place.

138 Telescoping series and their convergence.

(1) The telescoping series $\sum_{n=1}^{\infty}(b_n-b_{n+1})$ is convergent iff (b_n) is convergent.

(2) Each series is a telescoping series of its own partial sums.

139 Telescoping series and their convergence, Example 1.

Example 1: Compute the sum of the following series:

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots=\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)}.$$

Extra material: notes with solved Example 1.

140 Telescoping series and their convergence, Example 2.

Example 2: Compute the sum of the following series:

$$\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\frac{1}{3\cdot 4\cdot 5}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)},$$

using the result of the problem from V36 and the theorem from V133.

Extra material: notes with solved Example 2.

141 Telescoping series and their convergence, Example 3.

Example 3: Compute the sum of the following series (the same as in V140), using partial fraction decomposition:

$$\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\frac{1}{3\cdot 4\cdot 5}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)}.$$

Note: You could apply the same method for the series from Videos 36 and 139; do it as an exercise.

Extra material: notes with solved Example 3.

142 Another way of showing that the harmonic series is divergent to plus infinity.

(1) Show that the harmonic series (from V33) is divergent to plus infinity, using subsequences of the sequence of partial sums.

(2) Show that the p-series for $p=\frac{1}{2}$ is divergent, without using a comparison with the harmonic series (V35).

Extra material: notes from the iPad.

143 A nice alternating series that shows to be divergent.

Show that the following series is divergent to plus infinity:

$$\frac{1}{\sqrt{2}-1}-\frac{1}{\sqrt{2}+1}+\frac{1}{\sqrt{3}-1}-\frac{1}{\sqrt{3}+1}+\frac{1}{\sqrt{4}-1}-\frac{1}{\sqrt{4}+1}+\cdots.$$

Comment the result in the light of the theorem from V136.

Extra material: notes with solved Example 3.

144 Turning sequences into infinite series.

Find the limit of:

$$0,\ 1,\ \frac{1}{2},\ \frac{3}{4},\ \frac{5}{8},\ \frac{11}{16},\ \ldots$$

the sequence formed by starting at 0, going right to 1, then going along the number axis in an alternating manner (to the left, then to the right, and so on), each time one half of the previous distance.

145 Turning sequences into infinite series, a geometric introduction to Euler's constant.

Let

$$S_n=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}-\ln(n+1),\quad n\geqslant 1.$$

Convert the sequence into an infinite series. Motivate with an illustration that the series is convergent to a number γ between 0 and 1. This number is called **Euler's constant**.

$$\gamma=\sum_{n=1}^{\infty}\left(\frac{1}{n}-\ln\frac{n+1}{n}\right)=\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}-\ln(n+1)\right).$$

146 Absolute and conditional convergence.

(1) If a series is *absolutely convergent*, then it is also convergent.

(2) The converse is not true: show an example of a convergent series that is not absolutely convergent (i.e., is just *conditionally convergent*).

(3) If almost all elements of a series are of the same sign, then the series is convergent if and only if it is absolutely convergent.

(4) The series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if and only if

$$\sum_{n=1}^{\infty} a_n^+ < +\infty \quad \text{and} \quad \sum_{n=1}^{\infty} a_n^- < +\infty,$$

where

$$r^+ = \begin{cases} r; & r \geqslant 0 \\ 0; & r < 0 \end{cases} \quad \text{and} \quad r^- = \begin{cases} 0; & r > 0 \\ -r; & r \leqslant 0 \end{cases}, \quad \text{i.e.,} \quad r = r^+ - r^- \quad \text{and} \quad |r| = r^+ + r^-.$$

(5) If the series $\sum_{n=1}^{\infty} a_n$ is conditionally convergent then

$$\sum_{n=1}^{\infty} a_n^+ = +\infty = \sum_{n=1}^{\infty} a_n^-.$$

147 Fun stuff: arithmetico-geometric series, Problem 1.

Problem 1: Find expressions for the partial sums of $\sum_{k=1}^{\infty} kx^{k-1}$ for some fixed $x \neq 1$. For which values of x is the series convergent?

Extra material: notes with solved Problem 1.

148 More fun stuff: using arithmetico-geometric series, Problem 2.

Problem 2: Find expressions for the partial sums of $\sum_{k=1}^{\infty} k^2x^{k-1}$ for some fixed $x \neq 1$. For which values of x is the series convergent?

Extra material: notes with solved Problem 2.

149 More fun stuff: using arithmetico-geometric series, Problem 3.

Problem 3: Find expressions for the partial sums of $\sum_{k=1}^{\infty} k(2^k - 1)$.

Extra material: notes with solved Problem 3.

150 Arithmetico-geometric series, Problem 4.

Problem 4: Compute $\sum_{n=1}^{\infty} \frac{n}{2^n}$. For now, assume that the series is convergent, so that it is OK to use the theorem from V133; in Section 10 you will learn how to motivate that the series is convergent.

Extra material: notes with solved Problem 4.

151 Using arithmetico-geometric series, Problem 5.

Problem 5: Compute $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$. For now, assume that the series is convergent, so that it is OK to use the theorem from V133; in Section 10 you will learn how to motivate that the series is convergent.

Extra material: notes with solved Problem 5.

S10 Number series: plenty of tests, even more exercises

You will learn: plenty of tests for convergence of number series (why they work and how to apply them): comparison tests, limit comparison test, ratio test (d'Alembert test), root test (Cauchy test), integral test.

Read along with this section: **Calculus book**: Chapter 9.3 *The Divergence and Integral Tests*, pages: from 1121 (9.3.1) to 1137 (9.3E.6); Chapter 9.4 *Comparison Tests*, pages: from 1138 (9.4.1) to 1151 (9.4E.6); Chapter 9.6 *Ratio and Root Tests*, pages: from 1168 (9.6.1) to 1184 (9.6E.7). **Lecture Notes from UC Davis**: Chapter 4 *Series*, pages: from 61 to 90.

152 Our roadmap.

153 Determining convergence, Exercise 1.

Exercise 1: Determine whether the following series are convergent:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n+5},\quad \sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{3n-2},\quad \sum_{n=4}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n-3}}.$$

Extra material: notes with solved Exercise 1.

154 Determining convergence, Exercise 2.

Exercise 2: Determine whether $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\ln n}{n}$ is convergent.

Extra material: notes with solved Exercise 2.

155 Determining convergence, Exercise 3.

Exercise 3: Determine whether $\sum_{n=0}^{\infty}\frac{5}{10^{3n}}$ is convergent. If possible, compute its sum.

Extra material: notes with solved Exercise 3.

156 Determining convergence, Exercise 4.

Exercise 4: Determine whether the following series are convergent:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt[5]{n^3}},\quad \sum_{n=1}^{\infty}\frac{1}{\sqrt[5]{n^3}},\quad \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt[3]{n^5}},\quad \sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n^5}}.$$

Extra material: notes with solved Exercise 4.

157 Series with positive elements: Comparison test.

Comparison test (CT): Given two sequences (a_n) and (b_n) such that *for almost all* n

$$0 \leqslant a_n \leqslant b_n.$$

* If $\sum_{n=1}^{\infty} b_n < +\infty$ then $\sum_{n=1}^{\infty} a_n < +\infty$.
* If $\sum_{n=1}^{\infty} a_n = +\infty$ then $\sum_{n=1}^{\infty} b_n = +\infty$.

158 Comparison test, Exercise 5.

Exercise 5: Determine whether $\sum_{n=1}^{\infty}\frac{2n+1}{3n^3+5}$ is convergent.

Extra material: notes with solved Exercise 5.

159 Comparison test, Exercise 6.

Exercise 6: Determine whether $\sum_{n=1}^{\infty}\frac{3n+5}{2n^2+1}$ is convergent.

Extra material: notes with solved Exercise 6.

160 Comparison test, Exercise 7.

Exercise 7: Determine whether $\sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n}$ is convergent.

Extra material: notes with solved Exercise 7.

161 Comparison test, Exercise 8.

Exercise 8: Determine whether $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ is convergent.

Extra material: notes with solved Exercise 8.

162 Comparison test, Exercise 9.

Exercise 9: Determine whether $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ is convergent.

Extra material: notes with solved Exercise 9.

163 Comparison test, Exercise 10.

Exercise 10: Determine whether $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ is convergent.

Extra material: notes with solved Exercise 10.

164 Comparison test, Exercise 11.

Exercise 11: Determine whether $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-3}}$ is convergent.

Extra material: notes with solved Exercise 11.

165 Comparison test, Exercise 12.

Exercise 12: Determine whether the following series are convergent:

$$\sum_{n=1}^{\infty} \frac{3^n}{2^n+5^n}, \quad \sum_{n=1}^{\infty} \frac{3^n}{2^n+3^n}.$$

Extra material: notes with solved Exercise 12.

166 Series with positive elements: Limit comparison test.

Limit comparison test (LCT): Given two positive sequences (a_n) and (b_n) such that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L,$$

where L is either a non-negative number or plus infinity.

* If $L < +\infty$ and $\sum_{n=1}^{\infty} b_n < +\infty$ then $\sum_{n=1}^{\infty} a_n < +\infty$.
* If $L > 0$ and $\sum_{n=1}^{\infty} b_n = +\infty$ then $\sum_{n=1}^{\infty} a_n = +\infty$.

In particular, if $L = 1$, the test is called **Asymptotic comparison test**, and the series corresponding to the sequences (a_n) and (b_n) converge or diverge at the same time, i.e.:

$$\sum_{n=1}^{\infty} a_n < +\infty \quad \Leftrightarrow \quad \sum_{n=1}^{\infty} b_n < +\infty.$$

Examples from V158, V159, and V164 are revisited in the context of the new criterion.

Extra material: an article with exercises from Videos 160–164 solved with help of the LCT.

167 Limit comparison test, Exercise 13.

Exercise 13: Determine whether $\sum_{n=1}^{\infty} \frac{n+5}{n^3-2n+3}$ is convergent.

Extra material: notes with solved Exercise 13.

168 Limit comparison test, Exercise 14.

Exercise 14: Determine whether $\sum_{n=1}^{\infty} \frac{n}{n^4 - 2}$ is convergent.

Extra material: notes with solved Exercise 14.

169 Limit comparison test, Exercise 15.

Exercise 15: Determine whether $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 + n + 1}$ is convergent.

Extra material: notes with solved Exercise 15.

170 Limit comparison test, Exercise 16.

Exercise 16: Determine whether $\sum_{n=1}^{\infty} \frac{1}{\ln 3n}$ is convergent.

Extra material: notes with solved Exercise 16.

171 Limit comparison test, Exercise 17.

Exercise 17: Determine whether $\sum_{n=1}^{\infty} \frac{1}{n \cdot \sqrt[n]{n}}$ is convergent.

Extra material: notes with solved Exercise 17.

172 An example where the CT works better than LCT, Exercise 18.

Exercise 18: Determine whether $\sum_{n=1}^{\infty} \frac{1 + \sin n}{n^2}$ is convergent.

Extra material: notes with solved Exercise 18.

173 Series with positive elements: Comparison test 2 (lemma).

Comparison test (CT2): Given two *positive* sequences (a_n) and (b_n) such that *for almost all* n

$$\frac{a_{n+1}}{a_n} \leqslant \frac{b_{n+1}}{b_n}.$$

* If $\sum_{n=1}^{\infty} b_n < +\infty$ then $\sum_{n=1}^{\infty} a_n < +\infty$.
* If $\sum_{n=1}^{\infty} a_n = +\infty$ then $\sum_{n=1}^{\infty} b_n = +\infty$.

174 Series with positive elements: d'Alembert criterion.

d'Alembert criterion: Given a *positive* sequence (a_n), we define

$$d_n = \frac{a_{n+1}}{a_n}.$$

* If there exists a number $0 < q < 1$ such that *for almost all* $n \in \mathbb{N}^+$ we have $d_n \leqslant q$ then $\sum_{n=1}^{\infty} a_n$ is convergent. In symbols:

$$(\exists \ 0 < q < 1 \quad \exists \ n_0 \quad \forall \ n > n_0 \quad d_n \leqslant q) \quad \Rightarrow \quad \sum_{n=1}^{\infty} a_n < +\infty.$$

* If *for almost all* $n \in \mathbb{N}^+$ we have $d_n \geqslant 1$ then $\sum_{n=1}^{\infty} a_n$ is divergent to plus infinity. In symbols:

$$(\exists \ n_0 \quad \forall \ n > n_0 \quad d_n \geqslant 1) \quad \Rightarrow \quad \sum_{n=1}^{\infty} a_n = +\infty.$$

175 d'Alembert criterion formulated with help of limit superior and limit inferior.

d'Alembert criterion (version 2): Given a *positive* sequence (a_n), we define

$$d_n = \frac{a_{n+1}}{a_n}.$$

Let

$$\overline{d} = \lim_{n\to\infty} \sup d_n = \sup S(d_n), \quad \underline{d} = \lim_{n\to\infty} \inf d_n = \inf S(d_n)$$

be the limit superior and the limit inferior of (d_n), i.e., the supremum and the infimum of the set of all the accumulation points of (d_n) (see V114).

* If $\overline{d} < 1$ then $\sum_{n=1}^{\infty} a_n < +\infty$.
* If $\underline{d} > 1$ then $\sum_{n=1}^{\infty} a_n = +\infty$.
* If $\overline{d} \geqslant 1$ or $\underline{d} \leqslant 1$ then the test is inconclusive.

Example: Show that d'Alembert criterion doesn't give any answer about convergence of

$$\sum_{n=1}^{\infty} (2 + (-1)^n) \cdot \frac{1}{2^n}.$$

Show in another way that the series is convergent and compute its sum. (Note: This example comes back in V190; we will show that it is possible to determine its convergence with our next test: the root test [Cauchy criterion].)

Extra material: notes with solved Example.

176 Series with positive elements: Ratio test.

Ratio test: Given a *positive* sequence (a_n), we define

$$d_n = \frac{a_{n+1}}{a_n}. \qquad \text{If} \quad \lim_{n\to\infty} d_n = d \quad \text{then}$$

* if $d < 1$ then $\sum_{n=1}^{\infty} a_n$ is convergent,
* if $d > 1$ then $\sum_{n=1}^{\infty} a_n$ is divergent to plus infinity,
* if $d = 1$ then the test is inconclusive (unless almost all $d_n \geqslant 1$, then we can conclude divergence).

177 Ratio test, Exercise 19.

Exercise 19: Determine whether $\sum_{n=1}^{\infty} \frac{n}{2^n}$ (see V150) is convergent.

Extra material: notes with solved Exercise 19.

178 Ratio test, Exercise 20.

Exercise 20: Determine whether $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ (see V151) is convergent.

Extra material: notes with solved Exercise 20.

179 Ratio test, Exercise 21.

Exercise 21: Determine whether $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is convergent (see V194 in Calc1p2).

Extra material: notes with solved Exercise 21.

180 Ratio test, Exercise 22.

Exercise 22: Show that the ratio test doesn't work for $\sum_{n=1}^{\infty} \frac{1}{n}$, $\sum_{n=1}^{\infty} \frac{1}{n^2}$, $\sum_{n=1}^{\infty} \frac{1}{n^3}$, etc.

Extra material: notes with solved Exercise 22.

181 Ratio test, Exercise 23.

Exercise 23: Determine whether $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ is convergent.

Extra material: notes with solved Exercise 23.

182 Ratio test, Exercise 24.

Exercise 24: Determine whether $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$ is convergent.

Extra material: notes with solved Exercise 24.

183 Ratio test, Exercise 25.

Exercise 25: Determine whether $\sum_{n=1}^{\infty} \frac{3^n \cdot n!}{n^n}$ is convergent.

Extra material: notes with solved Exercise 25.

184 Ratio test, Exercise 26.

Exercise 26: Determine whether the following series is convergent: $1 + \frac{1 \cdot 2}{1 \cdot 3} + \frac{1 \cdot 2 \cdot 3}{1 \cdot 3 \cdot 5} + \frac{1 \cdot 2 \cdot 3 \cdot 4}{1 \cdot 3 \cdot 5 \cdot 7} + \dots$

Extra material: notes with solved Exercise 26.

185 Ratio test, Exercise 27.

Exercise 27: Determine whether the following series (see V160) is convergent: $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} + \frac{1}{4 \cdot 2^4} + \dots$

Extra material: notes with solved Exercise 27.

186 Ratio test, Exercise 28.

Exercise 28: Determine whether the following series is convergent: $2 + \frac{3}{2} \cdot \frac{1}{4} + \frac{4}{3} \cdot \frac{1}{4^2} + \frac{5}{4} \cdot \frac{1}{4^3} + \dots$

Extra material: notes with solved Exercise 28.

187 Ratio test, Exercise 29.

Exercise 29: Determine whether the following series is convergent: $1 + \frac{2^2+1}{2^3+1} + \frac{3^2+1}{3^3+1} + \frac{4^2+1}{4^3+1} + \dots$

Extra material: notes with solved Exercise 29.

188 Ratio test, Exercise 30.

Exercise 30: Determine whether $\sum_{n=1}^{\infty} \frac{e^n n!}{n^n}$ is convergent.

Extra material: notes with solved Exercise 30.

189 Series with positive elements: Cauchy criterion.

Cauchy criterion: Given a *positive* sequence (a_n), we define

$$c_n = \sqrt[n]{a_n}.$$

* If there exists a number $0 \leqslant q < 1$ such that *for almost all* $n \in \mathbb{N}^+$ we have $c_n \leqslant q$ then $\sum_{n=1}^{\infty} a_n$ is convergent. In symbols:

$$(\exists \ 0 \leqslant q < 1 \quad \exists \ n_0 \quad \forall \ n > n_0 \quad c_n \leqslant q) \quad \Rightarrow \quad \sum_{n=1}^{\infty} a_n < +\infty.$$

* If (**Note: different compared to d'A!**) *for infinitely many* $n \in \mathbb{N}^+$ we have $c_n \geqslant 1$ then $\sum_{n=1}^{\infty} a_n$ is divergent to plus infinity. In symbols:

$$(\forall \ n_0 \quad \exists \ n > n_0 \quad c_n \geqslant 1) \quad \Rightarrow \quad \sum_{n=1}^{\infty} a_n = +\infty.$$

190 Cauchy criterion formulated with help of limit superior and limit inferior.

Cauchy criterion (version 2): Given a *positive* sequence (a_n), we define

$$c_n = \sqrt[n]{a_n}.$$

Let

$$\overline{c} = \lim_{n\to\infty} \sup c_n = \sup S(c_n)$$

be the limit superior of (c_n), i.e., the supremum of the set of all the accumulation points of (c_n) (see V114). (**Note**: different compared to d'A!: Only **limit superior** is involved.)

* If $\overline{c} < 1$ then $\sum_{n=1}^{\infty} a_n < +\infty$.
* If $\overline{c} > 1$ then $\sum_{n=1}^{\infty} a_n = +\infty$.
* If $\overline{c} = 1$ then the test is inconclusive (unless infinitely many $c_n \geqslant 1$, then we can conclude divergence).

Example: In V175 we have seen that d'Alembert criterion doesn't give any answer about convergence of

$$\sum_{n=1}^{\infty}(2+(-1)^n)\cdot\frac{1}{2^n}.$$

We have also shown that the series is convergent and that its sum is $\frac{5}{3}$. Confirm with help of Cauchy criterion that this series is convergent.

Extra material: notes with solved Example.

191 Series with positive elements: Root test.

Root test: Given a *positive* sequence (a_n), we define

$$c_n = \sqrt[n]{a_n}. \qquad \text{If} \quad \lim_{n\to\infty} c_n = c \quad \text{then}$$

* if $c < 1$ then $\sum_{n=1}^{\infty} a_n$ is convergent,
* if $c > 1$ then $\sum_{n=1}^{\infty} a_n$ is divergent to plus infinity,
* if $c = 1$ then the test is inconclusive (unless infinitely many $c_n \geqslant 1$, then we can conclude divergence).

Example: The criterion doesn't work for the p-series for $p = 1, 2, 3, \ldots$; compare with V180.

192 Root test, Exercise 31.

Exercise 31: Determine whether $\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}$ is convergent.

Extra material: notes with solved Exercise 31.

193 Root test, Exercise 32.

Exercise 32: Determine whether $\sum_{n=1}^{\infty}\left(\frac{x}{n}\right)^{n}$ (for some $x > 0$) is convergent.

Extra material: notes with solved Exercise 32.

194 Root test, Exercise 33.

Exercise 33: Determine whether $\sum_{n=1}^{\infty}\frac{1}{3^n}\cdot\left(\frac{n+1}{n}\right)^{n^2}$ is convergent.

Extra material: notes with solved Exercise 33.

195 Root test, Exercise 34.

Exercise 34: Determine whether $\sum_{n=1}^{\infty}\frac{n^4\cdot(\sqrt{5}+(-1)^n)^n}{4^n}$ is convergent.

Extra material: notes with solved Exercise 34.

196 Root test, Exercise 35.

Exercise 35: Determine whether $\sum\limits_{n=1}^{\infty}\left(\frac{3n+1}{2n+100}\right)^n$ and $\sum\limits_{n=1}^{\infty}\left(\frac{2n+100}{3n+1}\right)^n$ are convergent.

Extra material: notes with solved Exercise 35.

197 Series with positive elements: Integral test.

Integral test: Suppose that $f : [1, +\infty) \to \mathbb{R}$ is a *positive* and *decreasing* function, i.e.:

$$0 \leqslant f(x) \leqslant f(y) \quad \text{for} \quad x \geqslant y \geqslant 1.$$

Let $a_n = f(n)$. Then

$$\sum_{n=1}^{\infty} a_n < +\infty \quad \Leftrightarrow \quad \int_1^{\infty} f(x)\,dx < +\infty.$$

Furthermore, the limit

$$D = \lim_{n\to\infty}\left[\sum_{k=1}^{n} a_n - \int_1^n f(x)\,dx\right]$$

exists and $0 \leqslant D \leqslant a_1$.

Conclusion: This gives a proper motivation for the intuitive definition of Euler's constant γ (see V145):

$$\lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{1}{k} - \int_1^n \frac{1}{x}\,dx\right] \in (0,1).$$

Warning: Observe, think, choose wisely! Two scary examples where you shouldn't use the integral test.

198 Integral test, Exercise 36.

Exercise 36: Confirm the result from V135 (convergence of p-series) with help of the integral test. Also: Why on Earth would you bother giving the cumbersome proof in V135???

199 Integral test, Exercise 37.

Exercise 37: Determine whether $\sum\limits_{n=1}^{\infty}\frac{1}{\sqrt{2n+1}}$ is convergent.

Extra material: notes with solved Exercise 37.

200 Integral test, Exercise 38.

Exercise 38: Determine whether $\sum\limits_{n=2}^{\infty}\frac{1}{n\ln n}$ is convergent.

Extra material: notes with solved Exercise 38.

201 Integral test, Exercise 39.

Exercise 39: Use the integral test to show that $\sum\limits_{n=1}^{\infty}\frac{1}{n^2+1}$ converges. Show that the sum of the series is less than $\frac{\pi}{2}$.

Extra material: notes with solved Exercise 39.

202 Integral test, Exercise 40.

Exercise 40: Determine whether $\sum\limits_{n=1}^{\infty}\frac{n}{n^2+1}$ is convergent.

Extra material: notes with solved Exercise 40.

203 Integral test, Exercise 41.

Exercise 41: Determine whether $\sum\limits_{n=1}^{\infty}\frac{1}{n(n+1)}$ is convergent.

Extra material: notes with solved Exercise 41.

204 Integral test, Exercise 42.

Exercise 42: Determine whether $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$ is convergent.

Extra material: notes with solved Exercise 42.

205 Integral test, Exercise 43.

Exercise 43: Determine whether $\sum_{n=2}^{\infty} \frac{1}{n \cdot \ln n \cdot (\ln \ln n)^2}$ is convergent.

Extra material: notes with solved Exercise 43.

206 There are more tests, there are plenty of books.

207 Convergence of series, Problem 1.

Problem 1: Determine whether $\sum_{n=1}^{\infty} \frac{1}{n} \left(\sqrt{n+1} - \sqrt{n-1}\right)$ is convergent.

Extra material: notes with solved Problem 1.

208 Convergence of series, Problem 2.

Problem 2: Determine whether $\sum_{n=1}^{\infty} n^2 \cdot \sin \frac{\pi}{2^n}$ is convergent.

Extra material: notes with solved Problem 2.

209 Convergence of series, Problem 3.

Problem 3: Back to the series from V165; apply to them all the tests you can:

$$\sum_{n=1}^{\infty} \frac{3^n}{2^n + 5^n}, \quad \sum_{n=1}^{\infty} \frac{3^n}{2^n + 3^n}.$$

Extra material: notes with solved Problem 3.

210 Back to Euler constant gamma, Problem 4.

Problem 4: Use the comparison test to show that

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \ln \frac{n+1}{n} \right)$$

is convergent to a number $\gamma \in (0,1)$, i.e., $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \ln(n+1)\right) = \gamma$. (The constant γ was introduced in V145, came back in V197, and now appears for the third time.)

Extra material: notes with solved Problem 4.

Extra material: notes with the proof that the sequence $c_n = (1 + \frac{1}{n})^{n+1}$ is converging to e in a *decreasing* way (almost identically as in V109 of Calc1p1).

211 Back to the alternating harmonic series, Problem 5.

Problem 5: We go back to the alternating harmonic series (that we had seen in V116, V117, V136), this time with help of the limit of (S_n) from the previous lecture. Confirm the result from V117, that:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots + (-1)^{n-1}\frac{1}{n} + \cdots = \ln 2.$$

Extra material: notes with solved Problem 5.

212 The last one, Problem 6.

Problem 6: Show that $\frac{(n!)^n}{n^{n^2}} \to 0$.

Extra material: notes with solved Problem 6.

S11 Various operations on series

You will learn: how the regular computational rules like commutativity and associativity work for series; Cauchy product of series; remainders, their various shapes and their role in approximating the sum of a series.

Read along with this section: **Calculus book**: Chapter 9.2 *Infinite Series*, pages: from 1098 (9.2.1) to 1120 (9.2E.8); Chapter 9.5 *Alternating Series*, pages: from 1152 (9.5.1) to 1167 (9.5E.8); **Lecture Notes from UC Davis**: Chapter 4 *Series*, pages: from 75 to 80.

213 Linear combinations of convergent series (see V133).

214 The associative law works fine for partial sums, but not for the entire series.

Example: The series $\sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{2n+1}{n}$ diverges (why?). Show that if we group the elements in pairs, we will get a convergent series.

215 Rearrangements.

Th.: If a series is absolutely convergent, then every rearrangement of the series converges to the same sum.

216 Riemann Rearrangement Theorem.

Riemann Rearrangement Theorem: If a series is conditionally convergent, then it has rearrangements that converge to an arbitrary real number and rearrangements that diverge to $+\infty$ or $-\infty$.

Example (UC Davis, p.78): Rearrange the alternating harmonic series so that the sum of the new series is $\sqrt{2}$.

217 A useful lemma that allows us solve the example in the next video.

Lemma: Let $a_n \to 0$ (i.e., the necessary condition for the convergence of $\sum_{n=1}^{\infty} a_n$ is satisfied), $S_n = \sum_{k=1}^{n} a_k$, and $d \in \mathbb{N}^+ \setminus \{1\}$. If the subsequence of every dth element of (S_n) tends to S, i.e., $S_{dn} \to S$, then also $S_n \to S$.

218 Rearrangements, an example.

Example: Compute the sum of the following rearrangement of the alternating harmonic series:

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\dots+\frac{1}{4n-3}+\frac{1}{4n-1}-\frac{1}{2n}+\dots$$

Note: I use the same approach as in V211. Study the same example solved with another method in the Calculus book (pdf pages 1158–1159). Another rearrangement is studied in the notes from UC Davis (p. 75).

Extra material: notes with solved Example.

Extra material: An article with a more general example showing that, for any $p, q \in \mathbb{N}^+$ such that $p > q$, the rearrangement of the alternating harmonic series that takes a group of the first p fractions with odd denominators, then (with minus) a group of the first q fractions with even denominators, then the second groups of next p and q fractions, etc, has the sum equal to $\ln\left(2\cdot\sqrt{\frac{p}{q}}\right)$. (In the example shown during the lecture we had $p = 2$ and $q = 1$.)

219 Products of series.

Theorem (Cauchy): If $\sum_{n=1}^{\infty} a_n = A$, $\sum_{n=1}^{\infty} b_n = B$, and both series are **absolutely convergent**, then every product of these series is also an absolutely convergent series; the sum of each such series is equal to AB.

220 Cauchy product of series.

Definition: For any two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, their *Cauchy product* is $\sum_{n=1}^{\infty} c_n$, where

$c_n = a_1 b_n + a_2 b_{n-1} + \dots + a_n b_1 = \sum_{k=1}^{n} a_k b_{n-k+1}$ (**Note**: (c_n) is called *convolution of sequences* (a_n) *and* (b_n)).

Corollary (follows from Cauchy Theorem in V219): If $\sum_{n=1}^{\infty} a_n = A$, $\sum_{n=1}^{\infty} b_n = B$, and both series are **absolutely convergent**, then their Cauchy product is also absolutely convergent, and its sum is equal to AB.

221 Cauchy product of series, Example 1.

Example 1: Compute Cauchy product of $\sum_{n=1}^{\infty} x^{n-1}$ (where $|x| < 1$) with itself. Compare to V147.

Extra material: notes with solved Example 1.

222 Cauchy product of series, Example 2.

Example 2: Compute Cauchy product of $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ with itself. This example shows that Cauchy product of two conditionally convergent series can be divergent. (Compare to Mertens' Theorem.)

Extra material: notes with solved Example 2.

223 Cauchy product of series, Example 3.

Example 3: Compute Cauchy product of the following divergent series:

$$1 - \sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n, \qquad 1 + \sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^{n-1} \cdot \left(2^n + \frac{1}{2^{n+1}}\right).$$

You will discover that this product is an (absolutely) convergent series.

Extra material: notes with solved Example 3.

224 Remainders and approximations.

You get a list of various cases for estimating errors when approximating the sum of series by a partial sum. You get references to the videos with the theory that motivates the methods, and references to videos where we see some solved examples.

225 Remainders and approximations: alternating series, Exercise 1.

Exercise 1: Estimate the error when the following series is approximated by its first 10 terms:

$$\sum_{n=1}^{\infty} a_n = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \dots$$

How many terms must be used to compute the value of the series with allowable error 0.05?

226 Remainders and approximations: positive series, Exercise 2.

Exercise 2: Estimate the error when $\sum_{n=1}^{\infty} \frac{1}{n!}$ is approximated by its first 12 terms.

227 Remainders and approximations: positive series, Exercise 3.

Exercise 3: Here you get two following (non-identical) twin problems:

* Estimate the error when the following series is approximated by its first 10 terms:

$$\sum_{n=1}^{\infty} a_n = \frac{2}{3} + \frac{1}{2} \cdot \left(\frac{2}{3}\right)^2 + \frac{1}{3} \cdot \left(\frac{2}{3}\right)^3 + \frac{1}{4} \cdot \left(\frac{2}{3}\right)^4 + \dots$$

* Estimate the error when the following series is approximated by its first 10 terms:

$$\sum_{n=1}^{\infty} a_n = \frac{1}{3} + \frac{2}{3^2} + \frac{3}{3^3} + \frac{4}{3^4} + \dots$$

228 Remainders and approximations: integral test, Exercise 4.

Exercise 4:

* Estimate the error when $\sum_{n=1}^{\infty} \frac{1}{4n^2}$ is approximated by its first 10 terms.
* Estimate the number of terms necessary to compute $\sum_{n=1}^{\infty} \frac{1}{n^5+1}$ with allowable error 0.00001.

S12 Sequences of functions (a very brief introduction)

You will learn: you will get a very brief introduction to the topic of sequences of functions; more will be covered in *Real Analysis: Metric spaces*; the concepts of point-wise convergence and uniform convergence are briefly introduced and illustrated with one example each; these concepts will be further developed in *Real Analysis: Metric spaces*.

Read along with this section: **Lecture Notes from UC Davis**: Chapter 9 *Sequences and Series of Functions*, pages: from 169 to 177.

229 Sequences of functions: two famous examples.

Two examples of sequences of functions f_n, $g_n : [-1,1] \to \mathbb{R}$: $f_n(x) = x^n$, $g_n(x) = \dfrac{x^n}{n}$.

230 Point-wise convergence of sequences of functions.

Example 1: Find the point-wise limit of the sequences (f_n) and (g_n) from the previous video.

231 Uniform convergence of sequences of functions.

Example 2: Is the convergence of (f_n) and (g_n) uniform?

Remark: Uniform convergence implies point-wise convergence, and the limit functions are equal to each other.

Theorem (UC Davis, p. 174): The uniform limit of a sequence of continuous functions is continuous.

S13 Infinite series of functions (a very brief introduction)

You will learn: you get a very brief introduction to the topic of series of functions: just enough to introduce the topic of power series in the next section.

Read along with this section: **Lecture Notes from UC Davis**: Chapter 9 *Sequences and Series of Functions*, pages: from 177 to 181.

232 Series of functions; four types of their convergence.

Example: $\sum_{n=0}^{\infty} x^n = \dfrac{1}{1-x}$ for $x \in (-1,1)$. Point-wise convergence is clear now, more later.

233 Cauchy conditions for sequences and series of functions.

234 All the regular criteria and a new one: Weierstrass test.

Weierstrass test: If $f_n : D \to \mathbb{R}$ are such that

$$|f_n(x)| \leqslant a_n \qquad \text{for } \ n = 1, 2, 3 \ldots \quad \text{and for all } \ x \in D$$

for some non-negative sequence (a_n) s.t. $\sum_{n=1}^{\infty} a_n < +\infty$, then $\sum_{n=1}^{\infty} f_n$ is absolutely uniformly convergent on D.

Example: $\sum_{n=1}^{\infty} \dfrac{\sin nx}{n^2}$ is absolutely uniformly convergent on $\mathbb{R}$.

S14 Power series and their properties

You will learn: the concept of a power series and different ways of thinking about this topic; radius of convergence; arithmetic operations on power series (addition, subtraction, scaling, multiplication); some words about differentiation and integration of power series term after term (optional).

Read along with this section: **Calculus book**: Chapter 10.1 *Power Series and Functions*, pages: from 1190 (10.1.1) to 1206 (10.1E.8); Chapter 10.2 *Properties of Power Series*, pages: from 1207 (10.2.1) to 1225 (10.2E.6). **Lecture Notes from UC Davis**: Chapter 10 *Power Series*, pages: from 183 to 190.

235 Power series and several ways of thinking about them.

Warning: The expression 0^0 is undefined. It is common to use this shortcut anyway: $\sum_{n=0}^{\infty} a_n x^n = a_0 + \sum_{n=1}^{\infty} a_n x^n$.

Example 0: Our returning champion from V232: $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $x \in (-1, 1)$. Comes back again in V252.

236 Power series we have seen earlier in this course.

237 Power series we have seen in Precalculus 3 and Precalculus 4.

238 Before we start, let's get prepared, Exercise 1.

Ex 1: Determine the values of x for which the series $\sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n+1}}$ converges (absolutely, conditionally), diverges.

Extra material: notes with solved Exercise 1.

239 Before we start, let's get prepared, Exercise 2.

Ex 2: Determine the values of x for which the series $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2 \cdot 2^{2n}}$ converges (absolutely, conditionally), diverges.

Extra material: notes with solved Exercise 2.

240 Before we start, let's get prepared, Exercise 3.

Ex 3: Determine the values of x for which $\sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^n}{2n+3}$ converges (absolutely, conditionally), diverges.

Extra material: notes with solved Exercise 3.

241 Before we start, let's get prepared, Exercise 4.

Ex 4: Determine the values of x for which $\sum_{n=1}^{\infty} \frac{1}{2n-1} \cdot \left(\frac{3x+2}{-5}\right)^n$ converges (absolutely, conditionally), diverges.

Extra material: notes with solved Exercise 4.

242 Before we start, let's get prepared, Exercise 5.

Ex 5: Determine the values of x for which the series $\sum_{n=2}^{\infty} \frac{x^n}{2^n \cdot \ln n}$ converges (absolutely, conditionally), diverges.

Extra material: notes with solved Exercise 5.

243 Before we start, let's get prepared, Exercise 6.

Ex 6: Determine the values of x for which the series $\sum_{n=1}^{\infty} \frac{(4x+1)^n}{n^3}$ converges (absolutely, conditionally), diverges.

Extra material: notes with solved Exercise 6.

244 Before we start, let's get prepared, Exercise 7.

Ex 7: Determine the values of x for which the series $\sum_{n=1}^{\infty} \frac{(2x+3)^n}{\sqrt[3]{n} \cdot 4^n}$ converges (absolutely, conditionally), diverges.

Extra material: notes with solved Exercise 7.

245 Does the ratio test always work? An example.

Example: Determine the values of x for which the series $\sum_{n=1}^{\infty} \frac{2+(-1)^n}{n} x^n$ converges absolutely, converges conditionally, or diverges. You will notice that the method from the previous exercises fails. Try Cauchy test. We did something similar in V175 and V190.

Extra material: notes with solved Example.

246 Radius of convergence, two methods.

Examples: with $0 < R < +\infty$ (V238–V245), $R = +\infty$ (V179), and $R = 0$: $\sum_{n=0}^{\infty} n! \cdot x^n$.

Derivation of two formulas for the radius (in extended reals): if the limits exist:

$$R = \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{|a_n|}}, \qquad R = \frac{1}{\lim\limits_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}}.$$

Example from V241 revisited (solved with help of the formulas derived in this lecture).

247 Theorems about convergence of power series.

Theorem: Given power series $\sum\limits_{n=0}^{\infty} a_n x^n$ with radius of convergence $R \neq 0$. For each $r \in (0, R)$ the series is absolutely uniformly convergent on $[-r, r]$.

Corollary: With assumptions as above, the function f represented by the series is continuous on $[-r, r]$.

Theorem (Weierstrass), no proof here: Given power series $\sum\limits_{n=0}^{\infty} a_n x^n$ with radius of convergence $0 < R < +\infty$. If $\sum\limits_{n=0}^{\infty} a_n R^n$ ($\sum\limits_{n=0}^{\infty} a_n (-R)^n$) is convergent then $\sum\limits_{n=0}^{\infty} a_n x^n$ is uniformly convergent on $[0, R]$ (on $[-R, 0]$).

248 Uniqueness of power-series representations.

Theorem (about the uniqueness): Given two series such that $\sum\limits_{n=0}^{\infty} a_n x^n = A(x)$ on the interval I_A and $\sum\limits_{n=0}^{\infty} b_n x^n = B(x)$ on the interval I_B. If

$$\exists\ \delta > 0 \quad \forall\ x \in (-\delta, \delta) \quad A(x) = B(x)$$

(the neighbourhood is contained in both I_A and I_B), then

$$a_n = b_n \quad \text{for } \ n = 0, 1, 2, \ldots$$

Moreover, $I_A = I_B = I$ and $A(x) = B(x)$ for all $x \in I$.

249 Power-series representations for even and odd functions.

- ⋆ If function f is **even**, i.e., $f(-x) = f(x)$ for all x, and f is represented by a power series in some non-trivial neighbourhood of zero, i.e., $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$, then $a_1 = a_3 = a_5 = \ldots = 0$.
- ⋆ If function f is **odd**, i.e., $f(-x) = -f(x)$ for all x, and f is represented by a power series in some non-trivial neighbourhood of zero, i.e., $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$, then $a_2 = a_4 = a_6 = \ldots = 0$.

250 Uniqueness of power-series representations, an example.

Find power series representations for the functions:

- ⋆ **Example 1**: $f(x) = x^3$.
- ⋆ **Example 2**: $f(x) = x^3$ about $x_0 = -1$.

251 Arithmetic operations on power series.

Theorem: Let

$$\sum_{n=0}^{\infty} a_n x^n = A(x) \qquad (\text{for } x \in I_A), \qquad \sum_{n=0}^{\infty} b_n x^n = B(x) \qquad (\text{for } x \in I_B)$$

be two power series with intervals of convergence I_A and I_B. Then for each pair of real numbers α and β we have

$$\sum_{n=0}^{\infty} (\alpha a_n + \beta b_n) x^n = \sum_{n=0}^{\infty} (\alpha a_n x^n + \beta b_n x^n) = \alpha A(x) + \beta B(x) \qquad (\text{for } x \in I_A \cap I_B).$$

Example: Assuming that we know the power-series representation of $f(x) = e^x$ (Calc1p2, V194), derive the representations for $g(x) = e^{-x}$, for the hyperbolic sine and cosine.

Example: Use the result from V147 to compute the sum of the series $\sum\limits_{n=0}^{\infty} (n+3) x^n$ (where $|x| < 1$).

252 Power series changes anyway if we change the centre.

Find power-series representation of the function $f(x) = \dfrac{1}{1-x}$ about $x_0 = -1$. Compare to V235.

Extra material: an article with two solved problems. Find power series representations for the functions:

- ⋆ **Extra problem 1**: $f(x) = \dfrac{1}{x^2 - 3x + 2}$.
- ⋆ **Extra problem 2**: $f(x) = \dfrac{2x}{1 + x - 2x^2}$.

Extra material: notes from the iPad.

253 Cauchy product of power series, an example.

Back to the example from V147 and V221: Compute Cauchy product of $\sum_{n=1}^{\infty} x^{n-1}$ (where $|x| < 1$) with itself.

254 Cauchy product of power series, another example.

Compute Cauchy product of $\sum_{n=0}^{\infty} x^n = A(x)$ with $\sum_{n=0}^{\infty} (-1)^n \cdot x^n = B(x)$ in the common interval of convergence of both series. Verify that $A(x) \cdot B(x) = C(x)$ for all x satisfying $|x| < 1$.

255 The most exciting example of Cauchy product.

Define the following function:

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

(1) Show that the series is absolutely convergent for each $x \in \mathbb{R}$.

(2) $f(0) = 1,\ f(1) = e,\ f(x_1)f(x_2) = f(x_1 + x_2)$ for all $x_1, x_2 \in \mathbb{R}$.

(3) For all $x \in \mathbb{R}$ we have $f(x + x) = [f(x)]^2$.

(4) For all $x \in \mathbb{R}$ and for all $n \in \mathbb{N}^+$ we have $f(nx) = [f(x)]^n$.

(5) For all $x \in \mathbb{R}$ we have $f(x) \neq 0$ and $f(-x) = \dfrac{1}{f(x)}$.

(6) For all $x \in \mathbb{R}$ and for all $k \in \mathbb{Z}$ we have $f(kx) = [f(x)]^k$.

(7) For all $x \in \mathbb{R}$ and for all $n \in \mathbb{N}^+$ we have $f(\frac{1}{n} \cdot x) = [f(x)]^{1/n}$.

(8) For all $x \in \mathbb{R}$ and for all $q \in \mathbb{Q}$ we have $f(qx) = [f(x)]^q$.

(9) In particular, for $x = 1$ and for all $q \in \mathbb{Q}$ we have $f(q) = e^q$.

Following (1) and the Corollary from V247, motivate that this function is continuous on the entire $\mathbb{R}$, and that, as a consequence, $f(x) = e^x$ for all real numbers x.

Compare to the videos from *Precalculus 4: Exponentials and logarithms*: Videos 22–33, V211, and V212.

256 Differentiation and integration of power series term after term.

Back to the example from V255: Define the following function:

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Show that $f'(x) = f(x)$ for all $x \in \mathbb{R}$, assuming that the theorem about differentiation of power series is true.

257 Differentiation of power series term after term, an example.

Back to the example from V147: knowing how to compute the sum of the geometric series with quotient x s.t. $|x| < 1$, compute $\sum_{n=0}^{\infty} (n+1)x^n$.

258 Differentiation and integration of power series term after term, more examples.

Examples: Find power-series representations for the functions:

$$f(x)=\frac{1}{(1-x)^2},\quad f(x)=\frac{1}{(1-x)^3},\quad f(x)=\ln(1+x),\quad f(x)=\arctan x.$$

In the next section you will see that you don't need to rely on the currently presented method. Compare the two last results to the ones presented in V117.

259 Playing with geometric series, Problem 1.

Problem 1: Starting with the power-series representation:

$$\frac{1}{1+x}=\sum_{n=0}^{\infty}x^n=1+x+x^2+x^3+\dots\qquad(-1<x<1)$$

determine power-series representations for the following functions.

(1) $\frac{1}{2-x}$ in powers of x (i.e., with centre at 0)

(2) $\frac{1}{(2-x)^2}$ in powers of x

(3) $\frac{1}{1+2x}$ in powers of x

(4) $\frac{1}{x}$ in powers of $x-1$ (i.e., with centre at 1)

(5) $\frac{1}{x^2}$ in powers of $x+2$ (i.e., with centre at -2)

(6) $\frac{1-x}{1+x}$ in powers of x

(7) $\frac{x^3}{1-2x^2}$ in powers of x.

On what interval is each representation valid?

260 Power series, Problem 2.

Problem 2: Consider the following power series:

$$(x-1)+\frac{2!(x-1)^2}{2^2}+\frac{3!(x-1)^3}{3^3}+\dots+\frac{n!(x-1)^n}{n^n}+\dots$$

Determine its radius of convergence. (Hint: You can get some help from V188.)

Extra material: notes with solved Problem 2.

261 Power series, Problem 3.

Problem 3: Compute Cauchy product of the following series:

$$\ln(x+1)=\sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{x^n}{n},\qquad\frac{1}{x+1}=\sum_{n=0}^{\infty}(-1)^n\cdot x^n.$$

(The sum of the first of them was established in V258 and will be confirmed in Section 15; the second one is a geometric series with quotient $-x$.) Determine its radius of convergence.

Extra material: notes with solved Problem 3.

S15 Taylor series and related topics: a continuation from Calc1p2

You will learn: (a continuation from Section 10 in *Calculus* 1*, part* 2 *of* 2*: Derivatives with applications*) Taylor- and Maclaurin polynomials (and series) of smooth functions; applications to computing limits of indeterminate expressions and to approximating stuff.

Read along with this section: **Calculus book**: Chapter 10.3 *Taylor and Maclaurin Series*, pages: from 1226 (10.3.1) to 1246 (10.3E.7); Chapter 10.4 *Working with Taylor Series*, pages: from 1247 (10.4.1) to 1267 (10.4E.8). **Lecture Notes from UC Davis**: Chapter 8.6 *Taylor's Theorem*, pages: from 156 to 158; Chapter 10 *Power Series*, pages: from 191 to 206; Chapter 12.7 *Taylor's theorem with integral remainder*, pages: from 270 to 271.

262 All roads lead to Taylor series.

263 Taylor's Theorem with proof and some conclusions.

Extra material: an article with Taylor's Theorem and its proof.

264 Finding Taylor- and Maclaurin polynomials.

Extra material: an article with a derivation of the Maclaurin polynomial of degree 5 for $f(x) = \arctan x$.

Extra material: an article with more solved problems on Taylor polynomials. In Problems 1–9 below determine the nth-order Taylor polynomial $p_n(x)$ for the function f about $x = x_0$, where:

- ⋆ **Extra problem 1**: $n = 5,\ \ f(x) = \ln(x+1),\ \ x_0 = 0$; the same for: $g(x) = \ln(1+x^2),\ h(x) = -\ln(1-x)$; a generalisation for *any* positive natural n can be done with help of mathematical induction,
- ⋆ **Extra problem 2**: $n = 3,\ \ f(x) = \ln x,\ \ x_0 = e,$
- ⋆ **Extra problem 3**: $n = 3,\ \ f(x) = \dfrac{1}{(1+x)^2},\ \ x_0 = 0,$
- ⋆ **Extra problem 4**: $n = 3,\ \ f(x) = \sqrt{x},\ \ x_0 = 4,$
- ⋆ **Extra problem 5**: $n = 2,\ \ f(x) = x^2 \cdot \ln x,\ \ x_0 = 1,$
- ⋆ **Extra problem 6**: $n = 2,\ \ f(x) = e^{\sin x},\ \ x_0 = 0,$
- ⋆ **Extra problem 7**: $n = 2,\ \ (x) = x + e^{x^2},\ \ x_0 = 1,$
- ⋆ **Extra problem 8**: $n = 2,\ \ f(x) = \arctan x \cdot e^{2x},\ \ x_0 = 0,$
- ⋆ **Extra problem 9**: $n = 3,\ \ f(x) = e^{x^2} \cdot \ln(1+x),\ \ x_0 = 0,$
- ⋆ **Extra problem 10**: A smooth function f satisfies the following conditions: $f(0) = 0$, $f'(0) = 1$, $f''(0) = 2$ and $f^{(3)}(0) = 2$. Write down the 3rd-order Maclaurin polynomial for f.
- ⋆ **Extra problem 11**: Determine the 3rd-order Maclaurin polynomial $p_3(x)$ for $f(x) = (1+x)^{1/3}$. Use then this polynomial to get the 7th-order Maclaurin polynomial $q_7(x)$ for $g(x) = (x^3 + x^5)^{1/3}$.

265 Smooth and analytic functions.

Extra material: an article with a famous example of function

$$f(x) = \begin{cases} e^{-x^{-2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

that is smooth, but not analytic in any open neighbourhood of zero.

266 New series from old series, some examples.

Examples: Using Maclaurin series from V264, find the power-series representations of the following functions:

$$e^{3x+1},\quad \cos(2x^3),\quad x^2 \cdot \sin\tfrac{x}{3},\quad \sin x \cos x,\quad \arctan(5x^2),\quad \frac{e^{2x^2}-1}{x^2},\quad \ln(2+x^2),\quad \frac{1+x^3}{1+x^2},\quad \arctan(e^x - 1).$$

267 Approximations: some reading recommendations.

Motivate that you can approximate the logarithm of any positive number using Maclaurin series of $\ln \dfrac{1+x}{1-x}$.

268 Monic monomials around zero and the big O.

Example: Show that $\lim_{x\to 0} \dfrac{e^x - 1}{x} = 1$ using Maclaurin series of $g(x) = e^x$ (or, equivalently, the definition of $f(x)$ as in V255). Compare this new proof to the one given in V161 of Calc1p1.

269 Limits of indeterminate expressions, some examples.

Examples: Using Maclaurin series from V264, find the limits at zero of the following indeterminate forms:

$$\frac{\sin x^2}{\sinh x}, \qquad \frac{\sinh x - \sin x}{\cosh x - \cos x}, \qquad \frac{1 - \cos x^2}{(1-\cos x)^2}, \qquad \frac{(x - \arctan x)(e^{2x} - 1)}{2x^2 - 1 + \cos 2x}.$$

270 Limits of indeterminate expressions, more examples.

Examples: Using Maclaurin series from V264, find the limits at zero of the following indeterminate forms:

$$\frac{2\sin 3x - 3\sin 2x}{5x - \arctan 5x}, \qquad \frac{(e^x - 1 - x)^2}{x^2 - \ln(1+x^2)}, \qquad \frac{\sin(\sin x) - x}{x[\cos(\sin x) - 1]}.$$

Extra material: an article with more solved problems with applications of Maclaurin polynomials.

- ⋆ **Extra problem 1**: Determine the 2nd-order Maclaurin polynomial for $f(x) = \cos(x) \cdot \ln(1+x)$.
- ⋆ **Extra problem 2**: Compute the following limits, using Maclaurin polynomials:

$$\lim_{x\to 0} \frac{\ln(1+x^2)}{1-\cos x}, \qquad \lim_{x\to 0} \frac{e^x - x - \cos x}{x^2}, \qquad \lim_{x\to 0+} \left(\frac{1}{x} - \frac{1}{\sin x}\right), \qquad \lim_{x\to\infty} (xe^{\frac{1}{x}} - x + 2e^{\frac{1}{x}}).$$

271 Cherry on top: the sine and the cosine defined with help of series.

Define the following functions:

$$S(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots, \qquad C(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

(1) Show that the series are absolutely convergent for each $x \in \mathbb{R}$. Following Corollary from V247, motivate that these functions are continuous on the entire $\mathbb{R}$.

(2) Function $S(x)$ is odd, function $C(x)$ is even.

(3) For all $x \in \mathbb{R}$ we have $S(x) \cdot C(x) = \frac{1}{2} S(2x)$.

272 Calculus, Advanced Calculus, and Real Analysis.

S16 Extras

You will learn: about all the courses we offer, and where to find discount coupons. You will also get a glimpse into our plans for future courses, with approximate (very hypothetical!) release dates.

B Bonus lecture.

Extra material 1: a pdf with all the links to our courses, and coupon codes.

Extra material 2: a pdf with an advice about optimal order of studying our courses.

Extra material 3: a pdf with information about course books, and how to get more practice.