Calculus 1, part 1 of 2: Limits and continuity[1]

Single variable calculus

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A short table of contents

[1]Recorded May–September 2023. Published on `www.udemy.com` on 2023-09-XX.

An extremely detailed table of contents; the videos (titles in green) are numbered

In blue: problems solved on an iPad (the solving process presented for the students; active problem solving)

In red: solved problems demonstrated during a presentation (a walk-through; passive problem solving)

In magenta: additional problems solved in written articles (added as resources).

In dark blue: *Read along with this section*: references for further reading and more practice problems in the *Calculus* book by Gilbert Strang and Edwin *Jed* Herman:

`https://math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)`

`https://math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)/zz%3A_Back_Matter/30%3A_Detailed_Licensing`

This book is added as a resource to Video 1, with kind permission of the LibreTexts Office (given on July 20th, 2023).

S1 Introduction to the course

You will learn: about the content of this course, and generally about Calculus and its topics.

1 Introduction to the course.

Extra material: this list with all the movies and problems.

Gilbert Strang & Edwin *Jed* Herman: **Calculus**, OpenStax, as described above.

2 What is Calculus and who needs it.

3 The difference between Calculus and Real Analysis; my choices.

4 The greatest names in Calculus.

5 Elementary functions and their superpowers.

6 How we cheated our way through the high school while discussing functions.

7 What we are going to learn in this series Calculus 1.

8 Limits are at the heart and soul of Calculus.

S2 Preliminaries: basic notions and elementary functions

You will learn: you will get a brief recap of the Precalculus stuff you are supposed to master in order to be able to follow Calculus, but you will also get some words of consolation and encouragement, I promise.

Read along with this section: **Calculus book**: Chapter 1 *Functions and graphs*; it gives you a quick repetition of the Precalculus stuff, if you don't have the time for watching the Precalculus series, or reading the *Precalculus book*.

9 How essential is it to master Precalculus before Calculus?

Extra material: *Precalculus*: Carl Stitz, Ph.D., Lakeland Community College; Jeff Zeager, Ph.D., Lorain County Community College; version from July 4, 2013.

10 The essence of Precalculus 1.

Extra material: the list with all the videos and problems in the course *Precalculus 1: Basic notions*.

11 The essence of Precalculus 2.

Example: Possible number of real zeros for polynomials, depending on degree:

a) $ax + b$, $a \neq 0$: always **one** zero,

b) $x^2 + x + 1$: **no** zeros, $x^2 + 2x + 1$: a **double** zero, $x^2 + 2x - 3$: **two** different zeros,

c) $(x-1)(x^2+x+1)$: **one** zero, $(x-1)(x-2)(x-3)$: **three** zeros, $(x-1)^2(x-2)$: **three** zeros, a double and a single, $(x-1)^3$: a **triple** zero,

d) $(x^2+x+1)^2$: **no** zeros, $(x-1)(x-2)(x^2+x+1)$: **two** zeros, $(x-1)^2(x^2+x+1)$: **double** zero, $(x-1)^2(x-3)^2$: **two double** zeros, $(x-1)(x-2)(x-3)(x-4)$: **four** different zeros.

Extra material: the list with all the videos and problems in the course *Precalculus 2: Polynomials and rational functions.*

12 The essence of Precalculus 3.

Extra material: the list with all the videos and problems in the course *Precalculus 3: Trigonometry.*

Extra material: the formula sheet from the course *Precalculus 3: Trigonometry.*

13 The essence of Precalculus 4.

Extra material: the list with all the videos and problems in the course *Precalculus 4: Exponentials and logarithms.*

Extra material: the formula sheet from the course *Precalculus 4: Exponentials and logarithms.*

14 Ask questions, use QA.

S3 Some reflections about the generalising of formulas

You will learn: how to generalise some formulas with or without help of mathematical induction.

15 Introduction to Section 3.

16 Induction: a brief repetition.

17 Generalisations of 3 basic laws (associativity, commutativity, distributivity).

- ∗ Associativity for 4 terms: $a+b+c+d = (a+b+c)+d = a+(b+c+d) = a+(b+c)+d = (a+b)+(c+d)$
- ∗ Commutativity for 3 terms: $a+b+c = a+c+b = b+a+c = b+c+a = c+a+b = c+b+a$
- ∗ Distributivity: $\left(\sum_{i=1}^{n} x_i\right) \cdot \left(\sum_{j=1}^{m} y_j\right) = \sum_{i,j} x_i y_j, \quad \left(\sum_{i=1}^{n} x_i\right) \cdot \left(\sum_{j=1}^{m} y_j\right) \cdot \left(\sum_{k=1}^{p} z_k\right) = \sum_{i,j,k} x_i y_j z_k.$

18 Some examples, before we get to the general method; Example 1.

Example 1: Let $a > 0$. We know from Precalculus 4 (V33), that

$$\forall\, x, y \in \mathbb{R} \quad a^{x+y} = a^x \cdot a^y.$$

Show that this formula can be generalised to:

$$\forall\, n \in \mathbb{N}^+ \quad \forall\, x_1, \ldots, x_n \in \mathbb{R} \quad a^{x_1+\cdots+x_n} = a^{x_1} \cdot \ldots \cdot a^{x_n}.$$

(The proof for $n = 3$ was shown in V146 in Precalculus 4; the induction step is based on the same idea.)

Note: The power rule $(a^n)^m$ for $n, m \in \mathbb{N}^+$ follows from the generalization of the product rule $a^n \cdot a^m = a^{n+m}$ (or $f(n) \cdot f(m) = f(n+m)$ if we denote $f(x) = a^x$) proven above. Also the quotient rule $\frac{a^n}{a^m} = a^{n-m}$ with $n > m$ is a special case of this product rule. This means that all the properties of exponential functions can be derived from the rule $f(x+y) = f(x)f(y)$ combined with the fact that the function is not constant, and that it is defined and continuous on the entire $\mathbb{R}$. (Compare to the result proven in Video 212 in Precalculus 4.)

Extra material: notes from the iPad, with the proof of Example 1.

19 Some examples, before we get to the general method; Example 2.

Example 2: Let $a > 0$ and $a \neq 1$. We know from Precalculus 4 (V118), that

$$\forall\, x, y \in \mathbb{R}^+ \quad \log_a(xy) = \log_a x + \log_a y.$$

Show that this formula can be generalised to:

$$\forall\, n \in \mathbb{N}^+ \quad \forall\, x_1, \ldots, x_n \in \mathbb{R}^+ \quad \log_a(x_1 \cdot \ldots \cdot x_n) = \log_a x_1 + \cdots + \log_a x_n.$$

(The proof for $n = 3$ was shown in V136 in Precalculus 4; the induction step is based on the same idea.)

Extra material: notes from the iPad, with the proof of Example 2.

20 Some examples, before we get to the general method; Example 3.

Example 3 (de Morgan's laws): We know from Precalculus 1 (V118), that for all pairs of statements p and q (regardless their logical value):

$$\neg(p \vee q) \;\Leftrightarrow\; \neg p \wedge \neg q,$$
$$\neg(p \wedge q) \;\Leftrightarrow\; \neg p \vee \neg q.$$

Show that these formulas can be generalised to any number of statements:

$$\forall\, n \in \mathbb{N}^+ \quad \forall\, p_1, \ldots, p_n \quad \neg(p_1 \vee \cdots \vee p_n) \;\Leftrightarrow\; \neg p_1 \wedge \cdots \wedge \neg p_n,$$
$$\forall\, n \in \mathbb{N}^+ \quad \forall\, p_1, \ldots, p_n \quad \neg(p_1 \wedge \cdots \wedge p_n) \;\Leftrightarrow\; \neg p_1 \vee \cdots \vee \neg p_n.$$

(I didn't show the generalisation in Precalculus 1, but I got a question about it; both question and my answer are under V119 in Precalculus 1. Both laws can be also formulated and generalised for sets and their unions and intersections.)

Extra material: notes from the iPad, with the proof of Example 3.

21 Types of formulas which expand easily because of the associativity of operations.

The General Method: If X is a set of elements with two binary operations: $\star$, $\diamond : X \times X \to X$ which are **associative**, and $f : X \to X$ is a function defined on X, and, moreover:

$$\forall\, x, y \in X \quad f(x \star y) = f(x) \diamond f(y),$$

then we can generalise the formula in the following way:

$$\forall\, n \in \mathbb{N}^+ \quad \forall\, x_1, \ldots, x_n \in X \quad f(x_1 \star \ldots \star x_n) = f(x_1) \diamond \ldots \diamond f(x_n).$$

Extra material: notes from the iPad, with the proof of The General Method.

22 An exercise: applying The General Method.

Exercise: Show how the formulas from Videos 18–20 match The General Method.

Extra material: notes from the iPad.

23 **Optional**: Linear transformations between vector spaces.

In Linear Algebra, function $T : V \to W$ (where V and W are so called *linear spaces* or *vector spaces*; think of them as sets of elements which can be added to each other in such a way that this addition is both commutative and associative) is called *additive* if

$$\forall\, x, y \in V \quad T(x + y) = T(x) + T(y)$$

(we say that T *preserves addition*). Show that for this type of functions we have:

$$\forall\, n \in \mathbb{N}^+ \quad \forall\, x_1, \ldots, x_n \in V \quad T(x_1 + \cdots + x_n) = T(x_1) + \cdots + T(x_n).$$

Extra material: notes from the iPad.

24 **Optional, Future**: Limits, differentiation, and other linear operations.

25 A word about the sigma symbol and The Binomial Theorem.

26 Squaring the sums.

Squaring-the-sum Formula: For each $n \in \mathbb{N}^+ \setminus \{1\}$ and each $a_1, a_2, \ldots, a_n \in \mathbb{R}$:

$$\left(\sum_{i=1}^{n} a_i\right)^2 = \sum_{i=1}^{n} a_i^2 + 2\sum_{j=2}^{n}\sum_{i=1}^{j-1} a_i a_j.$$

Extra material: notes from the iPad.

27 Less obvious, but still possible: when the LHS is a function of a sum or of a product.

28 **Future**: The derivative of a product formula.

Product Formula for Derivatives: The following formula holds for differentiable functions: $(fg)' = f'g + fg'$. (This formula will be proven in Calculus 1, part 2 of 2.) Generalise this formula for more than two functions.

Extra material: notes from the iPad.

29 Two trigonometric formulas to remember; a complex-numbers trick.

A trick: Knowing that multiplication of two complex numbers by each other gives a new complex number, whose module is equal to the product of the moduli of the factors, and whose argument is equal to the sum of the arguments of the factors, re-create the formulas for the cosine and the sine of the sum of two angles.

Extra material: notes from the iPad.

30 Generalisations of the formulas to remember for the sum of more arguments.

Exercise: Using the sum identities

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta, \quad \sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha,$$

derive the formulas for $\cos(\alpha + \beta + \gamma)$ and $\sin(\alpha + \beta + \gamma)$.

Extra material: notes from the iPad.

31 How to get more trigonometric formulas for free.

Exercise: Using the sum identities $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ and $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$, derive the Pythagorean Identity and the formulas for $\sin 2\alpha$, $\cos 2\alpha$, $\cos(\alpha - \beta)$, $\sin(\alpha - \beta)$, $\sin x + \sin y$, etc.

Extra material: notes from the iPad.

32 Derivation versus proof by induction; Formula 1.

Formula 1: Derive the following formula for the sum of first n positive natural numbers, and show a geometrical illustration of this formula:

$$S_n = \sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

33 Derivation versus proof by induction; Formula 2.

Formula 2: Prove (and derive) the following formula for $n \in \mathbb{N}^+$:

$$S_n^{(2)} = \sum_{k=1}^{n} k^2 = 1 + 4 + 9 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Extra material: notes from the iPad with proof of Formula 2 by induction.

Extra material: an article notes from the iPad with a derivation of Formula 2.

34 **Future**: Riemann integral and area under the graph.

35 Derivation versus proof by induction; Formula 3.

Formula 3: Prove the following formula for $n \in \mathbb{N}^+$:

$$S_n^{(3)} = \sum_{k=1}^{n} k^3 = 1 + 8 + 27 + 64 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Extra material: notes from the iPad with proof of Formula 3 by induction.

36 **Extremely important**: a generalisation of the triangle inequality.

Triangle Inequality: We know from Precalculus 1 (V34 and V38), that

$$\forall\, x, y \in \mathbb{R} \quad |x + y| \leqslant |x| + |y|.$$

Show that this inequality can be generalised to:

$$\forall\, n \in \mathbb{N}^+ \quad \forall\, x_1, \ldots, x_n \in \mathbb{R} \quad |x_1 + \cdots + x_n| \leqslant |x_1| + \cdots + |x_n|.$$

(The proof could be repeated for any metric space (X, d). Here I will also show an illustration completing the proof of the triangle inequality for two arguments in $\mathbb{R}$ with the absolute value.)

Extra material: notes from the iPad, with the proof of a general triangle inequality.

37 Bernoulli's inequality.

Bernoulli's inequality: For each $n \in \mathbb{N}^+$ and for each $d > -1$: $(1 + d)^n \geqslant 1 + nd$.

Extra material: notes from the iPad with a proof of the inequality.

38 An interesting lemma for the future (for the proof of Stolz–Cesàro Theorem).

Lemma: Let $A, B \in \mathbb{R}$; $m \in \mathbb{N}^+$; $k_i \in \mathbb{R}$ for $i = 1, \ldots, m$; $n_i > 0$ for $i = 1, \ldots, m$. If

$$\forall\ 1 \leqslant i \leqslant m \quad A \leqslant \frac{k_i}{n_i} \leqslant B,$$

then

$$A \leqslant \frac{k_1 + \cdots + k_m}{n_1 + \cdots + n_m} \leqslant B.$$

Extra material: notes from the iPad with a proof of the lemma.

S4 The nature of the set of real numbers

You will learn: about the structure and properties of the set of real numbers as an ordered field with the Axiom of Completeness, and consequences of this definition.

39 How to use this section; good news first!

40 The theory of real numbers justifies the practical consequences.

An article with the following theory, which will be discussed in this section and in the later sections:

a) Definitions: upper bound / supremum / maximum; lower bound / infimum / minimum. [S4]
b) Theorem 1: A statement equivalent with the Axiom of Completeness. [S4]
c) Theorem 2 (**Weierstrass**): Convergence of a monotone sequence of real numbers. [S5]
d) Lemma: About separation. [S8 and S9]
e) Theorem 3: The Boundedness Theorem. [S9]
f) Theorem 4: The Max-Min Theorem. [S9]
g) Theorem 5 (**Darboux**): The Intermediate-Value Theorem (IVT). [S9]

41 Where in the Precalculus series you find information about real numbers.

42 Algebra, orders, and completeness.

43 About deriving rules directly from the axioms.

44 **Optional**: Deriving rules directly from the axioms; addition.

Prove directly from the axioms, and from previously proven facts:

1. **Uniqueness of zero**. There exists exactly one zero, i.e., such number a that for each $x \in \mathbb{R}$ we have $x + a = x$. (This unique number is denoted 0.) [A2, A4]
2. **Uniqueness of additive inverse**. Each real number x has exactly one additive inverse (opposite), i.e., such a number y that $x + y = 0$. (This unique number is denoted $-x$.) [A2, A3, A4, A5]
3. The previous property allows us define the **difference** between two real numbers: For each pair of real numbers x, y there exists exactly one real number d such that $x + d = y$. (This unique number is denoted $y - x$.) [A2, A3, A4, 2]

4. **Cancellation**. If $x+y=x+z$ then $y=z$. [3]; In Precalculus 1 it was proven in a different way in Video 213 (Property 1: proven by applying A2, A3, A4, A5).
5. **Double minus makes plus**. For each real number x we have $-(-x)=x$. [2]
6. **Opposite of the sum is the sum of the opposites**. For each pair of real numbers x, y we have $-(x+y)=(-x)+(-y)$. [2]

Extra material: notes from the iPad.

45 **Optional**: Deriving rules directly from the axioms; multiplication.

Prove directly from the axioms, and from previously proven facts:

7. **Uniqueness of 1**. There exists exactly one neutral element of multiplication, i.e., such number a that for each $x \in \mathbb{R}$ we have $xa=x$. (This unique number is denoted 1.) [M2, M4]
8. **Uniqueness of multiplicative inverse**. Each real number $x \neq 0$ has exactly one multiplicative inverse (reciprocal), i.e., such a number y that $xy=1$. (This number is denoted $x^{-1}=\frac{1}{x}$.) [M2, M3, M4, M5]
9. The previous property allows us define the **quotient** between two real numbers: For each pair of real numbers $x \neq 0$ and y there exists exactly one real number q such that $xq=y$. (This unique number is denoted $\frac{y}{x}$.) [M2, M3, M4, 8]
10. **Cancellation**. If $x \neq 0$ and $xy=xz$ then $y=z$. [9]

10.1 For each real number x we have $x \cdot 0=0$. [A2, A4, M4, D, 3]

10.2 **Zero-Product Property**. If $xy=0$ then $x=0$ or $y=0$. [9, 10.1]; In Precalculus 1 it was proven in a different way in Video 214.

10.3 If $x \neq 0$ then also $x^{-1} \neq 0$. [8, 10.1, M4]

11. **Taking the reciprocal twice gives the original number**. For each real number $x \neq 0$ we have $(x^{-1})^{-1}=x$. [9]
12. **Reciprocal of the product is the product of the reciprocals**. For each pair of real numbers $x, y \neq 0$ we have $(xy)^{-1}=x^{-1}y^{-1}$. [9]

Extra material: notes from the iPad.

46 **Optional**: Deriving rules directly from the axioms; multiplication (continued).

Prove directly from the axioms, and from previously proven facts:

13. For each pair of real numbers x, y we have $(-x)y=x(-y)=-(xy)$. [2, 10.1, D, M2]
14. For each pair of real numbers x, y we have $(-x)(-y)=xy$. [5, 13]
15. For each real number x we have $-x=(-1)x$. [M4, M2, 13]

Extra material: notes from the iPad.

47 **Optional**: Deriving rules directly from the axioms; inequalities.

Prove directly from the axioms, and from previously proven facts:

16. **Adding inequalities side-wise**: If $x_1<x_2$ and $y_1<y_2$ then $x_1+y_1<x_2+y_2$. [O2, O3, A2]
17. **Opposite to negative is positive**: If $x<0$ then $0<-x$. (Also: If $0<x$ then $-x<0$.) [16, 2]
18. **Plus times plus gives plus**: If $0<x$ and $0<y$ then $0<xy$. [O4, 10.1]
19. **Plus times minus gives minus**: If $0<x$ and $y<0$ then $xy<0$. [O4, 10.1]
20. **Minus times minus gives plus**: If $x<0$ and $y<0$ then $0<xy$. [14, 17, 18]
21. **The square of a non-zero number is positive**: If $x \neq 0$ then $0<x^2$. [18, 20]
22. **One is greater than zero**: $0<1$ (Also: $-1<0$). [M4, 21, 17]
23. **Multiplying inequalities (between positive numbers) side-wise**: If $0<x_1<x_2$ and $0<y_1<y_2$ then $x_1y_1<x_2y_2$. [O2, O4, M2]
24. **Inverse to positive is positive**: If $0<x$ then $0<x^{-1}$. (Also: If $x<0$ then $x^{-1}<0$.) [10.3, 19, 22]
25. **Inverses on opposite sides of 1**: If $1<x$ then $0<x^{-1}<1$. (If $0<x<1$ then $1<x^{-1}$.) [8, 22]

Extra material: notes from the iPad.

48 **Optional**: Some examples of fields: $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{C}$, $\mathbb{Z}_p$.

Some examples of fields:

- ⋆ **Example 0**: The set of real numbers $\mathbb{R}$ is an ordered and complete field.
- ⋆ **Example 1**: The set of rational numbers $\mathbb{Q}$ is an ordered field, but **not** a complete field.
- ⋆ **Example 2**: The set of complex numbers $\mathbb{C}$ is **not** an ordered field, but it is a complete field.
- ⋆ **Example 3**: The set $\{0, 1, \ldots, p-1\}$ (where p is a prime number) with addition and multiplication modulo p is a field, but it is **not** ordered.

49 **Just for fun**: There are plenty of ordered number fields between $\mathbb{Q}$ and $\mathbb{R}$.

Example 4: Choose a positive natural number d which is **not** a square of a natural number (ex: 2, 3, 5, 6, 7, 8, 10,...). The following set

$$\{\alpha + \beta\sqrt{d};\ \alpha, \beta \in \mathbb{Q}\}$$

with addition and multiplication as in $\mathbb{R}$ is an ordered number field $\mathbb{F}_d$ such that $\mathbb{Q} \subset \mathbb{F}_d \subset \mathbb{R}$.

Extra material: notes with solved Example 4.

50 Absolute value, distances, and Triangle Inequality.

51 Some definitions with examples: supremum, infimum, maximum, minimum.

52 Axiom of Completeness and its reformulation in terms of supremums.

53 **Hardcore**: Existence of roots.

Extra material: An article with some theorems and proofs:

a) Theorem: Existence of roots. Each equation $x^n = a$ (where $a \geqslant 0$ and $n \in \mathbb{N}^+$) has a solution in $\mathbb{R}$.
b) Corollary 1: The equation $x^n = a$ (with $a > 0$ and $n \in \mathbb{N}^+$, $n \geqslant 2$) has exactly one positive solution. This solution is called the nth root of a.
c) Corollary 2: If the equation $x^n = a$ (with $a > 0$ and $n \in \mathbb{N}^+$, $n \geqslant 2$) has a negative solution, then this negative solution is unique, too.
d) Corollary 3: If n is odd, then the equation $x^n = a$ has exactly one solution.
e) Corollary 4: If n is even, then the equation $x^n = a$ has exactly two solutions: c and $-c$.
f) Corollary 5: If n is odd (and only then), then the Theorem also works if $a < 0$.

54 A word about Peano axioms defining the set of natural numbers.

55 The Minimum Principle for natural numbers.

The Minimum Principle: Each non-empty subset of $\mathbb{N}$ contains a least element (minimum).

Extra material: notes from the iPad.

56 A word about construction of integer, rational, and real numbers.

Show (in two different ways) that the set $L = \{x \in \mathbb{Q};\ \ x^2 < 2 \ \vee \ x < 0\}$ has no greatest element.

57 The floor function, or the greatest integer function.

The greatest integer: For each real number x there is a unique integer N such that $N \leqslant x < N + 1$. This unique number is denoted $N = [x]$ or $N = \lfloor x \rfloor$ (different in different sources, books, countries).

Extra material: notes with a proof of existence and uniqueness of *the greatest integer* (*floor function*).

58 Three equivalent properties following from the Axiom of Completeness.

The following three properties are equivalent, and they follow from the Axiom of (Dedekind) Completeness:

1. The set $\mathbb{N}$ is not bounded above;
2. The Archimedean Principle;
3. The density of $\mathbb{Q}$ in $\mathbb{R}$.

59 Density of $\mathbb{Q}$ in $\mathbb{R}$ and why we need to know about it.

60 **Hardcore**: There is just one set of real numbers.

Lemma: If $\mathbb{K}$ is an Archimedean field, then each element $x \in \mathbb{K}$ can be perfectly approximated (below) by rational numbers, i.e.:

$$x = \sup\{q \in \mathbb{Q};\ \ q < x\}.$$

Theorem: There exists, up to isomorphism, exactly one ordered and complete number field.

61 Supremum, infimum, etc; Example 1.

Example 1: Find minimum, infimum, supremum, maximum of the following sets: $[0,1]$, $(0,1)$, $[0,1)$, $(0,1]$.

Extra material: notes with solved Example 1.

62 Supremum, infimum, etc; Example 2.

Example 2: Find minimum, infimum, supremum, maximum of the following sets:

a) $\left\{\frac{1}{n};\quad n \in \mathbb{N}^+\right\}$

b) $\left\{(-1)^n\frac{1}{n};\quad n \in \mathbb{N}^+\right\}$

c) $\left\{1+(-1)^n\frac{1}{n};\quad n \in \mathbb{N}^+\right\}$

d) $\left\{\frac{n+1}{n};\quad n \in \mathbb{N}^+\right\}$

e) $\left\{\frac{4n}{n+1};\quad n \in \mathbb{N}^+\right\}$

f) $\left\{\frac{3n+2}{2n+3};\quad n \in \mathbb{N}^+\right\}$

g) $\left\{(-1)^n+\frac{1}{n};\quad n \in \mathbb{N}^+\right\}$

h) $\left\{(-1)^n+(-1)^{n+1}\frac{1}{n};\quad n \in \mathbb{N}^+\right\}$.

Extra material: notes with solved Example 2.

63 A preparation for some subtleties in the definitions of limits and continuity.

64 The magical power of leading themes.

65 Accumulation points (cluster points) and isolated points; derived sets.

Example 3: For the following sets A, show that the derived set is A':

a) $A = \left\{\frac{1}{n};\quad n \in \mathbb{N}^+\right\}, \quad A' = \{0\}$

b) $A = \left\{(-1)^n\frac{1}{n};\quad n \in \mathbb{N}^+\right\}, \quad A' = \{0\}$

c) $A = \left\{1+(-1)^n\frac{1}{n};\quad n \in \mathbb{N}^+\right\}, \quad A' = \{1\}$

d) $A = \left\{\frac{n+1}{n};\quad n \in \mathbb{N}^+\right\}, \quad A' = \{1\}$

e) $A = \left\{\frac{4n}{n+1};\quad n \in \mathbb{N}^+\right\}, \quad A' = \{4\}$

f) $A = \left\{\frac{3n+2}{2n+3};\quad n \in \mathbb{N}^+\right\}, \quad A' = \{\frac{3}{2}\}$

g) $A = \left\{ (-1)^n + \frac{1}{n}; \quad n \in \mathbb{N}^+ \right\}, \quad A' = \{-1, 1\}$

h) $A = \left\{ (-1)^n + (-1)^{n+1}\frac{1}{n}; \quad n \in \mathbb{N}^+ \right\}, \quad A' = \{-1, 1\}$

i) $A_1 = [0, 1],\ A_2 = (0, 1),\ A_3 = [0, 1),\ A_4 = (0, 1], \quad A'_k = [0, 1]$.

Show that the sets described in a)–h) are *discrete*, i.e., contain only *isolated points*.

Extra material: notes with solved (some parts of) Example 3.

66 Your first encounter with our leading theme.

Our leading theme: Given the set:

$$D_f = (-2, -1.5) \cup (-1.5, -1] \cup \{1\} \cup (2, 2.5) \cup (2.5, 3].$$

Which real numbers are accumulation points of D_f? Which real numbers are isolated points of D_f? Determine the derived set D'_f.

67 Various relations between the concepts of supremum, infimum, accumulation point.

Exercise: Answer the following questions using examples (or counterexamples) from our earlier videos:

a) Is the supremum (infimum) of a set automatically also its accumulation point?
b) Is each accumulation point of a set automatically also its supremum or infimum?
c) True or false: If supremum (infimum) of A does **not** belong to A, it must be an accumulation point of A.
d) Must all accumulation points of a set belong to this set?
e) If an accumulation point of A belongs to A, does it need to be a supremum or infimum of A?
f) Must all isolated points of a set belong to this set?
g) Must infimum/supremum of a set belong to this set?
h) Can an isolated point be an accumulation point?
i) Can an isolated point be a supremum (infimum)?

Extra material: notes from the iPad.

S5 Sequences and their limits

You will learn: the concept of a number sequence, with many examples and illustrations; subsequences, monotone sequences, bounded sequences; the definition of a limit (both proper and improper) of a number sequence, with many examples and illustrations; arithmetic operations on sequences and The Limit Laws for Sequences; accumulation points of sequences; the concept of continuity of arithmetic operations, and how The Limit Laws for Sequences will serve later in Calculus for computing limits of functions and for proving continuity of elementary functions; Squeeze Theorem for Sequences; Weierstrass' Theorem about convergence of monotone and bounded sequences; extended reals and their arithmetic; determinate and indeterminate forms and their importance; some first insights into comparing infinities (*Standard Limits in the Infinity*); a word about limits of sequences in metric spaces; Cauchy sequences (fundamental sequences) and a sketch of the construction of the set of real numbers using an equivalence relation on the set of all Cauchy sequences with rational elements.

Read along with this section: **Calculus book**: Chapter 9 *Sequences and Series*; note that the authors work with functions (both limits and continuity) **before** they discuss sequences, so they use some methods that are not (yet) available for us, for example l'Hôpital's Rule. We will come back to these examples in *Calculus* 2*, part* 2 *of* 2*: Sequences and series*; also series will be left for this later course. Here we cover just enough about sequences for a good understanding of continuous functions.

68 Sequences in Precalculus 1.

69 More reasons (than given in V8) to study sequences now.

70 What is a sequence? Notation and terminology.

Example: Why notation matters: $((-1)^n)_{n=1}^{\infty}$ versus $\{(-1)^n;\ \ n \in \mathbb{N}^+\}$.

71 Sequences as functions; various ways of defining sequences.

- By picture.
- Explicit, by a (closed) formula. $a_n = \frac{1}{n}$.
- By writing down **enough** terms, with or without a table. $2,\ 5,\ 8,\ 11,\ 14,\ 17,\ \ldots$; $1,\ 2,\ 4,\ \ldots$.
- By a recursive description, i.e. showing how to get the next elements from the previous one(s). $a_1 = 2,\ a_{n+1} = 3a_n$ for $n \in \mathbb{N}^+$.
- By verbal description. a_n is the nth prime number; b_n is the nth digit in the decimal expansion of π.

Extra material: notes from the iPad.

72 Exercise 1: Reading formulas (The one with MANIM for multiple purposes).

Exercise 1: Write eleven first elements of the following sequences:

a) $a_n = \dfrac{1}{n}$

b) $b_n = (-1)^n \dfrac{1}{n}$

c) $c_n = 1 + (-1)^n \dfrac{1}{n}$

d) $d_n = \dfrac{n+1}{n}$

e) $e_n = \dfrac{4n}{n+1}$

f) $f_n = \dfrac{3n+2}{2n+3}$

g) $g_n = (-1)^n + \dfrac{1}{n}$

h) $h_n = (-1)^n + (-1)^{n+1} \dfrac{1}{n}$

i) $i_n = (-1)^n$.

Use the formulas above to show that $c_{2k} = g_{2k},\ \ c_{2k+1} = -g_{2k+1}$, and $g_{2k} \cdot g_{2k+1} = -1$ for $k \in \mathbb{N}^+$.

Extra material: notes with solved (some parts of) Exercise 1.

73 Exercise 2: Finding formulas.

Exercise 2: Find the formula for a_n if the first elements are given by:

- $2,\ \frac{3}{2},\ \frac{4}{3},\ \frac{5}{4},\ \frac{6}{5},\ \frac{7}{6},\ \frac{8}{7},\ \frac{9}{8},\ \frac{10}{9},\ \frac{11}{10},\ \frac{12}{11},\ \ldots$
- $0,\ \frac{3}{2},\ \frac{2}{3},\ \frac{5}{4},\ \frac{4}{5},\ \frac{7}{6},\ \frac{6}{7},\ \frac{9}{8},\ \frac{8}{9},\ \frac{11}{10},\ \frac{10}{11},\ \ldots$
- $1,\ 9,\ 25,\ 49,\ 81,\ 121,\ \ldots$

Extra material: notes with solved Exercise 2.

74 Exercise 3: Guess and prove.

Exercise 3: Find the formula for a_n if $a_0 = 1$ and $a_{n+1} = 2a_n$ for $n = 0, 1, 2, \ldots$

Extra material: notes with solved Exercise 3.

75 Exercise 4: Guess and prove.

Exercise 4: Find the formula for a_n if $a_1 = 3,\ a_2 = 7$ and $a_n = 3a_{n-1} - 2a_{n-2}$ for $n = 3, 4, 5, \ldots$

Extra material: notes with solved Exercise 4.

76 Problem 1: Simplify the formula.

Problem 1: Simplify the following formula for $n \in \mathbb{N}^+$, and write down the values of S_n for $1 \leqslant n \leqslant 5$:

$$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)}.$$

Extra material: notes with solved Problem 1.

77 **Optional**: Problem 2: Recursive to explicit (or: closed) formula.

Problem 2: The Fibonacci sequence is defined as: $F_1 = 0,\ F_2 = 1,\ F_{n+2} = F_n + F_{n+1}$ for $n \in \mathbb{N}^+$. Show that

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{n-1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n-1} \right].$$

Extra material: notes with solved Problem 2.

78 Number sequences versus functions defined for all positive arguments.

- Linear function on $\mathbb{R}^+ \cup \{0\}$. $f(x) = mx + b \ \ (x \geqslant 0); \quad a_n = a + dn.$
- Exponential function on $\mathbb{R}^+ \cup \{0\}$. $f(x) = a^x \ \ (x \geqslant 0,\ a > 1); \quad a_n = q^n.$
- Exponential function on $\mathbb{R}^+ \cup \{0\}$. $f(x) = a^x \ \ (x \geqslant 0,\ 0 < a < 1); \quad a_n = q^n.$
- Power functions on $\mathbb{R}^+$. $f(x) = x^\alpha \ \ (x > 0); \quad a_n = \frac{1}{n},\ \ b_n = \sqrt{n}.$
- (Some) rational functions. $f(x) = \dfrac{3x+2}{2x+3} \ \ (x > 0); \quad a_n = \dfrac{3n+2}{2n+3}.$
- Not always possible in the other direction (from a sequence to a function on $\mathbb{R}^+$). a_n is the nth prime number; b_n is the nth digit in the decimal expansion of π; $c_n = (-1)^n$.

79 Bounded sequences.

- All the sequences from V72 are bounded.
- a_n is the nth prime number is bounded below, but unbounded above.
- b_n is the nth digit in the decimal expansion of π is bounded.
- $a_n = n$ is bounded below, but unbounded above.
- $a_n = (-1)^n n$ is unbounded below, and unbounded above.

80 Monotone sequences versus monotone functions.

Which sequences from V72 are monotone? Verify that $f_{n+1} > f_n$ for each $n \in \mathbb{N}^+$.

You get a reference slide (slide 25) with a complete summary of monotonic behaviour of geometric sequences depending on the first element and the quotient. We come back to this example in the second part of *Calculus* 2.

If $f : \mathbb{R}^+ \to \mathbb{R}$ is defined for all positive real numbers and monotone (or: bounded), then the corresponding sequence $a_n = f(n)$ has the same property:

- arithmetic progressions
- geometric progressions
- sequences defined by power functions.

Converse is not true, though; this can be shown with the following examples:

- $a_n = \sin(n\pi)$ is constant (i.e., monotone), while $f(x) = \sin(\pi x)$ is not
- $a_n = n + \sin(n\pi)$ is strictly increasing, while $f(x) = x + \sin(\pi x)$ is not
- $a_n = \tan(n\pi)$ is constant (i.e., bounded), while $f(x) = \tan(\pi x)$ is not (maybe not really fair, because f is not defined on entire $\mathbb{R}^+$).

Extra material: notes from the iPad.

81 Arithmetic operations on sequences.

- Scaling of a sequence. $k = 2, \;\; a_n = \dfrac{1}{n}, \;\; m_n = ka_n = \dfrac{2}{n}.$
- Sum / difference of two sequences. $b_n = (-1)^n \dfrac{1}{n}, \;\; d_n = \dfrac{n+1}{n}, \;\; m_n = b_n + d_n.$
- Product of two sequences. $e_n = \dfrac{4n}{n+1}, \;\; f_n = \dfrac{3n+2}{2n+3}, \;\; m_n = e_n f_n.$
- Quotient of two sequences. $a_n = \dfrac{1}{n}, \;\; i_n = (-1)^n, \;\; m_n = \dfrac{i_n}{a_n}.$
- Absolute value of a sequence. $h_n = (-1)^n + (-1)^{n+1}\dfrac{1}{n}, \;\; m_n = |h_n|.$

82 Two ways of depicting sequences; prelude to convergence.

Example: Illustrate the following sequences as functions (see MANIM in V72):

a) $a_n = \dfrac{1}{n}$

b) $b_n = (-1)^n \dfrac{1}{n}$

c) $c_n = 1 + (-1)^n \dfrac{1}{n}$

d) $d_n = \dfrac{n+1}{n}$

e) $e_n = \dfrac{4n}{n+1}$

f) $f_n = \dfrac{3n+2}{2n+3}$

g) $g_n = (-1)^n + \dfrac{1}{n}$

h) $h_n = (-1)^n + (-1)^{n+1}\dfrac{1}{n}$

i) $i_n = (-1)^n$.

We look at the accumulation points in a different way.

83 Playing with symbols.

Let $(a_n)_{n=1}^{\infty}$ be a number sequence, and let L be a real number. Write with help of logical symbols:

1. (Fix some $\varepsilon > 0$.) Infinitely many elements of $(a_n)_{n=1}^{\infty}$ are *inside* the ε-neighbourhood of L.
2. Each ε-neighbourhood of L contains infinitely many elements of $(a_n)_{n=1}^{\infty}$.
3. (Fix some $\varepsilon > 0$.) *Almost all* elements of $(a_n)_{n=1}^{\infty}$ are *inside* the ε-neighbourhood of L. (Meaning: Only a finite number of elements of $(a_n)_{n=1}^{\infty}$ are outside the ε-neighbourhood of L.)
4. Each ε-neighbourhood of L contains *almost all* elements of $(a_n)_{n=1}^{\infty}$.

Extra material: notes from the iPad.

84 Accumulation points of sequences.

Example: The sequence $((-1)^n)_{n=1}^{\infty}$ and the set $\{(-1)^n; \;\; n \in \mathbb{N}^+\}$ don't have the same accumulation points.

85 Accumulation points of sequences, Exercise 5.

Exercise 5: Construct a sequence that has:

1. one accumulation point,
2. two accumulation points,
3. three accumulation points,

4. each positive natural number as accumulation point,
5. each real number as accumulation point.

Extra material: notes from the iPad.

86 What is a subsequence? Some examples.

87 Limit of a sequence, definition and notation.

Example 1: All constant sequences. i.e., $a_n = c$ for all $n \in \mathbb{N}^+$, are *convergent*, and $\lim\limits_{n\to\infty} a_n = c$.

Example 2: The sequence $a_n = n$ is divergent.

88 It doesn't matter what happens with the first m elements.

Example: Let $N \geqslant 2$ be any natural number. Sequence $a_n = c$ for all $n \geqslant N$, and *any* values of a_n for $1 \leqslant n \leqslant N-1$, is *convergent*, and $\lim\limits_{n\to\infty} a_n = c$.

89 Accumulation points are limits of subsequences.

90 If a sequence is convergent then it has exactly one accumulation point.

Uniqueness of limit (or: limit as the only accumulation point): If $\lim\limits_{n\to\infty} a_n = L$, then L is the only accumulation point of (a_n). In particular, the limit is unique.(This can be used for proving that some sequence does not have a limit: if we can indicate its two subsequences with different limits.)

Example 1: The converse $(q \Rightarrow p)$ is **not** true, as shown by the sequence:

$$1,\ 1,\ 2,\ \tfrac{1}{2},\ 3,\ \tfrac{1}{3},\ 4,\ \tfrac{1}{4},\ 5,\ \tfrac{1}{5},\ 6,\ \tfrac{1}{6},\ 7,\ \tfrac{1}{7},\ 8,\ \tfrac{1}{8},\ \ldots$$

The following is true, though: If a sequence is **bounded**, and it has exactly one accumulation point, then it is convergent. This is a consequence of Bolzano–Weierstrass' Theorem, which will be discussed in *Calculus 2, part 2 of 2: Sequences and series.*

Example 2: Sequence $i_n = (-1)^n$ is *divergent*, as $\lim\limits_{n\to\infty} i_{2n} = 1$ and $\lim\limits_{n\to\infty} i_{2n-1} = -1$.

Extra material: notes from the iPad.

91 Limit of a sequence; playing with epsilons.

Example: Prove directly from the definition that $\lim\limits_{n\to\infty} \frac{1}{n} = 0$. Use the following values of ε: $2,\ 1,\ \frac{1}{2},\ \frac{3}{10},\ \frac{7}{50},\ \frac{11}{100}$, and, finally, *any* ε. In each case, establish the least possible n_ε, and make an appropriate illustration.

92 Computing limits from the definition, Exercise 6.

Exercise 6: Prove directly from the definition that

$$\lim_{n\to\infty} \frac{2n+1}{3n+2} = \frac{2}{3}.$$

Extra material: notes with solved Exercise 6.

93 Computing limits from the definition, Exercise 7.

Exercise 7: Prove directly from the definition that

$$\lim_{n\to\infty} \frac{2n^2+n}{3n^2+2} = \frac{2}{3}.$$

Extra material: notes with solved Exercise 7.

94 Computing limits from the definition, Exercise 8.

Exercise 8: Prove directly from the definition that

$$\lim_{n\to\infty} (2+3^{-n}) = 2.$$

Extra material: notes with solved Exercise 8.

95 Why you need proofs with epsilons; some preparations for them.

The following rules can be good to keep in mind:

R1. **Triangle inequality**: $|x+y| \leqslant |x| + |y|$ for all $x, y \in \mathbb{R}$. (Proven in V36.)

R2. $||x| - |y|| \leqslant |x - y|$ for all $x, y \in \mathbb{R}$.

R3. Let $x, y \in \mathbb{R}$ and $c > 0$. Then $|x - y| < c \ \Rightarrow \ |x| < c + |y|$.

R4. The trick of adding and subtracting the same term (as done in the proof of R3) is very useful; will be applied many times in Calculus and beyond.

R5. Traditionally, in the ε-proofs, one tries to end up with $< \varepsilon$. This is why we do mysterious things before we get there (like applying the definition of limit for $\varepsilon/2$). The thing is that if something can get arbitrarily small (say: less than $\sqrt{23}\varepsilon$ for any $\varepsilon > 0$), it can be less than *any* $\varepsilon > 0$. Still, I will try to end my proofs with $< \varepsilon$, as this is how I was brought up ;-)

Extra material: notes from the iPad.

96 Properties of convergent sequences.

Prove the following properties of convergent sequences:

P0. Uniqueness of limit for convergent sequences. (Proven in V90.)

P1. Each convergent sequence is bounded. Construct an example showing that the converse is not true.

P2. Each subsequence of a convergent sequence is convergent (to the same limit).

P3. $\lim_{n\to\infty} a_n = a \ \Leftrightarrow \ \lim_{n\to\infty} |a_n - a| = 0$. In particular, when $a = 0$: $\lim_{n\to\infty} a_n = 0 \ \Leftrightarrow \ \lim_{n\to\infty} |a_n| = 0$.

P4. If $\lim_{n\to\infty} a_n = 0$ and $c \in \mathbb{R}$, then $\lim_{n\to\infty} (ca_n) = 0$.

P5. If $\lim_{n\to\infty} a_n = 0$ and (b_n) is bounded, then $\lim_{n\to\infty} (a_n b_n) = 0$.

P6. If $a_n \leqslant b_n$ for *almost all* $n \in \mathbb{N}^+$, $a_n \to a$ and $b_n \to b$, then $a \leqslant b$. Construct an example showing that it can be $a = b$ even if $a_n < b_n$ for all n.

P7. If $\lim_{n\to\infty} a_n = a \neq 0$, then $a_n \neq 0$ for *almost all* $n \in \mathbb{N}^+$.

P8. If $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} (a_n + b_n) = 0$. (This will be formulated and proven as **Lemma** in V104, in the proof of T1.)

Extra material: notes from the iPad.

97 Squeeze Theorem for sequences.

Squeeze Theorem: Given three number sequences: $(a_n), (b_n), (c_n)$ such that $b_n \leqslant a_n \leqslant c_n$ for all $n \geqslant N$, where $N \in \mathbb{N}^+$. If $\lim_{n\to\infty} b_n = L = \lim_{n\to\infty} c_n$, where $L \in \mathbb{R}$, then also $\lim_{n\to\infty} a_n = L$.

Examples: Use Squeeze Theorem to prove that: $\lim_{n\to\infty} \frac{\sin n}{n} = 0$, $\lim_{n\to\infty} \frac{[nx]}{n} = x$, where $x \in \mathbb{R}$ is any number, and $[a]$ denotes *the greatest integer* (or: *floor*) for $a \in \mathbb{R}$, i.e., such integer $[a]$ that $[a] \leqslant a < [a] + 1$.

98 Squeeze Theorem for sequences, Exercise 9.

Exercise 9: Use Squeeze Theorem and property P5 from V96, in combination with Bernoulli's Inequality or Binomial Theorem, to prove that $\lim_{n\to\infty} q^n = 0$ if $|q| < 1$. (Obviously, if $q = 1$, we get a constant sequence convergent to 1, and if $q = -1$, we get the alternating sequence $a_n = (-1)^n$, which is divergent; see V90.)

Extra material: notes with solved Exercise 9.

99 Squeeze Theorem for sequences, Exercise 10.

Exercise 10: Show that

Q1: $\lim_{n\to\infty} \sqrt[n]{n} = 1$

Q2: for each $a > 0$ we have $\lim_{n\to\infty} \sqrt[n]{a} = 1$.

Q3: if $\lim_{n\to\infty} a_n = a$ (where $a > 0, a_n > 0$ for each $n \in \mathbb{N}^+$), then $\lim_{n\to\infty} \sqrt[n]{a_n} = 1$.

Extra material: notes with solved Exercise 10.

100 **Extremely important**: New limits from old limits.

Theorem (Limit Laws for Sequences): Given two convergent number sequences (a_n) and (b_n) s.t. $a_n \to a$ and $b_n \to b$. Prove that:

T1. if $\alpha,\ \beta \in \mathbb{R}$ then $\lim\limits_{n\to\infty} (\alpha a_n + \beta b_n) = \alpha a + \beta b$.

T2. $\lim\limits_{n\to\infty} (a_n b_n) = ab$.

T3. if $k \in \mathbb{N}^+$ then $\lim\limits_{n\to\infty} a_n^k = a^k$.

T4. if $b_n \neq 0$ for all $n \in \mathbb{N}^+$ and $b \neq 0$ then $\lim\limits_{n\to\infty} \dfrac{1}{b_n} = \dfrac{1}{b}$.

T5. if $b_n \neq 0$ for all $n \in \mathbb{N}^+$ and $b \neq 0$ then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{a}{b}$.

T6. if $k \in \mathbb{N}^+ \setminus \{1\}$ then $\lim\limits_{n\to\infty} \sqrt[k]{a_n} = \sqrt[k]{a}$. If k is even, then we need the restriction: $a_n \geqslant 0$ for all $n \in \mathbb{N}^+$; no restrictions needed for odd k.

T7. if $q \in \mathbb{Q}$ and $a_n, a > 0$, then $\lim\limits_{n\to\infty} a_n^q = a^q$.

T8. $\lim\limits_{n\to\infty} |a_n| = |a|$. Find an example showing that the converse implication is not true, i.e., that convergence of $(|a_n|)$ does not imply convergence of (a_n).

(See V21 for generalization of T1 (and T2) for the sum (product) of any number of sequences.)

101 New limits from old limits, Exercise 11.

Exercise 11: Compute the limits of sequences we have seen earlier, using the Limit Laws for Sequences and other theorems we have learned until now (from videos: 81, 82, 92, 93, 94, 34):

$$\frac{1}{n};\quad (-1)^n\frac{1}{n},\quad 1+(-1)^n\frac{1}{n},\quad \frac{n+1}{n},\quad \frac{4n}{n+1},\quad \frac{3n+2}{2n+3}$$

$$2+3^{-n},\quad \frac{2n+1}{3n+2},\quad \frac{2n^2+n}{3n^2+2},\quad \frac{4n}{n+1}\cdot\frac{3n+2}{2n+3},$$

$$\frac{a_0n^k+a_1n^{k-1}+\cdots+a_{k-1}n+a_k}{b_0n^m+b_1n^{m-1}+\cdots+b_{m-1}n+b_m},\quad (a_0, b_0 \neq 0;\ k, m \in \mathbb{N}^+),$$

$$\frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}.$$

Extra material: notes with solved Exercise 11.

102 New limits from old limits, Exercise 12.

Exercise 12: Compute limits of the following sequences:

$$\left(2-\frac{1}{n^2}\right)\left(5+\frac{3}{n}\right),\quad \left(\frac{2n+1}{3n-5}\right)^3,\quad \frac{(-1)^n}{\sqrt{n}}\cdot\sin(\sqrt{n}\cdot\tfrac{\pi}{2}).$$

Extra material: notes with solved Exercise 12.

103 New limits from old limits, Exercise 13.

Exercise 13: Compute limits of the following sequences:

$$\frac{7\cdot 4^{n+1}-8}{2\cdot 4^n+2^n},\quad \frac{\sqrt[3]{n^2+1}+\sqrt{n}}{\sqrt[4]{n^3}+\frac{3}{\sqrt{n}}+1}\cdot\cos(2n-1),\quad n-\sqrt{n^2+5n}.$$

Extra material: notes with solved Exercise 13.

104 New limits from old limits, proof part 1.

Proof of parts T1, T2, and T3 from V100.

Lemma: If $x_n \to 0$ and $y_n \to 0$ then $x_n + y_n \to 0$. (Prove directly from the ε-definition of limit!)

Extra material: notes from the iPad.

105 New limits from old limits, proof part 2.

Proof of parts T4 and T5 from V100.

Extra material: notes from the iPad.

106 New limits from old limits, proof part 3.

Proof of parts T6 and T7 from V100.

Extra material: notes from the iPad.

107 New limits from old limits, proof part 4.

Proof of part T8 from V100.

Extra material: notes from the iPad.

108 Weierstrass' Theorem about convergence of bounded monotone sequences.

Theorem (Weierstrass): Each increasing (decreasing) and bounded above (below) sequence has a limit, and this limit is equal to the supremum (infimum) of the set of all the elements in this sequence.

109 Weierstrass' Theorem, Example 0.

Example 0: Read the document attached to this video.

* Show that the sequence $a_n = \left(1 + \frac{1}{n}\right)^n$ is increasing, using Bernoulli's inequality.
* Assume that the following theorem is true (the proof is very technical; it will be presented in *Calculus* 2, *part* 2 *of* 2*: Sequences and series*): If the sequence (a_n) is s.t. $a_n \to 0$, $a_n \neq 0$ and $a_n > -1$ for all $n \in \mathbb{N}^+$, then
$$\lim_{n\to\infty} (1 + a_n)^{\frac{1}{a_n}} = e.$$
Compute the following limits:
$$\lim_{n\to\infty} \left(1 + \frac{1}{\sqrt{n}}\right)^{\sqrt{n}}, \quad \lim_{n\to\infty} \left(1 + \frac{2}{n}\right)^{n}, \quad \lim_{n\to\infty} \left(1 + \frac{1}{2n}\right)^{n}.$$

Extra material: notes with solved Example 0.

Extra material: theorems and proofs from the course *Precalculus* 4*: Exponentials and logarithms*:

a) Definitions: upper bound / supremum / maximum; lower bound / infimum / minimum.
b) Theorem 1: A statement equivalent with the Axiom of Completeness.
c) Theorem 2: Convergence of a monotone sequence of real numbers.
d) Theorem 3: Convergence of geometric sequences and series with quotient $0 < q < 1$.
e) Theorem 4: About two increasing sequences with the limit in e.
f) Theorem 5: Number e is irrational.

110 Weierstrass' Theorem, Exercise 14.

Exercise 14: Show that the following sequences are monotone and bounded, and compute their limits:

* $a_1 = q,\ a_{n+1} = qa_n$ for $n \geqslant 1$, where $0 < q < 1$.
* $a_1 = \sqrt{2},\ a_{n+1} = \sqrt{2 + a_n}$ for $n \geqslant 1$.

Extra material: notes with solved Exercise 14.

Extra material: an article about nested roots, from the course *Mathematics: Fun tricks for everybody.*

111 Monotone, bounded, convergent sequences; a test.

Test: Answer the following questions using examples (or counterexamples) from our earlier videos:

a) Is each convergent sequence bounded?
b) Is each bounded sequence convergent?
c) Does each convergent sequence have exactly one accumulation point?
d) If a sequence has exactly one accumulation point, does it need to be convergent? Bounded?
e) Can a sum of two bounded sequences be unbounded?
f) Can a sum of two unbounded sequences be bounded?
g) Can a sum of two convergent sequences be divergent?
h) Can a quotient of two bounded sequences be unbounded?
i) Can a quotient of two unbounded sequences be bounded?
j) Can a quotient of two convergent sequences be divergent?
k) Does a monotone sequence need to be bounded/unbounded?
l) Does a convergent sequence need to be monotone?
m) Does a monotone and bounded sequence need to be convergent?

Extra material: notes from the iPad.

112 Extended reals.

113 What does it mean that arithmetic operations are continuous?

114 Improper limits.

Examples:

a) $a_n = n;\quad \lim_{n\to\infty} a_n = +\infty$
b) $a_n = -n;\quad \lim_{n\to\infty} a_n = -\infty$
c) If $d > 0$ and $a_n = dn + 1$ then $\lim_{n\to\infty} a_n = +\infty$
d) If $a_n \leqslant b_n$ for *almost all* $n \in \mathbb{N}^+$ and $a_n \to +\infty$, then $b_n \to +\infty$
e) If $a_n \leqslant b_n$ for *almost all* $n \in \mathbb{N}^+$ and $b_n \to -\infty$, then $a_n \to -\infty$
f) If $q > 1$ then $q^n \to \infty$.

115 Extending arithmetic to extended reals.

How we extend arithmetic to extended reals, in a continuous way:

A) $a + \infty = +\infty \quad \text{for} \quad -\infty < a \leqslant +\infty$
S) $a - \infty = -\infty \quad \text{for} \quad -\infty \leqslant a < +\infty$
M1) $a \cdot (\pm\infty) = \pm\infty \quad \text{for} \quad 0 < a \leqslant +\infty$
M2) $a \cdot (\pm\infty) = \mp\infty \quad \text{for} \quad -\infty \leqslant a < 0$
D1) $\dfrac{a}{\pm\infty} = 0 \quad \text{for} \quad a \in \mathbb{R}$
D2) $\dfrac{a}{0^+} = +\infty \quad \text{for} \quad 0 < a \leqslant +\infty$
D3) $\dfrac{a}{0^-} = -\infty \quad \text{for} \quad 0 < a \leqslant +\infty$
D4) $\dfrac{a}{0^+} = -\infty \quad \text{for} \quad -\infty \leqslant a < 0$
D5) $\dfrac{a}{0^-} = +\infty \quad \text{for} \quad -\infty \leqslant a < 0.$

Lemma: If $a_n \to \infty$ then (a_n) is bounded below. (Prove directly from the E-definition of limit!)

116 Powers involving extended reals.

How we extend taking powers for extended reals, in a continuous way:

Po1) $a^{+\infty} = +\infty \quad \text{for} \quad 1 < a \leqslant +\infty$

Po2) $a^{-\infty} = 0 \quad \text{for} \quad 1 < a \leqslant +\infty$

Po3) $a^{+\infty} = 0 \quad \text{for} \quad 0 \leqslant a < 1$

Po4) $a^{-\infty} = +\infty \quad \text{for} \quad 0^+ \leqslant a < 1$

Po5) $\infty^b = +\infty \quad \text{for} \quad 0 < b \leqslant +\infty$

Po6) $\infty^b = 0 \quad \text{for} \quad -\infty \leqslant b < 0.$

117 Some important examples, Exercise 15.

Exercise 15: compute limits of the sequences:

a) $b_n = a_0 n^k + a_1 n^{k-1} + \cdots + a_{k-1} n + a_k, \quad a_0 \neq 0, \quad k \in \mathbb{N}^+$

b) $c_n = \dfrac{a_0 n^k + a_1 n^{k-1} + \cdots + a_{k-1} n + a_k}{b_0 n^m + b_1 n^{m-1} + \cdots + b_{m-1} n + b_m}, \quad (a_0, b_0 \neq 0; \ k, m \in \mathbb{N}^+, \quad k > m)$

c) $a_n = n + \sin n$.

Extra material: notes with solved Exercise 15.

118 **Very important**: Indeterminate forms.

Indeterminate forms: $\infty - \infty, \ 0 \cdot \infty, \ \frac{0}{0}, \ \frac{\infty}{\infty}, \ 1^\infty, \ \infty^0, \ 0^0$; somewhat indeterminate: $\frac{1}{0}$.

119 Comparing infinities: Not always intuitive; Exercise 16.

Standard Limits in the Infinity: Let $a > b > 1$ and $k > m > 0$. Then:

$$n^n \gg n! \gg a^n \gg b^n \gg n^k \gg n^m \gg \ln n \qquad \text{where} \qquad a_n \gg b_n (> 0) \quad \Leftrightarrow \quad \frac{a_n}{b_n} \to \infty \quad \Leftrightarrow \quad \frac{b_n}{a_n} \to 0.$$

Example: If $a_n = \dfrac{1.01^n}{n^{10}}$, then $a_n \approx 0$ for $n \leqslant 8{,}800$, and still $a_n \to \infty$ when $n \to \infty$! For $n = 10{,}000$ we have $a_n = 1{,}636$. All this shows that what happens for (relatively) small n does not matter.

Exercise 16:

a) Show that if $a > b > 1$ then $a^n \gg b^n$.

b) Show that if $k > m > 0$ then $n^k \gg n^m$.

c) Use the theorem about *Standard Limits in the Infinity* to compute limits of the following sequences:

$$a_n = \frac{n^2 - n + 1}{1 + 2n^2}, \quad b_n = \frac{3^n - n^3}{5 \cdot 2^n + \ln n}, \quad c_n = \frac{2n \cdot n! + 2^n}{n^3 + (n+1)!}, \quad d_n = \frac{n \cdot n! + 2^n + n^4}{(n+1)! + 3^n + n^6}.$$

Extra material: notes with solved Exercise 16.

Extra material: an article with more solved problems on comparing infinities. In all problems, compute the limits of a_n with n tending to infinity, where:

- ⋆ **Extra problem 1**: $a_n = \dfrac{3n^3 + n^2}{n^2 - n^3}$
- ⋆ **Extra problem 2**: $a_n = \dfrac{5n^3 - 2n^2}{n^4 - n + 1}$
- ⋆ **Extra problem 3**: $a_n = \dfrac{e^n}{1 - n}$
- ⋆ **Extra problem 4**: $a_n = \dfrac{n}{e^n + 1}$
- ⋆ **Extra problem 5**: $a_n = \dfrac{e^{2n} + n^2 - 1}{2e^{2n} + n^{35}}$

- ⋆ **Extra problem 6**: $a_n = \dfrac{e^n - \ln n + 3n^2}{n^3 + e^{n+1}}$
- ⋆ **Extra problem 7**: $a_n = \dfrac{n^5 + 2^{n+1}}{2^n - n^3}$
- ⋆ **Extra problem 8**: $a_n = \dfrac{n}{n + \sin n}$
- ⋆ **Extra problem 9**: $a_n = \dfrac{e^{-n} + 3n^4 - n}{n^4 + 2n^2}$
- ⋆ **Extra problem 10**: $a_n = \dfrac{e^{-n} + 3n^3 + 4}{2n^3 + n - 1}$
- ⋆ **Extra problem 11**: $a_n = (1 + e^{-n}) \cdot \dfrac{n^2 + 3n}{n^2 + 1}$
- ⋆ **Extra problem 12**: $a_n = \dfrac{8e^n - \ln n + 19 \cdot 4^n + n^{2023}}{7e^n - 100 \cdot 2^{2n} + n^{1973}}$
- ⋆ **Extra problem 13**: $a_n = \dfrac{\sqrt{n^4 + 3n^3 + 2}}{n^2 + 1}$
- ⋆ **Extra problem 14**: $a_n = \dfrac{\sqrt{n^2 + n + 3}}{2n + 1}$
- ⋆ **Extra problem 15**: $a_n = \dfrac{\sqrt{n^2 - n} + n}{n + 1}$
- ⋆ **Extra problem 16**: $a_n = \dfrac{3n^2 + 3n + 5}{n^2 + \sqrt{n^4 - 7n + 1}}$
- ⋆ **Extra problem 17**: $a_n = \sqrt{n^2 + n + 1} - n$
- ⋆ **Extra problem 18**: $a_n = \sqrt{n^2 + n} - \sqrt{n^2 - 2n + 5}$.

120 Comparing infinities: An important limit for the exponential function.

Standard limit in the infinity: If $a > 1$ then:

$$\lim_{n\to\infty} \frac{a^n}{n!} = 0.$$

Two different proofs will be given. (Actually, the result is valid for *any* $a \in \mathbb{R}$; this will also be shown in the video, together with the motivation why this example is important for the exponential function $f(x) = e^x$.)

Extra material: notes from the iPad.

121 Comparing infinities: More quotients.

Standard limits in the infinity: If $a > 1$ and $k > 0$ then:

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0, \qquad \lim_{n\to\infty} \frac{n^k}{a^n} = 0, \qquad \lim_{n\to\infty} \frac{n^k}{\ln n} = \infty.$$

Example: MANIM animation showing the following functions:

$$f_1(x) = \frac{x}{e^x},\ \ f_2(x) = \frac{x^2}{e^x},\ \ f_3(x) = \frac{x^3}{e^x},\ \ f_4(x) = \frac{x^4}{e^x},\ \ f_5(x) = \frac{x^5}{e^x},\ \ f_7(x) = \frac{x^7}{e^x}.$$

Extra material: notes from the iPad.

122 **Optional**: Cauchy sequences, sequences in metric spaces, and completeness.

Theorem: If (a_n) is a convergent sequence in a metric space, then it is a Cauchy sequence.

S6 Limit of a function in a point

You will learn: the concept of a finite limit of a real-valued function of one real variable in a point: Cauchy's definition, Heine's definition (aka *Sequential condition*), and their equivalence; accumulation points (limit points, cluster points) of the domain of a function; one-sided limits; the concept of continuity of a function in a point, and continuity on a set; limits and continuity of elementary functions as building blocks for all the other functions you will meet in your Calculus classes; computational rules: limit of sum, difference, product, quotient of two functions; limit of a composition of two functions; limit of inverse functions; Squeeze Theorem; *Standard limits in zero* and other methods for handling indeterminate forms of the type $\frac{0}{0}$ (factoring and cancelling, using conjugates, substitution).

Read along with this section: **Calculus book**: Chapter 2 *Limits*. If you need to learn how to prove The Limit Laws for functions with help of the epsilon-delta definition (we don't need it in this course, as we have studied sequences already, and The Limit Laws for sequences fix all our problems!), or motivate continuity of elementary functions using this method, you might want to read Section 2.5. *The Precise Definition of a Limit* on pages 232–248 (the page numbers in the pdf; they correspond to the following side numbers in the book: 2.5.1–2.5E.4); you can ask me questions about this material via our QA. [You get the epsilon-delta proofs from me, too! In V143.]

123 It's all about proximity, also for functions.

124 In what kind of points is it meaningful to examine limits of a function?

125 An example from Precalculus 1.

Example 1: Function $f(x) = \dfrac{x^2-1}{x-1}$ is not defined in $a = 1$, but we can estimate the value that $f(x)$ approaches as x approaches 1.

126 An example from Precalculus 2.

Example 2: Function $f(x) = \dfrac{x-1}{x^2-1}$ is not defined in $a = 1$, but we can estimate the value that $f(x)$ approaches as x approaches 1.

127 An example from Precalculus 3.

Example 3: Function $f(x) = \dfrac{\sin x}{x}$ is not defined in $a = 0$, but we can estimate the value that $f(x)$ approaches as x approaches 0.

128 An example from Precalculus 4.

Example 4: Function $f(x) = \dfrac{e^x-1}{x}$ is not defined in $a = 0$, but we can estimate the value that $f(x)$ approaches as x approaches 0.

129 Formal definition of a limit in a point, with an illustration.

An explanation why limits only can be examined in accumulation points of the domain.

130 Limit, if exists, is unique.

Let $f : \mathbb{R} \to \mathbb{R}$ be defined in some neighbourhood of $a \in \mathbb{R}$. If $\lim\limits_{x\to a} f(x)$ exists, then it is unique.

Extra material: notes from the iPad.

131 Limits of some functions, Example 1.

Example 1: If $c \in \mathbb{R}$ is some constant and $f(x) = c$ for all $x \in \mathbb{R}$ then $\lim\limits_{x\to a} f(x) = c$ for all $a \in \mathbb{R}$.

132 Limits of some functions, Example 2.

Example 2: If $f(x) = x$ for all $x \in \mathbb{R}$ then $\lim\limits_{x\to a} f(x) = a$ for all $a \in \mathbb{R}$.

133 Limits of some functions, Example 3.

Example 3: Let $A = [0, \infty) \setminus \{9\}$ and $f : A \to \mathbb{R}$ is defined as $f(x) = \dfrac{x-9}{\sqrt{x}-3}$. Show that $\lim\limits_{x\to 9} f(x) = 6$.

Extra material: notes from the iPad.

134 Some (earlier) examples where the limit does not exist.

The following examples were introduced in Precalculus 1:

- If $f(x) = \lfloor x \rfloor$ and $a \in \mathbb{Z}$ then $\lim_{x\to a} f(x)$ does not exist.
- If $f(x) = \chi_{\mathbb{Q}}$ restricted to $[0,1]$, and $a \in [0,1]$ is any number, then $\lim_{x\to a} f(x)$ does not exist.
- Some examples of piece-wise functions presented in Precalculus 1.

135 One-sided limits.

136 One-sided limits, examples.

We analyse the following examples with respect to limits and one-sided limits:

- If $f(x) = \lfloor x \rfloor$ and $a \in \mathbb{Z}$ then $\lim_{x\to a^-} f(x) = a - 1$ and $\lim_{x\to a^+} f(x) = a$.
- Some examples of piece-wise functions presented in Precalculus 1.

137 One-sided limits, more examples.

Compare the following functions with respect to limits and one-sided limits: $f(x) = \mathrm{sgn}(x)$ and $g(x) = (\mathrm{sgn}(x))^2$.

Extra material: notes from the iPad.

138 **Extremely important**: New limits from old limits.

Theorem (Limit Laws for Functions): Given two functions $f, g : \mathbb{R} \to \mathbb{R}$ defined in some neighbourhood of $a \in \mathbb{R}$, and s.t. $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$; $k \in \mathbb{R}$ is a constant. Prove that:

L1. **Limit of a sum**: $\lim_{x\to a}[f(x) + g(x)] = L + M$.

L2. **Limit of a multiple**: $\lim_{x\to a} kf(x) = kL$.

L3. **Limit of a difference**: $\lim_{x\to a}[f(x) - g(x)] = L - M$.

Le1 **Lemma 1, about boundedness**: If $\lim_{x\to a} g(x) = M$ then there exists such $\delta_1 > 0$ that if $0 < |x - a| < \delta_1$ (and $x \in D_g$) then $|g(x)| < 1 + |M|$.

L4. **Limit of a product**: $\lim_{x\to a}[f(x) \cdot g(x)] = L \cdot M$.

Le2 **Lemma 2, about a lower bound (separation from zero)**: If $\lim_{x\to a} g(x) = M$ and $M \neq 0$, then there exists such $\delta_0 > 0$ that if $0 < |x - a| < \delta_0$ (and $x \in D_g$) then $|g(x)| > |M|/2$.

L5. **Limit of a quotient**: if $M \neq 0$ then $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$.

L6. **Limit of a power**: if m is an integer and n is a positive integer, then $\lim_{x\to a}[f(x)]^{m/n} = L^{m/n}$, provided $L > 0$ if n is even, and $L \neq 0$ if $m < 0$.

L7. **Order is preserved**: if $f(x) \leqslant g(x)$ on an interval containing a in its interior, then $L \leqslant M$.

Rules L1–L6 are also valid for right limits and left limits. So is rule L7, under the assumption that $f(x) \leqslant g(x)$ on an open interval extending from a in the appropriate direction.

(See V21 for generalization of L1 (and L4) for the sum (product) of any number of functions with a limit in a.)

The proof will be given to you in V143.

139 New limits from old limits, an exercise.

Exercise: Knowing that $\lim_{x\to 5} f(x) = 3$ and $\lim_{x\to 5} g(x) = 2$, compute the following limits:

$$\lim_{x\to 5}(2f(x) - g(x)), \quad \lim_{x\to 5} \frac{f(x)g(x)}{3g(x) - 1}, \quad \lim_{x\to 10}(\sqrt{f(x)} + 3g(x)).$$

Extra material: notes with solved Exercise 12.

140 Another approach to the topic of limits.

Let $a_n = \frac{1}{n}$, $b_n = (-1)^n \frac{1}{n}$, $c_n = 1 - \frac{1}{n}$, $d_n = 1 + (-1)^n \frac{1}{n}$. You get to see $(f(a_n))_{n=1}^{\infty}$, $(f(b_n))_{n=1}^{\infty}$, $(f(c_n))_{n=1}^{\infty}$, $(f(d_n))_{n=1}^{\infty}$ (well, maybe just up to $n = 70$...) for:

- $f(x) = x^2 - 1$,
- $f(x) = \lfloor x \rfloor$,
- $f(x) = \dfrac{1}{x}$ (in this case (c_n) and (d_n) start from $n = 2$).

141 Heine's definition of limits, and a remark about notation.

Show that the following limits don't exist, using Heine's definition:

- If $f(x) = \lfloor x \rfloor$ and $a \in \mathbb{Z}$ then $\lim\limits_{x \to a} f(x)$ does not exist.
- If $f(x) = \sin \frac{1}{x}$, then $\lim\limits_{x \to 0} f(x)$ does not exist.

142 Equivalence of Cauchy's and Heine's definitions of limits.

Proof of equivalence of both definitions of a limit of a function in some point.

A remark (similar to P3 in V96): $\lim\limits_{x \to a} f(x) = L \quad \Leftrightarrow \quad \lim\limits_{x \to a} |f(x) - L| = 0.$

Extra material: notes from the iPad.

143 Two proofs of the powerful theorem from V138.

How to prove the Theorem from V138 with help of the corresponding results for sequences.

Extra material: notes from the iPad, with proofs of L1–L5, L7 and Lemmas 1 and 2 formulated in V138. For L6 we rely on the results obtained for sequences (V100: T6 and T7 proven in V106).

144 What does it mean that a function is continuous?

The following fact follow immediately from the definition of continuity:

- Each function is continuous at each isolated point of its domain.
- The sequential condition (Heine) is equivalent with the $\varepsilon - \delta$ definition of continuity.
- Given two continuous functions f and g; their sum, product, scalings, quotient (with some restrictions for zeros in the denominator) are all continuous functions on $D_f \cap D_g$.

145 A handy test for continuity.

146 Examining continuity, some examples.

Some examples we have seen before:

- Back to our example from Videos: 66, 124, 135.
- $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = c$ (for some $c \in \mathbb{R}$) is continuous.
- $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = x$ is continuous.
- $f : [a, b] \cup [c, d] \to \mathbb{R}$ defined as $f(x) = x$ is continuous. (**Horizontal** jumps are OK.)
- $f(x) = \lfloor x \rfloor$ is discontinuous at each $a \in \mathbb{Z}$ and continuous at each $a \notin \mathbb{Z}$. (**Vertical** jumps are **not** OK.)
- A piece-wise function doesn't need to be discontinuous; three examples from Precalculus 1.

147 Some warnings about different definitions of continuity and discontinuity.

148 Continuity of polynomials, rational functions, and power functions.

Example: Compute $\lim\limits_{x \to a} f(x)$ where:

- $f(x) = 3x^4 - 7x^3 + 2x - 1, \ a = 1$
- $f(x) = \dfrac{x^2 - 1}{4x^2 + 3}, \ a = 2.$
- $f(x) = \sqrt{x} + x^{3/2} - \dfrac{x^2 - 77}{x - 5} + 2x - 5, \ a = 9.$

Extra material: notes from the iPad.

149 Continuity of trigonometric functions.

Prove continuity of the following functions:

* $f(x) = \sin x$
* $f(x) = \cos x$
* $f(x) = \tan x,\ f(x) = \cot x,\ f(x) = \csc x,\ f(x) = \sec x.$

Extra material: notes from the iPad.

150 An important lemma about one-sided limits of monotone functions.

Lemma: If $f : I \to \mathbb{R}$ is monotone on an interval I, then the left and right limits of f exist at every interior point $a \in I$.

* $\lim\limits_{x\to a^-} f(x)$ exists.
* $\lim\limits_{x\to a^+} f(x)$ exists.

Corollary: Every (possible) discontinuity of a **monotonic** function $f : I \to \mathbb{R}$ at an **interior** point of the interval I is a **jump discontinuity**.

151 Continuity of exponential functions.

Prove continuity of $f(x) = a^x$ for any $a \in \mathbb{R}^+ \setminus \{1\}$.

Corollary: Continuity of $f(x) = e^x$ in combination with the result formulated in V144 gives us a proof of continuity of the following functions:

$$\sinh(x) = \frac{e^x - e^{-x}}{2};\ \cosh(x) = \frac{e^x + e^{-x}}{2};\ \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}};\ \coth(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}}\ \ (x \neq 0).$$

Extra material: notes from the iPad.

152 Compositions of continuous functions.

Theorem: Let $f : D_f \to \mathbb{R}$ and $g : D_g \to \mathbb{R}$ where $f(D_f) \subset D_g$. If f is continuous at $a \in D_f$ and g is continuous at $f(a) \in D_g$, then $g \circ f : D_f \to \mathbb{R}$ is continuous at a. (The theorem generalises in a natural way for a composition of *any* number of functions satisfying the necessary conditions of continuity and inclusions.)

Extra material: notes from the iPad.

153 Compositions of continuous functions: some examples.

Examine continuity of the following functions:

* $f(x) = e^{x^2-2}$
* $f(x) = \begin{cases} \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$
* $f(x) = \begin{cases} x\sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Extra material: notes from the iPad.

154 Continuity of inverse functions.

Theorem: Let D_f and V_f be two intervals and $f : D_f \to V_f$ be a continuous, invertible, and strictly increasing (decreasing) function. Then the inverse $f^{-1} : V_f \to D_f$ is also continuous and strictly increasing (decreasing).

Extra material: notes from the iPad.

155 Continuity of inverse functions: why the assumption about interval is important.

The following example shows that the assumption that f is continuous **on an interval** is important: Function

$$f(x) = \begin{cases} x & \text{if } x \in [0, 1] \\ x - 1 & \text{if } x \in (2, 3] \end{cases}$$

is continuous, increasing, and invertible, but its inverse is not continuous.

Extra material: notes from the iPad.

156 Some important consequences of the theorems in V152 and 154.

The following functions are continuous:

$$f(x)=\arcsin x,\quad f(x)=\arccos x,\quad f(x)=\arctan x,\quad f(x)=\ln x,\quad f(x)=\log_a x,\quad f(x)=x^x.$$

Moreover, if $a_n \to a$ (where $a, a_n > 0$) and $b_n \to b$ then $a_n^{b_n} \to a^b$.

Inverse functions to the hyperbolic functions are continuous, too:

$$\operatorname{arsinh}(x)=\ln(x+\sqrt{x^2+1});\quad \operatorname{arcosh}(x)=\ln(x+\sqrt{x^2-1})\ \ (x\geqslant 1).$$

Extra material: notes from the iPad.

157 From determinate to indeterminate forms: something needs to be done.

Some examples of dealing with indeterminate forms:

- Factor and simplify: Compute $\lim\limits_{x\to 1}\dfrac{x^2-1}{x^3-1}$
- Use the conjugate: Compute $\lim\limits_{x\to 0}\dfrac{1-\sqrt{1+x}}{x}$
- Simplify the expression: Compute $\lim\limits_{x\to 1}\dfrac{\frac{1}{x+1}-\frac{1}{2}}{x-1}$.

Extra material: notes from the iPad.

Extra material: an article with more solved problems on computing limits of function in points:

- ⋆ **Extra problem 1**: $\lim\limits_{x\to 2}\dfrac{x^2+2x-8}{x^2-2x}$
- ⋆ **Extra problem 2**: $\lim\limits_{x\to -4}\dfrac{4x^2+15x-4}{x^2-16}$
- ⋆ **Extra problem 3**: $\lim\limits_{x\to 2}\dfrac{x^2-5x+6}{x^2-3x+2}$
- ⋆ **Extra problem 4**: $\lim\limits_{x\to -3}\dfrac{x^2-9}{x^2+4x+3}$
- ⋆ **Extra problem 5**: $\lim\limits_{x\to -4}\dfrac{2x^2+7x-4}{x^2+3x-4}$
- ⋆ **Extra problem 6**: $\lim\limits_{x\to 2}\dfrac{x^2-x-2}{x^2-5x+6}$
- ⋆ **Extra problem 7**: $\lim\limits_{x\to 5}\dfrac{x^2-3x-10}{x^2-4x-5}$
- ⋆ **Extra problem 8**: $\lim\limits_{x\to 3}\dfrac{x^2-2x-3}{x-3}$
- ⋆ **Extra problem 9**: $\lim\limits_{x\to -4}\dfrac{2x^2+7x-4}{x+4}$
- ⋆ **Extra problem 10**: $\lim\limits_{x\to 1}\dfrac{x^2+2x-3}{x^2-x}$
- ⋆ **Extra problem 11**: $\lim\limits_{x\to -2}\dfrac{3x^2+7x+2}{x+2}$

⋆ **Extra problem 12**: $\lim_{x\to 1} \frac{x^2+6x-7}{x^2-x}$

⋆ **Extra problem 13**: $\lim_{x\to 5} \frac{x^2-2x-15}{x-5}$

⋆ **Extra problem 14**: $\lim_{x\to 1} \frac{e^x(x^2+6x-7)}{x-1}$

⋆ **Extra problem 15**: $\lim_{x\to 0} \frac{\cos x \cdot (x^2-2x-3)}{x-3}$

⋆ **Extra problem 16**: $\lim_{x\to 1} \frac{x^2-1}{x^2-2x+1}$. **Warning**: This problem is different than the previous ones, so don't worry if you can't solve it yet! But you can give it a try; if you do, remember how arithmetic on extended reals works. If it's too hard now, come back here after you have studied Section 7.

⋆ **Extra problem 17**: $\lim_{x\to -1} \frac{\sqrt{x+2}-1}{x+1}$

⋆ **Extra problem 18**: $\lim_{x\to 5} \frac{\sqrt{x-1}-2}{x-5}$

⋆ **Extra problem 19**: $\lim_{h\to 0} \frac{\sqrt{5h^2+11h+6}-\sqrt{6}}{h}$.

You can also plot the given functions and observe what happens with their graphs around the given point. Many types of software will not mark *holes* in the graphs, but you know that they *should* be there! (To see what I mean, look at the image I have prepared for you under the solution to Problem 7 in the file; there you see that the point with coordinates $(5, \frac{7}{6})$ does **not** belong to the graph; the software I used didn't mark this hole, but I did...)

158 Squeeze Theorem for functions.

Squeeze Theorem: Given three functions: $f, g, h : \mathbb{R} \to \mathbb{R}$ defined in some neighbourhood of $a \in \mathbb{R}$ and such that $g(x) \leqslant f(x) \leqslant h(x)$ for $x \in (a-\Delta, a+\Delta)$, where $\Delta > 0$. If $\lim_{x\to a} g(x) = L = \lim_{x\to a} h(x)$, where $L \in \mathbb{R}$, then also $\lim_{x\to a} f(x) = L$.

The following variant of the Squeeze Theorem is often used: **Corollary**: If

$$0 \leqslant f(x) \leqslant h(x) \quad (\text{alt.: } |f(x)| \leqslant h(x))$$

and $h(x) \to 0$ as $x \to a$, then $f(x) \to 0$ as $x \to a$.

In particular: Product of a bounded function and a function tending to zero (at some point) is tending to zero (at this point). (This is also a consequence of Heine's definition of limit, and of P5 from V96.)

Example: $|x \sin \frac{1}{x}| \leqslant |x|$ shows that $x \sin \frac{1}{x} \to 0$ when $x \to 0$.

159 An application of Squeeze Theorem: a standard limit in zero.

The first standard limit in zero: $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

160 Now we can finally motivate the formula for the area of a disk.

The area of a disk with radius R is expressed by the formula $A = \pi R^2$.

Extra material: notes from the iPad.

161 More standard limits in zero.

More standard limit in zero: $\lim_{x\to 0} \frac{\ln(x+1)}{x} = 1$, $\lim_{x\to 0} \frac{e^x-1}{x} = 1$.

Extra material: notes from the iPad.

162 Problem solving, Problem 1.

Problem 1: Compute the following limits:

$$\lim_{x\to 5}\frac{\sin(x-5)}{x-5},\quad \lim_{x\to -1}\frac{\ln(2+x)}{x+1},\quad \lim_{x\to 0}\frac{e^{x^3}-1}{x^2},\quad \lim_{x\to 0}\frac{\ln(1+2x)}{\tan 5x}.$$

Extra material: notes with the solution of Problem 1.

Extra material: an article with more solved problems on computing limits of indeterminate forms, using *Standard Limits in Zero.*

- ⋆ **Extra problem 1**: $\lim_{x\to 0}\frac{\sin(7x)}{x}$
- ⋆ **Extra problem 2**: $\lim_{x\to 0}\frac{e^{5x}-1}{x}$
- ⋆ **Extra problem 3**: $\lim_{x\to 0}\frac{12x}{\sin(3x)}$
- ⋆ **Extra problem 4**: $\lim_{x\to 0}\frac{e^{2x}-1}{\sin(3x)}$
- ⋆ **Extra problem 5**: $\lim_{x\to 1}\frac{\ln x}{x-1}$
- ⋆ **Extra problem 6**: $\lim_{x\to 0}\frac{x}{\arctan x}$
- ⋆ **Extra problem 7**: $\lim_{x\to 1}\frac{\sin(\ln x)}{\ln x}$
- ⋆ **Extra problem 8**: $\lim_{x\to 0}\frac{(x^2-x)\cdot\sin x}{x^2}$
- ⋆ **Extra problem 9**: $\lim_{x\to 0}\frac{\sin(x^2+2x)}{x}$
- ⋆ **Extra problem 10**: $\lim_{x\to 0}\frac{\sin(3x^5+5x^4)}{2x^4}$
- ⋆ **Extra problem 11**: $\lim_{x\to 0}\ln\left(\frac{\sin x+x^2}{x}\right)$
- ⋆ **Extra problem 12**: $\lim_{x\to 0}\arcsin\left(\frac{\sin x}{\sqrt{2}\cdot x}\right)$
- ⋆ **Extra problem 13**: $\lim_{x\to 0}\arctan\left(\frac{\sin x}{x}\right)$

163 Problem solving, Problem 2.

Problem 2: Compute the following limits:

$$\lim_{x\to 0}\frac{\tan x}{x},\quad \lim_{x\to 0}\frac{1-\cos x}{x^2},\quad \lim_{x\to 0}\frac{\tan x-\sin x}{\sin^3 x},\quad \lim_{x\to 0}\frac{\sqrt{1+\tan x}-\sqrt{1+\sin x}}{x^3}.$$

Extra material: notes with the solution of Problem 2.

164 Problem solving, Problem 3.

Problem 3: Compute the following limits: $\lim_{x\to\frac{\pi}{6}}\frac{2\sin^2 x+\sin x-1}{2\sin^2 x-3\sin x+1}$, $\lim_{x\to\frac{\pi}{4}}(\tan 2x\cdot\tan(\frac{\pi}{4}-x))$.

Extra material: notes with the solution of Problem 3.

165 Problem solving, Problem 4.

Problem 4: Compute the following limits:

$$\lim_{x\to 0}\frac{e^x-e^{-x}}{\sin x},\quad \lim_{x\to 0}\frac{e^{x^2}-\cos x}{x^2},\quad \lim_{x\to 0}(1+\sin x)^{1/x},\quad \lim_{x\to\frac{1}{2}}\frac{\arcsin(1-2x)}{4x^2-1}.$$

Extra material: notes with the solution of Problem 4.

166 Problem solving, Problem 5.

Problem 5: Compute the following limits:

$$\lim_{x\to 4}\frac{x-\sqrt{3x+4}}{4-x},\quad \lim_{x\to 0}\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+25}-5}.$$

Extra material: notes with the solution of Problem 5.

167 Problem solving, Problem 6.

Problem 6: Compute the following limits:

$$\lim_{x\to 1}\frac{e^{\sin(\pi x)}+x\arctan x}{x-\ln(2x^3-1)},\quad \lim_{x\to 0}e^{\sin x}\cdot\frac{\cos x^2+\arctan x}{2+\ln(1+x)}.$$

Extra material: notes with the solution of Problem 6.

168 Problem solving, Problem 7.

Problem 7: Compute the following limits:

a) $\lim_{x\to 0}|x|$

b) $\lim_{x\to 0}\frac{\sin x}{|x|}$

c) $\lim_{x\to 2}\frac{|x-2|}{x-2}$

d) $\lim_{x\to 2}\frac{\sqrt{4-4x+x^2}}{x-2}$

e) $\lim_{x\to 1}\frac{x-4\sqrt{x}+3}{x^2-1}$.

Extra material: notes with the solution of Problem 7.

169 Problem solving, Problem 8.

Problem 8: Knowing that $\lim_{x\to 0+}f(x)=A$ and $\lim_{x\to 0-}f(x)=B$, compute:

a) $\lim_{x\to 0+}f(x^3-x)$

b) $\lim_{x\to 0-}f(x^3-x)$

c) $\lim_{x\to 0+}f(x^2-x^4)$

d) $\lim_{x\to 0-}f(x^2-x^4)$.

S7 Infinite limits and limits in the infinities

You will learn: define and compute infinite limits and limits in infinities for functions, and how these concepts relate to vertical and horizontal asymptotes for functions; as we already have learned the arithmetic on extended reals in Section 5, we don't need much theory here; we will perform a thorough analysis of limits of indeterminate forms involving rational functions in both zero and the infinities.

Read along with this section: **Calculus book**: Chapter 2 *Limits*.

170 An example showing (almost) all the concepts from this section.

Example: Consider $f(x) = \frac{1}{x}$:

a) The line $x = 0$ is a vertical asymptote to $y = f(x)$.

b) The line $y = 0$ is a horizontal asymptote to $y = f(x)$.

171 We do have all the theory needed.

172 What about all the warnings?

173 Infinite limits in the infinities.

174 Infinite limits in the infinities, a sequential condition.

The following functions have infinite limits in plus infinity: $f(x) = x$, $f(x) = x^k$ $(k > 0)$, $f(x) = a^x$ $(a > 1)$.

If $a_0 \neq 0$ and $p(x) = a_0x^k + a_1x^{k-1} + \cdots + a_{k-1}x + a_k$ then

$$\lim_{x\to\infty} p(x) = \begin{cases} +\infty & \text{if } \ a_0 > 0 \\ -\infty & \text{if } \ a_0 < 0 \end{cases}$$

$$\lim_{x\to-\infty} p(x) = \begin{cases} +\infty & \text{if } \ a_0 > 0 \ \text{ and } \ k \ \text{ is even} \\ -\infty & \text{if } \ a_0 < 0 \ \text{ and } \ k \ \text{ is even} \\ -\infty & \text{if } \ a_0 > 0 \ \text{ and } \ k \ \text{ is odd} \\ +\infty & \text{if } \ a_0 < 0 \ \text{ and } \ k \ \text{ is odd} \end{cases}$$

175 Finite limits in the infinities.

Example: Show that $\lim_{x\to\infty} x\sin\frac{1}{x} = 1 = \lim_{x\to-\infty} x\sin\frac{1}{x}$. (BTW, we can extend f to a continuous function on $\mathbb{R}$ by defining $\hat{f}(0) = 0$.)

176 Finite limits in the infinities, a sequential condition.

Motivate the following:

$$\lim_{x\to\infty}\frac{1}{x} = 0, \quad \lim_{x\to\infty} q^x = 0 \ (0 < q < 1), \quad \lim_{x\to-\infty} q^x = 0 \ (q > 1), \quad \lim_{x\to\infty}\frac{3x+2}{2x+3} = \frac{3}{2}.$$

(You have plenty of possible ways of proving these: directly from the definition; from the sequential condition; using the corresponding sequences and their limits and the fact that the functions are monotone; using arithmetic on extended reals as introduced and discussed in V115 and V116.)

Extra material: notes from the iPad.

177 Horizontal asymptotes.

Consider a rational function

$$f(x) = \frac{a_0x^k + a_1x^{k-1} + \cdots + a_{k-1}x + a_k}{b_0x^m + b_1x^{m-1} + \cdots + b_{m-1}x + b_m}, \quad (a_0, b_0 \neq 0;\ k, m \in \mathbb{N}^+).$$

$$\lim_{x\to\infty} f(x) = \begin{cases} \frac{a_0}{b_0} & \text{if } \ k = m \\ 0 & \text{if } \ k < m \\ +\infty & \text{if } \ k > m \ \text{ and } \ a_0b_0 > 0 \\ -\infty & \text{if } \ k > m \ \text{ and } \ a_0b_0 < 0 \end{cases}$$

$$\lim_{x\to-\infty} f(x) = \begin{cases} \frac{a_0}{b_0} & \text{if } \ k = m \\ 0 & \text{if } \ k < m \\ +\infty & \text{if } \ k > m \ \text{ and } \ a_0b_0 > 0 \ \text{ and } \ k-m \ \text{ is even} \\ -\infty & \text{if } \ k > m \ \text{ and } \ a_0b_0 < 0 \ \text{ and } \ k-m \ \text{ is even} \\ -\infty & \text{if } \ k > m \ \text{ and } \ a_0b_0 > 0 \ \text{ and } \ k-m \ \text{ is odd} \\ +\infty & \text{if } \ k > m \ \text{ and } \ a_0b_0 < 0 \ \text{ and } \ k-m \ \text{ is odd} \end{cases}$$

Examples: Evaluate the following limits:

$$\lim_{x\to\infty} \frac{4x^5 - 5x^3 + x^2 - 7}{8x^5 - 9x^4 - x^3 + 4x^2 - 9}, \qquad \lim_{x\to-\infty} \frac{5x^3 + 6x^2 - 3}{-8x^2 - 5}, \qquad \lim_{x\to\infty} \frac{-x^7 - 8x^2 - 1}{5x^{10} - 3x^4 - x}.$$

Extra material: notes with solved Examples.

178 Infinite limits at accumulation points of the domain (outside the domain).

179 Infinite limits at accumulation points of the domain, a sequential condition.

Examples: Evaluate the following limits:

$$\lim_{x\to 0} \frac{4x^2 - 3}{3x^2 - 9}, \qquad \lim_{x\to 0} \frac{12x^2 + 6x}{4x^2 - 5x}, \qquad \lim_{x\to 0} \frac{x^5 - 3x^4}{-x^2 + 7x}, \qquad \lim_{x\to 0} \frac{-x^2 + 7x}{x^5 - 3x^4}, \qquad \lim_{x\to 0} \frac{x^5 - 3x}{-x^5 + 7x^3}.$$

Extra material: notes with solved Examples.

180 Vertical asymptotes.

Examples: Determine all the vertical asymptotes for:

$$y = \frac{4x^2 - 3}{3x^2 - 9}, \qquad y = \frac{12x^2 + 6x}{4x^2 - 5x}, \qquad y = \frac{x^5 - 3x^4}{-x^2 + 7x}, \qquad y = \frac{-x^2 + 7x}{x^5 - 3x^4}, \qquad y = \frac{x^5 - 3x}{-x^5 + 7x^3}.$$

Extra material: notes with solved Examples.

181 Tangent and arctangent, and their asymptotes.

Example: Determine the asymptotes for:

a) $y = \tan x$.

b) $y = \arctan x$.

Extra material: notes from the iPad.

182 Comparing infinities: as in V119.

Standard Limits in the Infinity: Let $a > b > 1$ and $k > m > 0$. Then:

$$x^x \gg a^x \gg b^x \gg x^k \gg x^m \gg \ln x \qquad \text{where} \qquad f(x) \gg g(x)(>0) \quad \Leftrightarrow \quad \frac{f(x)}{g(x)} \to \infty \quad \Leftrightarrow \quad \frac{g(x)}{f(x)} \to 0.$$

Let $a > 1$. Use the fact that $\lim_{n\to\infty} \frac{a^n}{n} = \infty$ proven in V121 to show that $\lim_{x\to\infty} \frac{a^x}{x} = \infty$. It follows automatically that $\lim_{x\to\infty} \frac{a^x}{x^k} = \infty$ and $\lim_{x\to\infty} \frac{x^k}{\ln x} = \infty$ for $k > 0$ (gets proved with exactly the same computation as in V121).

All these *Standard Limits in the Infinity* will come back in *Calculus 1, part 2 of 2: Derivatives with applications* and you will see yet another proof of the same theorem.

Extra material: notes from the iPad.

Extra material: an article with more solved problems on computing limits of function in the infinities.

Note: You can practice with all the 18 problems from the article attached to Video 119: just replace n by x and work with functions instead of sequences. Everything will be just identical! This is why you now only get some limits in *minus* infinity: *plus* infinity can be practiced using the old stuff!

- ⋆ **Extra problem 1**: $\lim_{x\to-\infty} \frac{e^x + 3x^4 - x}{x^4 + 2x^2}$
- ⋆ **Extra problem 2**: $\lim_{x\to-\infty} \frac{e^x + 3x^3 + 4}{2x^3 + x - 1}$
- ⋆ **Extra problem 3**: $\lim_{x\to-\infty} (1 + e^x) \cdot \frac{x^2 + 3x}{x^2 + 1}$

⋆ **Extra problem 4**: $\lim_{x\to-\infty}\left(e^x+\frac{x^2+3x^3}{x^3-x+1}\right)$

⋆ **Extra problem 5**: $\lim_{x\to-\infty}\frac{\sqrt{x^2+1}-x-1}{x+3}$

⋆ **Extra problem 6**: $\lim_{x\to-\infty}\frac{\sqrt{x^2-x}+x}{x+1}$

⋆ **Extra problem 7**: $\lim_{x\to-\infty}\frac{\sqrt{x^2+x+3}}{2x+1}$.

183 Different horizontal asymptotes in plus and minus infinity.

Example: Show with the earlier computations that the following function has different asymptotes on both sides: $f(x)=\frac{\sqrt{x^2+x+3}}{2x+1}$.

184 Limits, Problem 1.

Problem 1: Compute $\lim_{x\to\infty}\sqrt{x(x-\sqrt{x^2-1})}$ and $\lim_{x\to\infty}(x-\sqrt{x^2+5x})$.

Extra material: notes with solved Problem 1.

185 Limits, Problem 2.

Problem 2: Compute $\lim_{x\to-\infty}\arctan\frac{x}{\sqrt{1+x^2}}$ and $\lim_{x\to\infty}\arccos(\sqrt{x^2+x}-x)$.

Extra material: notes with solved Problem 2.

186 Limits, Problem 3.

Problem 3: Compute $\lim_{x\to\infty}\left(\frac{x-3}{x+2}\right)^{2x+1}$ and $\lim_{x\to\infty}x(\ln(1+x)-\ln x)$.

Extra material: notes with solved Problem 3.

187 Limits, Problem 4.

Problem 4:

- Show that $\lim_{x\to0^+}x^\alpha\ln x=0$ if $\alpha>0$
- Compute $\lim_{x\to0^+}x^x$.

Extra material: notes with solved Problem 4.

188 Limits, Problem 5.

Problem 5: Show that the following limits don't exist:

$$\lim_{x\to\infty}\tan x,\quad \lim_{n\to\infty}\tan n,\quad \lim_{x\to\infty}\sin x.$$

Extra material: notes with solved Problem 5.

189 Limits, Problem 6.

Problem 6: Compute $\lim_{x\to2}\frac{x-3}{x^2-4x+4}$ and $\lim_{x\to2}\frac{x-2}{x^2-4x+4}$.

Extra material: notes with solved Problem 6.

Extra material: an article with more solved problems on limits and continuity of trigonometric and inverse trigonometric functions.

⋆ **Extra problem 1**: $\lim_{x\to3}\sin\left(\frac{\pi x-3\pi}{x^2-4x+3}\right)$

⋆ **Extra problem 2**: $\lim_{x\to\infty} \cos\left(\frac{5x^3-2x^2}{x^4-x+1}\right)$

⋆ **Extra problem 3**: $\lim_{x\to-\infty} \tan\left(\frac{x^3+5x^2}{x^4}\right)$

⋆ **Extra problem 4**: $\lim_{x\to\infty} \arctan\left(\frac{17+2x-4x^2+\sqrt{3}x^3}{23-5x^2+x^3}\right)$

⋆ **Extra problem 5**: $\lim_{x\to\infty} \arcsin\left(\frac{x^3-4x^2+13}{\sqrt{2}x^3+7x^2}\right)$

⋆ **Extra problem 6**: $\lim_{x\to\infty} \arctan\left(\frac{x^3-2x}{2^{-x}+1}\right)$

⋆ **Extra problem 7**: $\lim_{x\to\infty} \arctan\left(\frac{\sqrt{x^4+3x^3+2}}{x^2+1}\right)$

⋆ **Extra problem 8**: $\lim_{x\to-\infty} \frac{x-\sqrt{x^2+1}}{x\cdot\arctan x}$

⋆ **Extra problem 9**: $\lim_{x\to\infty} \frac{x+\arctan e^x}{x^2-x}$.

S8 Continuity and discontinuities

You will learn: continuous extensions and examples of removable discontinuity; piece-wise functions and their continuity or discontinuities.

Read along with this section: **Calculus book**: Chapter 2 *Limits*.

190 Three types of discontinuities, continuous extensions, and some warnings.

191 Continuity and discontinuities, Problem 1.

Problem 1: Determine the value of $k \in \mathbb{R}$ for which the following function is continuous:

$$f(x) = \begin{cases} 2-x^2\,, & x \geqslant 2 \\ k-3x\,, & x < 2 \end{cases}$$

Extra material: notes with solved Problem 1.

192 Continuity and discontinuities, Problem 2.

Problem 2: Function f has a removable discontinuity at $x=0$. Assign it a new value in zero making the function continuous:

$$f(x) = \begin{cases} \frac{\ln(1+3x)}{x}\,, & x \neq 0 \\ 2\,, & x = 0 \end{cases}$$

Extra material: notes with solved Problem 2.

193 Continuity and discontinuities, Problem 3.

Problem 3: Is it possible to choose such numbers A and B that the following function is continuous?

$$f(x) = \begin{cases} \frac{\sin 3x}{x}\,, & x < 0 \\ Ax+B\,, & 0 \leqslant x \leqslant \frac{\pi}{2} \\ \sin x\,, & x > \frac{\pi}{2} \end{cases}$$

Extra material: notes with solved Problem 3.

194 Continuity and discontinuities, Problem 4.

Problem 4: Let $f(x) = \dfrac{x^2 - 2x + 1}{x^3 + 2x - 3}$ for $x \neq 1$. Is it possible to find a continuous extension of f to the entire $\mathbb{R}$?

Extra material: notes with solved Problem 4.

195 Continuity and discontinuities, Problem 5.

Problem 5: Determine the values of the constants A and B for which $\lim_{x\to 0} f(x) = 3$ and $\lim_{x\to\infty} f(x) = \pi$, where

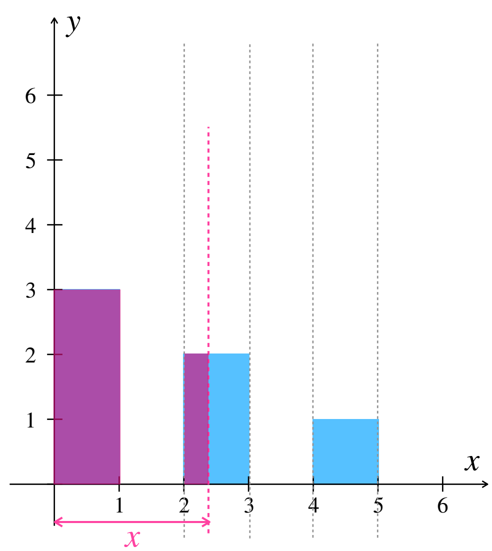
$$f(x) = \frac{\sin 2x}{x} + A \arctan x + B.$$

Extra material: notes with solved Problem 5.

196 Continuity and discontinuities, Problem 6.

Problem 6: Three rectangles: one 1×3, one 1×2, and one 1×1, are situated in the coordinate system as in the picture below. We define the following function $f : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$: for each $x \in \mathbb{R}^+ \cup \{0\}$ the value of $f(x)$ is equal to the total area of these rectangles between the y-axis and the vertical line through the point $(x, 0)$, i.e., the shadowed area in the picture. Write the formula for f and determine whether the function is continuous or not.

Extra material: notes with solved Problem 6.



197 Continuity and discontinuities, Problem 7.

Problem 7: Prove the following lemma and analyze the following function:

* **Lemma**: Let $a, b > 0$. Show that $\lim_{n\to\infty} \sqrt[n]{a^n + b^n} = \max\{a, b\}$.
* Plot the following function and discuss its continuity: $f(x) = \lim_{n\to\infty} \sqrt[n]{1 + x^{2n}}$.

Extra material: notes with solved Problem 7.

198 Continuity and discontinuities, Problem 8.

Problem 8: Plot the following function and discuss its continuity:

$$f(x) = \lim_{n\to\infty} \frac{x}{1 + (2\sin x)^{2n}}.$$

Extra material: notes with solved Problem 8.

199 **Optional**: Examples of functions with various numbers of discontinuity points.

Examples:

- Function with discontinuity points in all integer numbers, and continuous at all the other points: $f(x) = \lfloor x \rfloor$.
- If $f(x) = \chi_{\mathbb{Q}}$ restricted to $[0,1]$, then f is discontinuous at **all** $a \in [0,1]$.
- To get a function with any natural number N of discontinuity points, pick any N real numbers (different from each other) and construct a piece-wise linear function on the $N+1$ intervals they divide the number axis into; do it in such a way that the ends don't meet at any of the N partition points.
- Thomae's function: continuous at all the irrational numbers, discontinuous at all the rational numbers.
- If $A = \{\frac{1}{n};\ n \in \mathbb{N}^+\}$ and (y_n) is any sequence of real numbers, then the function $f : A \to \mathbb{R}$ defined as $f(\frac{1}{n}) = y_n$ is continuous. It can be extended to a continuous function on $A \cup \{0\}$ only if (y_n) is convergent (we put then $f(0) = y_0$ where $y_0 = \lim\limits_{n\to\infty} y_n$), otherwise not.

200 Piece-wise functions where the ends will always meet.

Problem 9: Plot the following function and discuss its continuity: $f(x) = |x+2| + |x| + |x-2| + |x-3|$.

Extra material: notes with solved Problem 9.

S9 Properties of continuous functions

You will learn: basic properties of continuous functions: The Boundedness Theorem, The Max-Min Theorem, The Intermediate-Value Theorem; you will learn the formulation and the meaning of these theorems, together with their proofs (in both written text and illustrations) and examples of their applications; we will revisit some old examples from the Precalculus series where we used these properties without really knowing them in a formal way (but well relying on our intuition, which is not that bad at a Precalculus level); uniform continuity; a characterisation of continuity with help of open sets.

Read along with this section: **Calculus book**: Chapter 2 *Limits*.

201 Briefly about important properties of functions continuous on intervals. (See the article under V40.)

202 Separation lemma.

Lemma: **About separation**: If $f : \mathbb{R} \to \mathbb{R}$ is continuous in a (where a is included in D_f together with some neighbourhood), then:

(1) If $\mu < f(a)$ then there exists $\delta_\mu > 0$ s.t. $f(x) > \mu$ for such x from D_f that $|x-a| < \delta_\mu$.

(2) If $\nu > f(a)$ then there exists $\delta_\nu > 0$ s.t. $f(x) < \nu$ for such x from D_f that $|x-a| < \delta_\nu$.

203 The Boundedness Theorem.

Theorem 3 (The Boundedness Theorem): If f is continuous on $[a,b]$, then f is bounded there, i.e., there exists a constant $M > 0$ such that $|f(x)| \leqslant M$ for all $x \in [a,b]$.

204 The Max-Min Theorem.

Theorem 4 (The Max-Min Theorem): Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then f attains its minimum and its maximum, i.e., there exist such $x_1, x_2 \in [a,b]$ that $f(x_1) = \sup V_f$ and $f(x_2) = \inf V_f$, where V_f denotes the range of f on $[a,b]$.

205 Halving intervals.

206 The Intermediate-Value Theorem.

Theorem 5 (Darboux): **The Intermediate-Value Theorem (IVT)**: If $f : [a,b] \to \mathbb{R}$ is continuous, and d is a real number between $f(a)$ and $f(b)$, then there exists at least one $c \in [a,b]$ s.t. $f(c) = d$.

Corollary 1: Each polynomial of an **odd** degree has at least one real zero.

Corollary 2 (Cauchy's Th.): If $f : [a,b] \to \mathbb{R}$ is continuous and $f(a)f(b) < 0$ then f has a zero on (a,b).

207 Some examples from Precalculus.

208 Properties of continuous functions, Problem 1.

Problem 1: **Fix-Point Th.**: If $f : [0,1] \to [0,1]$ is continuous, then there exists $x_0 \in [0,1]$ s.t. $f(x_0) = x_0$.

Extra material: notes with solved Problem 1.

209 Properties of continuous functions, Problem 2.

Problem 2: If $f, g : [a,b] \to \mathbb{R}$ are continuous and $f(a) < g(a)$ and $f(b) > g(b)$, then there exists $x_0 \in [a,b]$ s.t. $f(x_0) = g(x_0)$, i.e., the equation $f(x) = g(x)$ has at least one solution in $[a,b]$.

Extra material: notes with solved Problem 2.

210 Properties of continuous functions, Problem 3.

Problem 3:

* Show that the equation $x^3 + 1 = 3x$ has 3 real roots.
* Show that the polynomial $p(x) = x^3 - 15x + 1$ has 3 real zeros in the interval $[-4, 4]$.

Extra material: notes with solved Problem 3.

211 Properties of continuous functions, Problem 4.

Problem 4: Prove the following statement: If $f : [a, +\infty) \to \mathbb{R}$ is continuous and such that $\lim\limits_{x\to\infty} f(x) = L (\in \mathbb{R})$ then f is bounded. (Compare with P1 proven in V96.)

Extra material: notes with solved Problem 4.

212 Properties of continuous functions, Problem 5.

Problem 5: Can $(0,1]$, $(0,1)$, $[0,+\infty)$ or $[0,1] \cup [2,3]$ be the range of a continuous $f : [a,b] \to \mathbb{R}$? (A more general formulation of the IVT from V206: all the values between min and max of f on $[a,b]$ are also attained.)

Extra material: notes with solved Problem 5.

213 Properties of continuous functions, Problem 6.

Test: Answer the following questions using examples (or counterexamples) from our earlier videos:

a) Is each continuous function $f : (a,b) \to \mathbb{R}$ bounded?
b) Is each continuous function $f : [a,b] \to \mathbb{R}$ bounded?
c) Is each bounded function continuous?
d) Does each bijection need to be continuous?
e) Can a sum of two unbounded functions be bounded?
f) Can a sum of two continuous functions be discontinuous?
g) Can a sum of two discontinuous functions be continuous?
h) Let $f(x) < g(x)$ in some neighbourhood of a, and $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} g(x) = M$. Is it necessary that $L < M$? (Compare with L7 in V138.)

Extra material: notes from the iPad.

214 Uniform continuity.

Prove the first three facts and wait with the fourth one for *Calculus 2, part 1 of 2: Integrals with applications*:

* Function $f(x) = 2x + 1$ is uniformly continuous.
* Function $f(x) = \sqrt{x}$ is uniformly continuous.
* Function $f(x) = \frac{1}{x}$ is **not** uniformly continuous.
* Each continuous function $f : [a,b] \to \mathbb{R}$ is uniformly continuous.

215 **Future**: Open, closed, compact, and connected sets in metric spaces.

216 **Future**: Reformulation of the three important theorems in new terms.

217 **Advanced**: Three characterisations of continuous functions.

S10 Starting graphing functions

You will learn: how to start the process of graphing real-valued functions of one real variable: determining the domain and its accumulation points, determining the behaviour of the function around the accumulation points of the domain that are not included in the domain, determining points of discontinuity and one-sided limits in them, determining asymptotes. We will continue working with this subject in *Calculus 1, part 2 of 2: Derivatives with applications.*

Read along with this section: **Calculus book**: Chapter 2 *Limits*.

218 A todo list for plotting functions: for now and for the future.

219 Slant asymptotes, and a clarity for rational functions.

An example with different slant asymptotes: $f(x) = \frac{3x}{10} + \arctan x$.

220 A rational function, Problem 1.

Problem 1: Determine the domain and asymptotes for $f(x) = \frac{x^2+2x+5}{x+1}$.

Extra material: notes with solved Problem 1.

221 Different horizontal asymptotes in the infinities, Problem 2.

Problem 2: Determine the domain and asymptotes for $f(x) = \sqrt{x^2+x} - \sqrt{x^2+1}$.

Extra material: notes with solved Problem 2.

222 Asymptotes, Problem 3.

Problem 3: Determine the domain and asymptotes for $f(x) = \arctan\left(\frac{x}{x+1}\right)$.

Extra material: notes with solved Problem 3.

223 Asymptotes, Problem 4.

Problem 4: Determine the domain and asymptotes for $f(x) = \frac{\arcsin x}{\tan(x+1)}$.

Extra material: notes with solved Problem 4.

224 One-sided vertical asymptote, Problem 5.

Problem 5: Determine the domain and asymptotes for $f(x) = e^{1/(1+x)}$.

225 Wrap-up Calculus 1, part 1 of 2.

S11 Extras

You will learn: about all the courses we offer, and where to find discount coupons. You will also get a glimpse into our plans for future courses, with approximate (very hypothetical!) release dates.

B Bonus lecture.

Extra material 1: a pdf with all the links to our courses, and coupon codes.

Extra material 2: a pdf with an advice about optimal order of studying our courses.

Extra material 3: a pdf with information about course books, and how to get more practice.